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ON MEASURABLE SIERPIŃSKI-ZYGMUND FUNCTIONS

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Abstract. It is proved that there exists a Sierpiński-Zygmund function, which is measurable with respect to a certain invariant extension of the Lebesgue measure on the real line $\mathbb{R}$.

Let E be a nonempty set and let $f\colon E \to \mathbb{R}$ be a function.

We say that f is absolutely nonmeasurable if f is nonmeasurable with respect to any nonzero σ-finite diffused (i.e., continuous) measure μ defined on a σ-algebra of subsets of E.

Recall that a set $X \subset \mathbb{R}$ is universal measure zero if, for every σ-finite diffused Borel measure μ on $\mathbb{R}$, the equality $\mu^*(X) = 0$ is satisfied, where μ^* denotes, as usual, the outer measure associated with μ.

It is well known that there exist uncountable universal measure zero subsets of $\mathbb{R}$ (see, e.g., [8]).

We have a characterization of absolutely nonmeasurable functions in terms of universal measure zero sets and preimages of singletons.

Theorem 1. *Let $f\colon E \to \mathbb{R}$ be a function. The following two assertions are equivalent:*

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1) *f is absolutely nonmeasurable;*
2) *the range of f is a universal measure zero subset of $\mathbb{R}$ and $\mathrm{card}(f^{-1}(t)) \leq \omega$ for each point $t \in \mathbb{R}$.*

For the proof of this statement, see [5].

Let $\mathbf{c}$ denote the cardinality of the continuum. We would like to mention the following two straightforward consequences of Theorem 1.

I. If $\mathrm{card}(E) > \mathbf{c}$, then there are no absolutely nonmeasurable functions $f\colon E \to \mathbb{R}$;
II. If the cardinality of any universal measure zero subset of $\mathbb{R}$ is strictly less than $\mathbf{c}$, then there are no absolutely nonmeasurable functions $f\colon \mathbb{R} \to \mathbb{R}$.

The latter consequence shows us that absolutely nonmeasurable functions $f\colon \mathbb{R} \to \mathbb{R}$ exist if and only if there exists a universal measure zero set $X \subset \mathbb{R}$ with $\mathrm{card}(X) = \mathbf{c}$. More generally, absolutely nonmeasurable functions $f\colon E \to \mathbb{R}$ exist if and only if there exists a universal measure zero subset X of $\mathbb{R}$ with $\mathrm{card}(X) = \mathrm{card}(E)$.

This also implies that the existence of an absolutely nonmeasurable function acting from $\mathbb{R}$ into $\mathbb{R}$ cannot be established within the theory **ZFC**. Indeed, there are models of set theory in which the negation of the Continuum Hypothesis holds and the cardinality of each universal measure zero subset of $\mathbb{R}$ does not exceed the first uncountable cardinal ω_1 (see, e.g., [8]). Clearly, in such a model we do not have absolutely nonmeasurable functions acting from $\mathbb{R}$ into $\mathbb{R}$.

At the same time, by assuming Martin's Axiom and applying some properties of so-called generalized Luzin subsets of $\mathbb{R}$, it can be proved that there are injective additive absolutely nonmeasurable functions acting from $\mathbb{R}$ into $\mathbb{R}$. In other words, under Martin's Axiom, there exist absolutely nonmeasurable solutions of the Cauchy functional equation (see, for example, [5]).

Sierpiński and Zygmund constructed in their classical work [11] a function $f\colon \mathbb{R} \to \mathbb{R}$ having the following property: for each subset Y of $\mathbb{R}$ with $\mathrm{card}(Y) = \mathbf{c}$, the restriction $f|Y$ is not continuous on Y.

The above-mentioned result of Sierpiński and Zygmund was essentially motivated by the theorem of Blumberg [1] stating that, for any function $g\colon \mathbb{R} \to \mathbb{R}$, there exists an everywhere dense subset D of $\mathbb{R}$ such that the restriction $g|D$ is continuous on D. Obviously, the set D being dense in $\mathbb{R}$ is infinite. The existence of a Sierpiński-Zygmund function shows that one cannot assert the uncountability of D.

There are many works devoted to Sierpiński-Zygmund functions. In those works various constructions are presented which yield further interesting examples of Sierpiński-Zygmund functions with additional properties important from the viewpoint of real analysis (see, e.g., [9] and references therein).

