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ON MULTIVALUED COSINE FAMILIES

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Abstract. Let K be a convex cone in a real Banach space. The main purpose of this paper is to show that for a regular cosine family $\{F_t\colon t \in \mathbb{R}\}$ of linear continuous multifunctions $F_t\colon K \to cc(X)$ there exists a linear continuous multifunction $H\colon K \to cc(K)$ such that

$$F_t(x) \subset \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} H^n(x).$$

Let X be a real normed vector space. We will denote by $n(X)$ the family of all nonempty subsets of X and by $cc(X)$ the family of all nonempty compact and convex subsets of X.

For $A, B \subset X$ and $t \in \mathbb{R}$ we introduce

$$A + B = \{a + b\colon a \in A,\ b \in B\}, \quad tA = \{ta\colon a \in A\}.$$

A subset K of X is called a *cone* if $tK \subset K$ for all $t \in (0, +\infty)$. A cone is said to be *convex* if it is a convex set.

Let A, B, C be sets of $cc(X)$. We say that the set C is the *Hukuhara difference* of A and B, i.e., $C = A - B$ if $B + C = A$. By the Rådström Lemma [12] it follows that if this difference exists, then it is unique.

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A multifunction $F\colon [a,b] \to cc(X)$ is called *concave* if

$$F(\lambda t + (1-\lambda)s) \subset \lambda F(t) + (1-\lambda)F(s)$$

for all $s, t \in [a,b]$ and $\lambda \in (0,1)$.

We say that a multifunction $F\colon J \to cc(X)$, where J denotes an interval in $\mathbb{R}$, is *increasing* if for all $s, t \in J$ such that $s < t$ we have $F(s) \subset F(t)$. If for $s < t$ we have the inverse inclusion $F(t) \subset F(s)$, then the multifunction is called *decreasing*.

We call $F\colon \mathbb{R} \to cc(X)$ *even* if $F(-t) = F(t)$ for every $t \in \mathbb{R}$.

Let K be a convex cone in X. A multifunction $F\colon K \to n(K)$ is called *linear* if

$$F(x+y) = F(x) + F(y), \quad F(\lambda x) = \lambda F(x)$$

for all $x, y \in K$ and $\lambda \geq 0$.

The *image* of a set $A \subset K$ by $F\colon K \to n(X)$ is the set

$$F(A) = \bigcup_{y \in A} F(y).$$

Let X, Y, Z be nonempty sets. The *superposition* $G \circ F$ of multifunctions $F\colon X \to n(Y)$ and $G\colon Y \to n(Z)$ we define by the formula

$$(G \circ F)(x) = G(F(x)) \quad \text{for } x \in X.$$

Let A, A_1, $A_2, \ldots$ be elements of the family $cc(X)$. We say that the sequence $(A_n)_{n \in \mathbb{N}}$ is convergent to A and we write $A_n \to A$ if $d(A, A_n) \to 0$, where d denotes the Hausdorff metric derived by the norm in X.

Lemma 1 ([11, Lemma 1])**.** *Let X be a real Banach space, $A, A_1, A_2, \ldots,$ $B, B_1, B_2, \ldots \in cc(X)$. If $A_n \to A$, $B_n \to B$ and there exist the Hukuhara differences $A_n - B_n$ in $cc(X)$ for $n \in \mathbb{N}$, then there exists the Hukuhara difference $A - B$ and $A_n - B_n \to A - B$.*

The norm $\|A\|$ of a bounded set $A \subset X$ is defined by

$$\|A\| := \sup\{\|a\|\colon a \in A\} = d(A, \{0\}).$$

Next we introduce the Hukuhara version of the Riemann integral of multifunction $F\colon [a,b] \to cc(X)$ (see [2]). We will denote by $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ a partition of the interval $[a,b]$, i.e., a sequence satisfying inequalities $a = \alpha_0 < \alpha_1 < \ldots < \alpha_n = b$. The number

$$\delta(\Delta) = \max\{\alpha_{i+1} - \alpha_i\colon i = 0, 1, \ldots, n-1\}$$

is said to be the *diameter* of Δ. Φ denotes the family of all pairs (Δ, τ), where $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is a partition of the interval $[a,b]$ and $\tau =$