It is not difficult to prove that every Sierpiński-Zygmund function f turns out to be nonmeasurable with respect to the completion of any nonzero σ-finite diffused Borel measure on $\mathbb{R}$ (see, for instance, [4, Chapter 6]). At the same time, we cannot assert that f is absolutely nonmeasurable because it may happen (in those models of set theory where the Continuum Hypothesis fails to be true) that $\mathbf{c} > \mathrm{card}(f^{-1}(t)) > \omega$ for some point $t \in \mathbb{R}$ and, by virtue of Theorem 1, in this case f will be measurable with respect to a certain nonzero σ-finite diffused measure on $\mathbb{R}$.

The present paper is devoted to related measurability properties of Sierpiński-Zygmund functions. The main goal of the paper is to demonstrate that there are Sierpiński-Zygmund functions measurable with respect to some invariant extensions of the standard Lebesgue measure λ on $\mathbb{R}$.

For this purpose, we need several auxiliary propositions.

Lemma 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the following condition: for each subset Y of $\mathbb{R}$ with* $\mathrm{card}(Y) = \mathbf{c}$*, the restriction $f|Y$ is not a Borel mapping. Then f is a Sierpiński-Zygmund function.*

This lemma is trivial. However, it plays an essential role for our considerations below. In this context, let us point out that in Chapter 5 of [4] a construction of a function f satisfying the above-mention condition is discussed in detail.

As usual, we denote by $\mathbb{T}$ the one-dimensional unit torus in the Euclidean plane $\mathbb{R}^2$. Recall that $\mathbb{T}$ is a commutative compact topological group canonically isomorphic to the group of all rotations of the plane $\mathbb{R}^2$ about its origin.

Lemma 2. *Let $\phi\colon \mathbb{R} \to \mathbb{T}$ be the canonical surjective continuous group homomorphism defined by*

$$\phi(x) = (\cos x, \sin x) \quad (x \in \mathbb{R}).$$

There exists a mapping $\psi\colon \mathbb{T} \to \mathbb{R}$ such that:

1) *$\phi \circ \psi$ coincides with the identity transformation of $\mathbb{T}$;*
2) *ψ is discontinuous only at one point of $\mathbb{T}$ (hence, ψ is a Borel mapping).*

We omit an easy proof of Lemma 2 (actually, this simple lemma is a very particular case of the theorem of Kuratowski and Ryll-Nardzewski [7] on measurable selectors).

The torus $\mathbb{T}$ being a compact topological group carries the Haar probability measure ν. In fact, the completion of ν coincides with the Lebesgue probability measure on $\mathbb{T}$ invariant under all translations of $\mathbb{T}$.

As usual, we denote by $\lambda \times \nu$ the product measure whose multipliers are λ and ν. For our further purposes, it will be convenient to denote by the same symbol $\lambda \times \nu$ the completion of the above-mentioned product measure.

We say that a function $f\colon \mathbb{R} \to \mathbb{T}$ has thick graph if every $(\lambda \times \nu)$-measurable set Z with $(\lambda \times \nu)(Z) > 0$ intersects the graph of f.

Lemma 3. *Let $f\colon \mathbb{R} \to \mathbb{T}$ be a group homomorphism whose graph is thick. For any set $Z \in \mathrm{dom}(\lambda \times \nu)$, let us put*

$$Z' = \{x \in \mathbb{R}\colon (x, f(x)) \in Z\}$$

and define the class of sets

$$\mathcal{S} = \{Z'\colon Z \in \mathrm{dom}(\lambda \times \nu)\}.$$

Finally, define a functional μ on $\mathcal{S}$ by the formula

$$\mu(Z') = (\lambda \times \nu)(Z) \quad (Z' \in \mathcal{S}).$$

Then the following relations hold:

1) *$\mathcal{S}$ is a σ-algebra of subsets of $\mathbb{R}$ containing $\mathrm{dom}(\lambda)$ and invariant under the group of all isometric transformations of $\mathbb{R}$;*
2) *the functional μ is well-defined;*
3) *μ is a measure on $\mathcal{S}$ extending λ and invariant under the same group of all isometries of $\mathbb{R}$;*
4) *the original homomorphism f is measurable with respect to μ.*

Proof. We use the argument given in [6] (cf. also [3]).

If $Z \in \mathrm{dom}(\lambda \times \nu)$ and $Z_i \in \mathrm{dom}(\lambda \times \nu)$ for each index i from a countable set I, then we can write

$$\mathbb{R} \setminus Z' = \{x \in \mathbb{R}\colon (x, f(x)) \in \mathbb{R}^2 \setminus Z\},$$
$$\bigcup\{Z'_i\colon i \in I\} = \{x \in \mathbb{R}\colon (x, f(x)) \in \bigcup\{Z_i\colon i \in I\}\},$$

which shows us that $\mathcal{S}$ is a σ-algebra of subsets of $\mathbb{R}$.