$(\tau_0, \dots, \tau_{n-1})$ is a sequence of points such that $\tau_i \in [\alpha_i, \alpha_{i+1}]$. $I(\Delta, \tau)$ denotes the set

$$I(\Delta, \tau) = \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i) F(\tau_i)$$

for $(\Delta, \tau) \in \Phi$. If $I(\Delta, \tau)$ converges to $I \in cc(X)$ with respect to the Hausdorff metric d when $\delta(\Delta) \to 0$, i.e.,

$$(\forall \varepsilon > 0)\ (\exists \eta > 0)\ (\forall (\Delta, \tau) \in \Phi)\, (\delta(\Delta) < \eta \implies d(I(\Delta, \tau), I) < \varepsilon)\,,$$

then we say that I is the *integral* of the multifunction F on the interval $[a, b]$ and we write

$$I = \int_a^b F(t)\, dt.$$

If there exists the integral of a multifunction $F\colon [a, b] \to cc(X)$, then we say that F is *integrable*.

Next lemmas describe some properties of the Riemann integral for multifunctions.

Lemma 2 ([2, p. 212])**.** *If $a < c < b$ and $F\colon [a, b] \to cc(X)$ is integrable on $[a, c]$ and on $[c, b]$, then F is integrable on $[a, b]$ and*

$$\int_a^b F(t)\, dt = \int_a^c F(t)\, dt + \int_c^b F(t)\, dt. \tag{1}$$

Lemma 3 ([2, p. 212])**.** *Let X be a real Banach space. If $F\colon [a, b] \to cc(X)$ is integrable on $[a, b]$, then for every $c \in (a, b)$ F is integrable on $[a, c]$ and on $[c, b]$ and formula* (1) *holds.*

Lemma 4 ([10, Lemma 1.3])**.** *If $F\colon [a, b] \to cc(X)$ is integrable, a', b', A, B are real numbers such that $a' < b'$, $Aa' + B = a$, $Ab' + B = b$, then*

$$\int_a^b F(t)\, dt = A \int_{a'}^{b'} F(Au + B)\, du.$$

Lemma 5 ([10, Lemma 1.4])**.** *Let $F, G\colon [a, b] \to cc(X)$ be integrable. If $F(t) \subset G(t)$ for all $t \in [a, b]$, then*

$$\int_a^b F(t)\, dt \subset \int_a^b G(t)\, dt.$$

Lemma 6. *If $F\colon \mathbb{R} \to cc(X)$ is integrable on each interval $[a,b]$ and even, then*

$$\int_a^b F(t)\,dt = \int_{-b}^{-a} F(t)\,dt. \tag{2}$$

Proof. Let $\Delta = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ be a partition of the interval $[a,b]$ and $\tau_i \in [\alpha_i, \alpha_{i+1}]$ for $i \in \{0, 1, \dots, n-1\}$. Then $\overline{\Delta} = \{\beta_0, \beta_1, \dots, \beta_n\}$, where $\beta_i = -\alpha_{n-i}$ is partition of $[-b, -a]$. By the evenness of the multifunction F we have the equality of integral sums

$$I(\Delta, \tau) = \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i) F(\tau_i) = \sum_{i=0}^{n-1} (-\alpha_i - (-\alpha_{i+1}))\, F(-\tau_i) = I(\overline{\Delta}, \overline{\tau}),$$

where $\overline{\tau_i} = -\tau_{n-1-i}$, $\overline{\tau} = (\overline{\tau_0}, \overline{\tau_1}, \dots, \overline{\tau_{n-1}})$. Let (Δ_p) be a normal sequence of partitions of the interval $[a,b]$ and $(\overline{\Delta_p})$ be corresponding normal sequence of partitions of the interval $[-b,-a]$. Then $I(\Delta_p, \tau) = I(\overline{\Delta_p}, \overline{\tau})$, hence and by integrability of F we have (2). □

Lemma 7. *Let X be a real Banach space. If $F\colon [0,\infty) \to cc(X)$ is integrable on each interval $[a,b] \subset [0,\infty)$ where $0 \le a \le b < \infty$ and there exist the Hukuhara differences $F(t) - F(s)$ for $0 \le s \le t$, then for $u \ge 0$ there exists the Hukuhara difference*

$$\int_{a+u}^{b+u} F(t)\,dt - \int_a^b F(t)\,dt.$$

Proof. Let $\Delta = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ be a partition of the interval $[a,b]$ and $\tau_i \in [\alpha_i, \alpha_{i+1}]$ for $i \in \{0, 1, \dots, n-1\}$. Then $\Delta^u = \{\alpha_0 + u, \alpha_1 + u, \dots, \alpha_n + u\}$ is the partition of the interval $[a+u, b+u]$ and $\tau_i + u \in [\alpha_i + u, \alpha_{i+1} + u]$. We consider integral sums corresponding to these partitions