The invariance of $\mathcal{S}$ under the group of all isometries of $\mathbb{R}$ follows directly from the invariance of $\mathrm{dom}(\lambda \times \nu)$ under all translations of the product group $\mathbb{R} \times \mathbb{T}$ and under the symmetry of this group.

If, for a set $Z' \in \mathcal{S}$, we have simultaneously $Z' = \{x \in \mathbb{R}\colon (x, f(x)) \in Z_1\}$ and $Z' = \{x \in \mathbb{R}\colon (x, f(x)) \in Z_2\}$, where Z_1 and Z_2 are some $(\lambda \times \nu)$-measurable sets, then

$$\{x \in \mathbb{R}\colon (x, f(x)) \in Z_1 \triangle Z_2\} = \emptyset.$$

In view of the thickness of the graph of f, we claim that $(\lambda\times\nu)(Z_1\triangle Z_2)=0$, which also implies that $(\lambda\times\nu)(Z_1)=(\lambda\times\nu)(Z_2)$. This establishes the correctness of the definition of μ.

In a similar way, we prove the countable additivity of μ and its invariance under the group of all isometries of $\mathbb{R}$.

Further, if X is an arbitrary λ-measurable set, then we can write

$$X=\{x\in\mathbb{R}\colon (x,f(x))\in X\times\mathbb{T}\},\quad X\in\mathcal{S},\quad \mu(X)=(\lambda\times\nu)(X\times\mathbb{T})=\lambda(X)$$

and, therefore, the measure μ extends λ.

Finally, for any Borel set $B\subset\mathbb{T}$, we have

$$f^{-1}(B)=\{x\in\mathbb{R}\colon (x,f(x))\in\mathbb{R}\times B\},\quad \mathbb{R}\times B\in\mathrm{dom}(\lambda\times\nu),\quad f^{-1}(B)\in\mathcal{S},$$

which shows that f is measurable with respect to μ.

This completes the proof of Lemma 3. □

Lemma 4. *Let $(G,+)$ be an infinite commutative divisible group such that, for any nonzero natural number n and for any element $g\in G$, the equation $nx=g$ has at most countably many solutions in G. Let A be an arbitrary subset of G. There exists a subgroup $[A]$ of G satisfying the following conditions:*

1) $A\subset[A]$;
2) $\mathrm{card}([A])=\mathrm{card}(A)+\omega$;
3) *for each nonzero natural number n and for each element $g\in[A]$, all solutions of the equation $nx=g$ belong to $[A]$ (in particular, $[A]$ is a divisible subgroup of G).*

Proof. Construct by recursion a sequence $(G_0,G_1,\dots,G_k,\dots)$ of subgroups of G. First of all, put:

$G_0=$ the group generated by A.

Suppose that the subgroup G_k has already been defined satisfying the equality $\mathrm{card}(G_k)=\mathrm{card}(A)+\omega$. Denote by X_k the set of all solutions of the equations

$$nx=g\quad(n=1,2,\dots;\ g\in G_k).$$

Taking into account the assumption of the lemma, we get

$$\mathrm{card}(X_k)=\mathrm{card}(G_k)+\omega=\mathrm{card}(A)+\omega.$$

Let us put:

$G_{k+1}=$ the group generated by the set $G_k\cup X_k$.

Proceeding in this way, we obtain the increasing sequence $(G_0,G_1,\dots,G_k,\dots)$ of subgroups of G.

Finally, denote $[A]=\bigcup\{G_k\colon k<\omega\}$. A straightforward verification shows that the group $[A]$ is the required one. □

Remark 1. Lemma 4 has a direct analog for infinite noncommutative divisible groups $(G, \cdot)$.

Lemma 5. *There exists a group homomorphism*

$$f \colon \mathbb{R} \to \mathbb{T}$$

possessing the following properties:

1) *the graph of f is thick;*
2) *for any uncountable Borel subset Y of $\mathbb{R}$ and for any Borel mapping $h \colon Y \to \mathbb{T}$, we have the inequality*

$$\mathrm{card}(\{x \in \mathbb{R} \colon f(x) = h(x)\}) < \mathbf{c}.$$

Proof. We shall construct the required homomorphism f by using the method of transfinite recursion.

Let α denote the least ordinal number of cardinality $\mathbf{c}$.

Let $\preceq$ be a well-ordering of $\mathbb{R}$ isomorphic to α.

Let $\{h_\xi \colon \xi < \alpha\}$ be an enumeration of all Borel mappings acting from uncountable Borel subsets of $\mathbb{R}$ into $\mathbb{T}$.