$$I(\Delta, \tau) = \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i) F(\tau_i)$$

and

$$I(\Delta^u, \tau_u) = \sum_{i=0}^{n-1} ((\alpha_{i+1} + u) - (\alpha_i + u))\, F(\tau_i + u),$$

where $\tau_u = (\tau_0 + u, \tau_1 + u, \dots, \tau_{n-1} + u)$. Since for $i \in \{0, 1, \dots, n-1\}$ the Hukuhara differences $F(\tau_i + u) - F(\tau_i)$ exist, it follows that there exists the

Hukuhara difference

$$I(\Delta^u, \tau_u) - I(\Delta, \tau) = \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i)(F(\tau_i + u) - F(\tau_i)).$$

Let (Δ_p) be a normal sequence of partitions of the interval $[a, b]$ and (Δ_p^u) be corresponding normal sequence of partitions of the interval $[a+u, b+u]$. Since F is integrable on each interval we have

$$I(\Delta_p, \tau) \to \int_a^b F(t)\, dt \quad \text{and} \quad I(\Delta_p^u, \tau_u) \to \int_{a+u}^{b+u} F(t)\, dt \quad \text{when } p \to \infty.$$

Hence and by Lemma 1 there exists the Hukuhara difference

$$\int_{a+u}^{b+u} F(t)\, dt - \int_a^b F(t)\, dt.$$

□

In next lemmas we assume that X is a real Banach space.

Lemma 8 ([2, p. 212])**.** *If $F\colon \mathbb{R} \to cc(X)$ is continuous, then it is integrable on each interval $[a, b] \subset \mathbb{R}$.*

Lemma 9 ([2, p. 211])**.** *If $F\colon [a, b] \to cc(X)$ is continuous, then*

$$\left\| \int_a^b F(t)\, dt \right\| \le \int_a^b \|F(t)\|\, dt.$$

Lemma 10. *If $F\colon \mathbb{R} \to cc(X)$ is continuous, then for every $a \in \mathbb{R}$ the multifunction*

$$H(t) = \int_a^t F(u)\, du \quad \text{for } t \ge a$$

is continuous.

Proof. Let $h > 0$ and $t \ge a$. By Lemmas 3 and 9 and properties of the Hausdorff metric we have

$$d(H(t), H(t+h)) = d\left(\int_a^t F(u)\, du, \int_a^t F(u)\, du + \int_t^{t+h} F(u)\, du \right)$$

$$= \left\| \int_t^{t+h} F(u)\,du \right\| \le \int_t^{t+h} \|F(u)\|\,du \le h \sup_{t\le u\le t+h} \|F(u)\|$$

and this tends to zero as $h \to 0^+$. So H is continuous. □

Lemma 11 ([2, p. 216])**.** *If $F\colon \mathbb{R} \to cc(X)$ is continuous, then*

$$\lim_{h\to 0^+} \frac{1}{h} \int_t^{t+h} F(u)\,du = F(t) \quad \textit{for } t \in \mathbb{R}.$$

Lemma 12. *If $F\colon [0,\infty) \to cc(X)$ is continuous, then*

$$\int_0^t \left(\int_0^s F(u)\,du \right) ds = \int_0^t (t-u)F(u)\,du \quad \textit{for } t \ge 0. \tag{3}$$

Proof. We define

$$\phi(t) := d\left(\int_0^t \left(\int_0^s F(u)\,du \right) ds, \int_0^t (t-u)F(u)\,du \right) \quad \text{for } t \ge 0.$$

Lemma 10 implies that ϕ is continuous. By Lemma 3 and properties of the Hausdorff metric for $h > 0$ we have

$$\begin{aligned}
&\phi(t+h) \\
&= d\left(\int_0^{t+h} \left(\int_0^s F(u)\,du \right) ds, \int_0^{t+h} (t+h-u)F(u)\,du \right) \\
&\le d\left(\int_0^{t} \left(\int_0^s F(u)\,du \right) ds, \int_0^{t} (t-u)F(u)\,du \right) \\
&+ d\left(\int_t^{t+h} \left(\int_0^s F(u)\,du \right) ds, \int_t^{t+h} (t+h-u)F(u)\,du + h\int_0^t F(u)\,du \right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\phi(t+h)-\phi(t)}{h} \\
&\le d\left(\frac{1}{h}\int_t^{t+h} \left(\int_0^s F(u)\,du \right) ds, \frac{1}{h}\int_t^{t+h} (t+h-u)F(u)\,du + \int_0^t F(u)\,du \right)
\end{aligned} \tag{4}$$

for $t \geq 0$, $h > 0$. Since F is continuous there exists $M > 0$ such that $\|F(u)\| \leq M$ for $u \in [t, t+1]$. Therefore by Lemma 9 we get

$$\left\| \frac{1}{h} \int_t^{t+h} (t+h-u)F(u)\,du \right\| \leq \frac{1}{h} \int_t^{t+h} (t+h-u)\|F(u)\|\,du \leq \frac{Mh}{2}$$

for $0 < h < 1$. This implies that

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} (t+h-u)F(u)\,du = \{0\}.$$

Using the last equality, (4), Lemmas 10 and 11 we have

$$\begin{aligned}
&\liminf_{h \to 0^+} \frac{\phi(t+h) - \phi(t)}{h} \\
&\leq \lim_{h \to 0^+} d\left(\frac{1}{h} \int_t^{t+h} \left(\int_0^s F(u)\,du \right) ds, \int_0^t F(u)\,du \right) \\
&+ \lim_{h \to 0^+} \left\| \frac{1}{h} \int_t^{t+h} (t+h-u)F(u)\,du \right\| \\
&= d\left(\int_0^t F(u)\,du, \int_0^t F(u)\,du \right) + 0 = 0.
\end{aligned}$$

According to a corollary from the Zygmund Lemma in [5, p. 174] the function ϕ is nonincreasing. Therefore

$$\phi(t) \leq \phi(0) = 0$$

for $t \geq 0$. This shows that equality (3) holds. □

Lemma 13. *If $F\colon [0,\infty) \to cc(X)$ is continuous, then*

$$\int_0^t \left(\frac{(t-s)^n}{n!} \int_0^s (s-u)F(u)\,du \right) ds = \int_0^t \frac{(t-u)^{n+2}}{(n+2)!} F(u)\,du \tag{5}$$

for $t \geq 0$, $n = 0, 1, 2 \ldots$.

Proof. For every nonnegative integer n we define

$$\phi_n(t) = d\left(\int_0^t \left(\frac{(t-s)^n}{n!} \int_0^s (s-u)F(u)\,du \right) ds, \int_0^t \frac{(t-u)^{n+2}}{(n+2)!} F(u)\,du \right).$$

For $n = 0$ we have

$$\phi_0(t) = d\left(\int_0^t \left(\int_0^s (s-u)F(u)\,du\right)\,ds,\ \int_0^t \frac{(t-u)^2}{2!}F(u)\,du\right)$$

and by Lemma 3 and properties of the Hausdorff metric we have

$$\begin{aligned}
&\phi_0(t+h) \\
&= d\left(\int_0^t \left(\int_0^s (s-u)F(u)\,du\right)\,ds + \int_t^{t+h} \left(\int_0^s (s-u)F(u)\,du\right)\,ds,\right. \\
&\int_0^t \frac{(t-u)^2}{2!}F(u)\,du + h\int_0^t (t-u)F(u)\,du \\
&\left.+\frac{h^2}{2}\int_0^t F(u)\,du + \int_t^{t+h} \frac{(t+h-u)^2}{2!}F(u)\,du\right) \\
&\le \phi_0(t) + d\left(\int_t^{t+h} \left(\int_0^s (s-u)F(u)\,du\right)\,ds,\ h\int_0^t (t-u)F(u)\,du\right. \\
&\left.+\frac{h^2}{2}\int_0^t F(u)\,du + \int_t^{t+h} \frac{(t+h-u)^2}{2!}F(u)\,du\right)
\end{aligned}$$

for $h > 0$. Therefore

$$\begin{aligned}
&\frac{\phi_0(t+h) - \phi_0(t)}{h} \\
&\le d\left(\frac{1}{h}\int_t^{t+h} \left(\int_0^s (s-u)F(u)\,du\right)\,ds,\ \int_0^t (t-u)F(u)\,du\right) \\
&+ \frac{h}{2}\left\|\int_0^t F(u)\,du\right\| + \left\|\frac{1}{h}\int_t^{t+h} \frac{(t+h-u)^2}{2!}F(u)\,du\right\|.
\end{aligned}$$