Let $\{Z_\xi \colon \xi < \alpha\}$ be an enumeration of all those Borel subsets of $\mathbb{R} \times \mathbb{T}$ whose $(\lambda \times \nu)$-measure is strictly positive. We may assume, without loss of generality, that the range of the partial family

$$\{Z_\xi \colon \xi < \alpha,\ \xi \text{ is an odd ordinal}\}$$

coincides with the range of the entire family $\{Z_\xi \colon \xi < \alpha\}$.

Under this notation, we are going to define four α-sequences

$$\{x_\xi \colon \xi < \alpha\}, \quad \{y_\xi \colon \xi < \alpha\},$$
$$\{V_\xi \colon \xi < \alpha\}, \quad \{f_\xi \colon \xi < \alpha\}$$

satisfying the following relations:

(a) $\{x_\xi \colon \xi < \alpha\}$ is a Hamel basis of $\mathbb{R}$;
(b) for each ordinal $\xi < \alpha$, the set V_ξ is the vector subspace of $\mathbb{R}$, over the field $\mathbb{Q}$ of all rationals, generated by $\{x_\zeta \colon \zeta \leq \xi\}$;
(c) for each ordinal $\xi < \alpha$, we have the group homomorphism $f_\xi \colon V_\xi \to \mathbb{T}$ such that $f_\xi(x_\xi) = y_\xi$;
(d) if $\zeta < \xi < \alpha$, then f_ξ extends f_ζ;
(e) for each odd ordinal $\xi < \alpha$, we have $(x_\xi, y_\xi) \in Z_\xi$;
(f) if $\xi < \alpha$, then we have

$$f_\xi(qx_\xi + v) \neq h_\zeta(qx_\xi + v)$$

for all $\zeta < \xi$, $q \in \mathbb{Q} \setminus \{0\}$, $v \in \bigcup\{V_\zeta \colon \zeta < \xi\}$, $qx_\xi + v \in \mathrm{dom}(h_\zeta)$.

Suppose that, for an ordinal $\xi < \alpha$, the partial ξ-sequences

$$\{x_\zeta : \zeta < \xi\}, \quad \{y_\zeta : \zeta < \xi\},$$
$$\{V_\zeta : \zeta < \xi\}, \quad \{f_\zeta : \zeta < \xi\}$$

have already been constructed. Let us put:

$$V' = \bigcup\{V_\zeta : \zeta < \xi\}, \quad f' = \bigcup\{f_\zeta : \zeta < \xi\}.$$

Applying (d), we claim that f' is a group homomorphism from V' into $\mathbb{T}$. Now, consider two possible cases.

1. The ordinal ξ is even. In this case, let x be the least element (with respect to $\preceq$) of $\mathbb{R} \setminus V'$ and let

$$A = \{h_\zeta(qx + v) : \zeta < \xi, q \in \mathbb{Q} \setminus \{0\}, v \in V', qx + v \in \mathrm{dom}(h_\zeta)\} + f'(V').$$

According to Lemma 4, we have

$$\mathrm{card}([A]) = \mathrm{card}(A) + \omega \le \mathrm{card}(\xi) + \omega < \mathbf{c}.$$

Choose an element $y \in \mathbb{T} \setminus [A]$ and put $x_\xi = x$, $y_\xi = y$. Taking into account the fact that $\mathbb{T}$ is a divisible group, we can extend f' to a group homomorphism

$$f_\xi : V_\xi \to \mathbb{T}$$

satisfying the condition $f_\xi(x_\xi) = y_\xi$.

2. The ordinal ξ is odd. In this case, we apply the classical Fubini theorem to the set Z_ξ and choose an element $x \in \mathbb{R} \setminus V'$ such that

$$\nu(\{t \in \mathbb{T} : (x, t) \in Z_\xi\}) > 0.$$

Further, we choose an element

$$y \in \{t \in \mathbb{T} : (x, t) \in Z_\xi\} \setminus [A]$$

(where A is defined as in the case 1) and put again $x_\xi = x$, $y_\xi = y$. Obviously, we have $(x_\xi, y_\xi) \in Z_\xi$. As above, we define a group homomorphism $f_\xi : V_\xi \to \mathbb{T}$ extending f' and satisfying the condition $f_\xi(x_\xi) = y_\xi$.

Proceeding in this manner, we will be able to construct the required α-sequences. Now, we put

$$f = \bigcup\{f_\xi : \xi < \alpha\}.$$

In view of (a), this f is a group homomorphism from $\mathbb{R}$ into $\mathbb{T}$.

In view of (e), the graph of f is thick in the product space $\mathbb{R} \times \mathbb{T}$.