Since

$$\left\|\frac{1}{h}\int_t^{t+h} \frac{(t+h-u)^2}{2!}F(u)\,du\right\|$$

$$\leq \frac{1}{h}\int_t^{t+h}\frac{(t+h-u)^2}{2!}\,du \sup_{t\leq u\leq t+h}\|F(u)\| = \frac{h^2}{6}\sup_{t\leq u\leq t+h}\|F(u)\|$$

we have

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}\frac{(t+h-u)^2}{2!}F(u)\,du = \{0\}$$

and

$$\liminf_{h\to 0^+}\frac{\phi_0(t+h)-\phi_0(t)}{h}\leq 0.$$

As we know the function ϕ_0 is continuous nonnegative and $\phi_0(0) = 0$, whence by the corollary from the Zygmund Lemma $\phi_0(t) = 0$ for $t \geq 0$. Thus

$$\int_0^t\left(\int_0^s (s-u)F(u)\,du\right)ds = \int_0^t \frac{(t-u)^2}{2!}F(u)\,du.$$

Let n be a nonnegative integer. Suppose that $\phi_n \equiv 0$. By Lemma 3 and properties of the Hausdorff metric for $h > 0$ we have

$$\begin{aligned}
&\phi_{n+1}(t+h)\\
&= d\left(\int_0^{t+h}\left(\frac{(t+h-s)^{n+1}}{(n+1)!}\int_0^s (s-u)F(u)\,du\right)ds,\right.\\
&\left.\int_0^{t+h}\frac{(t+h-u)^{n+3}}{(n+3)!}F(u)\,du\right)\\
&\leq \phi_{n+1}(t) + d\left(\sum_{i=0}^{n}\binom{n+1}{i}\left(\int_0^t\frac{(t-s)^i h^{n+1-i}}{(n+1)!}\int_0^s (s-u)F(u)\,du\right)ds\right.\\
&+\int_t^{t+h}\left(\frac{(t+h-s)^{n+1}}{(n+1)!}\int_0^s (s-u)F(u)\,du\right)ds,\\
&\left.\sum_{j=0}^{n+2}\binom{n+3}{j}\int_0^t\frac{(t-u)^j h^{n+3-j}}{(n+3)!}F(u)\,du + \int_t^{t+h}\frac{(t+h-u)^{n+3}}{(n+3)!}F(u)\,du\right).
\end{aligned}$$

Therefore

$$\frac{\phi_{n+1}(t+h)-\phi_{n+1}(t)}{h}$$

$$\leq d\left(\sum_{i=0}^{n}\binom{n+1}{i}\left(\int_0^t \frac{(t-s)^i h^{n-i}}{(n+1)!}\int_0^s (s-u)F(u)\,du\right)\,ds\right.$$
$$+\frac{1}{h}\int_t^{t+h}\left(\frac{(t+h-s)^{n+1}}{(n+1)!}\int_0^s (s-u)F(u)\,du\right)\,ds,$$
$$\sum_{j=0}^{n+2}\binom{n+3}{j}\int_0^t \frac{(t-u)^j h^{n+2-j}}{(n+3)!}F(u)\,du$$
$$\left.+\frac{1}{h}\int_t^{t+h}\frac{(t+h-u)^{n+3}}{(n+3)!}F(u)\,du\right).$$

Since

$$\left\|\frac{1}{h}\int_t^{t+h}\left(\frac{(t+h-s)^{n+1}}{(n+1)!}\int_0^s (s-u)F(u)\,du\right)\,ds\right\| \leq M\frac{h^{n+1}(t+h)^2}{2(n+2)!}$$

and

$$\left\|\frac{1}{h}\int_t^{t+h}\frac{(t+h-u)^{n+3}}{(n+3)!}F(u)\,du\right\| \leq M\frac{h^{n+3}}{(n+4)!}$$

for $0<h<1$, where $M=\sup\{\|F(u)\|\colon t\leq u\leq t+1\}$, we have

$$\liminf_{h\to 0^+}\frac{\phi_{n+1}(t+h)-\phi_{n+1}(t)}{h}$$
$$\leq d\left(\int_0^t\left(\frac{(t-s)^n}{n!}\int_0^s (s-u)F(u)\,du\right)\,ds,\int_0^t \frac{(t-u)^{n+2}}{(n+2)!}F(u)\,du\right)$$
$$=\phi_n(t)=0.$$

Applying the corollary from the Zygmund Lemma we obtain that $\phi_{n+1}\equiv 0$ and equality (5) holds for every $n\geq 0$ and $t\geq 0$. □

It is not difficult to check that the following lemma is true.

Lemma 14. *Let K be a convex cone in X. If $F\colon K\to cc(X)$ is linear continuous, $G\colon [a,b]\to cc(K)$ is continuous, then*

$$\int_a^b F(G(t))\,dt = F\left(\int_a^b G(t)\,dt\right).$$

Let $F, G\colon K \to cc(K)$. We can define the multifunctions $F+G$ and $F-G$ on K as follows

$$(F+G)(x) := F(x) + G(x) \quad \text{for } x \in K$$

and

$$(F-G)(x) := F(x) - G(x)$$

if the Hukuhara differences $F(x) - G(x)$ exist for all $x \in K$.

Lemma 15 ([11, Lemma 2])**.** *For each set $A \subset K$ the inclusion*

$$(F+G)(A) \subset F(A) + G(A) \tag{6}$$

holds. Moreover, if there exist the Hukuhara difference $F(A) - G(A)$ and the multifunction $F - G$, then

$$F(A) - G(A) \subset (F-G)(A). \tag{7}$$

Let K be a closed convex cone in a real Banach space. Applying Theorem 4 in [14] we define the *norm* of a linear continuous multifunction $F\colon K \to n(K)$, denoted by $\|F\|$, to be the smallest element of the set

$$\{M \geq 0\colon \|F(x)\| \leq M\|x\|,\ x \in K\}.$$

Theorem 1. *Let X be a real Banach space and let K be a closed convex cone in X. Assume that $H\colon K \to cc(K)$ is a linear continuous multifunction. Then for every $x \in K$ and $t \geq 0$ the series*

$$F_t(x) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} H^n(x) \tag{8}$$

is convergent in the metric space $(cc(K), d)$. Moreover, the multifunctions F_t, $t \geq 0$ are linear and

$$(2F_t \circ F_s)(x) \subset F_{t+s}(x) + F_{t-s}(x)$$

for $x \in K$, $t \geq s \geq 0$.

The proof is similar to the proof of Theorem in [9].

It is obvious that for single-valued functions we have the equality instead of the inclusion in the assertion of Theorem 1 (see also Theorem 3.3 in [15]).

Theorem 2. *Let X be a real Banach space and let K be a closed convex cone in X. If $H\colon K \to cc(K)$ is a linear continuous multifunction such that $x \in H(x)$ and $H^2(x) = H(x)$ for $x \in K$, then the multifunction given by* (8) *satisfies the equation*

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)) \tag{9}$$

for $x \in K$, $s, t \in \mathbb{R}$.

Proof. By the assumption $H^2(x) = H(x)$ we have

$$F_t(x) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} H^n(x) = x + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} H(x) = x + (\cosh t - 1)H(x).$$

This equality yields (9). □

For example $H\colon [0,\infty) \to cc([0,\infty))$ defined by $H(x) = [0,x]$ or $H\colon [0,\infty)^2 \to cc([0,\infty)^2)$ defined by $H((x,y)) = [0,x] \times [0,y]$ satisfy the assumptions of Theorem 2.

Definition. Let $(K,+)$ be a semigroup. A one-parameter family $\{F_t\colon t \in \mathbb{R}\}$ of multifunctions $F_t\colon K \to n(K)$ is said to be a *cosine family* if

$$F_0(x) = \{x\} \quad \text{for } x \in K$$

and

$$F_{t+s} + F_{t-s} = 2F_t \circ F_s \tag{10}$$

on K for $s,t \in \mathbb{R}$.

Let X be a real normed space. A cosine family $\{F_t\colon t \in \mathbb{R}\}$ is *regular* if

$$\lim_{t \to 0} d(F_t(x), \{x\}) = 0.$$

Cosine families of single-valued functions was considered by many authors. These families are relate to second order differential equations (for example see [3], [16]). J. Kisyński proved that solution of some second order differential problem is a cosine family (see Lemma 1.3.3 in [3]).