Finally, relation (f) implies that the inequality

$$\mathrm{card}(\{x \in \mathbb{R} : f(x) = h_\xi(x)\}) < \mathbf{c}$$

holds for all ordinals $\xi < \alpha$. This also yields that, for any set $X \subset \mathbb{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restriction of f to X is not a Borel mapping. To show the last fact, suppose otherwise, i.e., suppose that $f|X$ is Borel. Then, according to a well-known statement of classical descriptive set theory (see, e.g., [4, Chapter 5]), the function $f|X$ can be extended to a $\mathbb{T}$-valued Borel function f^* defined on a Borel subset of $\mathbb{R}$. Since $\mathrm{card}(\mathrm{dom}(f^*)) = \mathbf{c}$, we claim that $f^* = h_\xi$ for some ordinal $\xi < \alpha$. But this is impossible in view of the above inequality.

Lemma 5 has thus been proved. □

Theorem 2. *There exists a Sierpiński-Zygmund function*

$$\chi\colon \mathbb{R} \to \mathbb{R}$$

measurable with respect to some invariant extension of Lebesgue measure λ*.*

Proof. Let f be as in Lemma 5. By virtue of Lemma 3, f is measurable with respect to a certain measure μ on $\mathbb{R}$ which extends λ and is invariant under the group of all isometries of $\mathbb{R}$. Let us put

$$\chi = \psi \circ f,$$

where $\psi\colon \mathbb{T} \to \mathbb{R}$ denotes the Borel mapping of Lemma 2.

We assert that χ is the required Sierpiński-Zygmund function.

Indeed, χ is a μ-measurable function as the composition of two functions, first of which is μ-measurable and the second one is Borel.

Let X be a subset of $\mathbb{R}$ with $\mathrm{card}(X) = \mathbf{c}$. It suffices to show that the restriction $\chi|X$ is not a Borel mapping. Suppose otherwise, i.e., suppose that $\chi|X$ is Borel. Then

$$\phi \circ \chi|X = \phi \circ \psi \circ f|X = f|X$$

must be Borel, too, which is impossible in view of Lemma 5.

The contradiction obtained ends the proof of our theorem. □

Remark 2. We cannot assert that the function χ of Theorem 1 is additive. In this context, the following question seems to be interesting: does there exist an additive Sierpiński-Zygmund function acting from $\mathbb{R}$ into $\mathbb{R}$ and measurable with respect to some invariant extension of λ?

Note that, by applying the method described above, it is possible to prove that there exists an additive Sierpiński-Zygmund function acting from $\mathbb{R}$ into $\mathbb{R}$ and measurable with respect to some quasiinvariant extension of λ.

Replacing $\mathbb{R}$ by a Polish topological vector space equipped with a nonzero σ-finite diffused Borel measure, we come to a certain analog of Theorem 2.

Theorem 3. *Let $E \neq \{0\}$ be a Polish topological vector space and let μ be a nonzero σ-finite diffused Borel measure on E. There exist a measure μ' on E extending μ and a Sierpiński-Zygmund function $\chi\colon E \to \mathbb{R}$ measurable with respect to μ'. Moreover, if μ is invariant (quasiinvariant) under a group $G \subset E$, then μ' can also be chosen invariant (quasiinvariant) under the same G.*

The proof of this statement is similar to the proof of Theorem 2.

Remark 3. It is well known that if E is an infinite-dimensional Polish topological vector space, then there exists no nonzero σ-finite Borel measure on E quasiinvariant under the group of all translations of E. On the other hand, for some E, it is possible to construct a nonzero σ-finite Borel measure on E invariant under an everywhere dense vector subspace of E (see, e.g., [2]).

Remark 4. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary Sierpiński-Zygmund function. By assuming that all sets in $\mathbb{R}$ of cardinality strictly less than $\mathbf{c}$ are of first category, it is not difficult to show that, for any second category set $X \subset \mathbb{R}$, the restriction $\chi|X$ cannot be extended to a function $\chi^*\colon \mathbb{R} \to \mathbb{R}$ possessing the Baire property.

Similarly, let $f\colon \mathbb{R} \to \mathbb{R}$ be a Sierpiński-Zygmund function. By assuming that all sets in $\mathbb{R}$ of cardinality strictly less than $\mathbf{c}$ are of Lebesgue measure zero, it is not hard to prove that, for any set $X \subset \mathbb{R}$ of strictly positive outer Lebesgue measure, the restriction $f|X$ cannot be extended to a function $f^*\colon \mathbb{R} \to \mathbb{R}$ measurable in the Lebesgue sense.

In this context, we would like to recall that the Baire property is a certain topological analog of measurability (see, for instance, [10]).

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