Lemma 16 ([11, Lemma 8])**.** *Let X be a Banach space and let K be a closed convex cone in X such that* int $K \neq \emptyset$. *Assume that $\{F_t\colon t \in \mathbb{R}\}$ is a regular cosine family of continuous additive set-valued functions $F_t\colon K \to cc(K)$ and $x \in F_t(x)$ for all $x \in K$ and $t \in \mathbb{R}$. Then there exist the Hukuhara differences $F_t(x) - F_s(x)$ for all $0 \leq s \leq t$ and $x \in K$.*

Theorem 3. *Let X be a real Banach space and let K be a closed convex cone in X such that int$K \neq \emptyset$. Assume that $\{F_t\colon t \in \mathbb{R}\}$ is a regular cosine family of continuous linear multifunctions $F_t\colon K \to cc(X)$ and $x \in F_t(x)$ for all $x \in K$ and $t \in \mathbb{R}$. Then there exists a continuous linear multifunction $H\colon K \to cc(K)$ such that*

$$F_t(x) \subset \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} H^n(x)$$

for $x \in K$ and $t \in \mathbb{R}$.

Proof. Let $x \in K$. Consider the multifunction $t \mapsto F_t(x)$ for $t \in \mathbb{R}$. We first show some properties of this multifunction. Putting $t = 0$ in (10) we have $F_s(x) + F_{-s}(x) = 2F_0(F_s(x)) = 2F_s(x) = F_s(x) + F_s(x)$. By Rådström Lemma we obtain $F_s(x) = F_{-s}(x)$. Thus the multifunction $t \mapsto F_t(x)$ is even. Hence and by (10) we have $F_t \circ F_s = F_s \circ F_t$. According to Theorem 2 in [13] the multifunction $t \mapsto F_t(x)$ is continuous on $[0, +\infty)$. As $t \mapsto F_t(x)$ is even it is continuous on $\mathbb{R}$. By Lemma 8 it is integrable on each interval $[a, b] \subset \mathbb{R}$. Setting $t = (v + u)/2$, $s = (v - u)/2$, $u, v \in \mathbb{R}$ in (10) we get

$$F_v(x) + F_u(x) = 2F_{(v+u)/2}(F_{(v-u)/2}(x)).$$

Since $x \in F_t(x)$ we have

$$F_{(v+u)/2}(x) \subset \frac{F_v(x) + F_u(x)}{2}.$$

Hence, by the continuity and Theorem 1 in [7] the multifunction $t \mapsto F_t(x)$ is concave. We observe that $t \mapsto F_t(x)$ is increasing in $[0, +\infty)$ and decreasing in $(-\infty, 0]$. Indeed, for $0 \leq s \leq t$ there exists $\lambda \in [0, 1]$ such that $s = (1 - \lambda)0 + \lambda t$. Hence $F_s(x) \subset (1 - \lambda)F_0(x) + \lambda F_t(x) = (1 - \lambda)x + \lambda F_t(x) \subset (1 - \lambda)F_t(x) + \lambda F_t(x) = F_t(x)$. For $t \leq s \leq 0$ we have $F_s(x) = F_{-s}(x) \subset F_{-t}(x) = F_t(x)$.

We next define some multifunctions. We use them later in the construction of the multifunction H and in proving of the inclusion in the assertion. To define these multifuctions we need to show the existence of some limits. Take arbitrary α and t such that $0 < \alpha < t$. Integrating over $s \in [0, \alpha]$ formula (10) we obtain

$$\int_0^\alpha F_{t+s}(x)\, ds + \int_0^\alpha F_{t-s}(x)\, ds = \int_0^\alpha 2F_t(F_s(x))\, ds.$$

Hence, by Lemmas 4, 6 and 14 we have

$$\int_t^{\alpha+t} F_w(x)\, dw + \int_{t-\alpha}^{t} F_w(x)\, dw = 2F_t\left(\int_0^\alpha F_s(x)\, ds\right).$$

The last equality and Lemma 2 lead to

$$\int_{t-\alpha}^{t+\alpha} F_w(x)\, dw = 2F_t\left(\int_0^\alpha F_s(x)\, ds\right). \tag{11}$$

Replacing in (11) t by $t + u$, where $u \in (0, \alpha)$, we get

$$\int_{t-\alpha+u}^{t+\alpha+u} F_w(x)\, dw = 2F_{t+u}\left(\int_0^\alpha F_s(x)\, ds\right). \tag{12}$$

Adding both the sides (11) and (12) we have

$$\int_{t-\alpha+u}^{t+\alpha+u} F_w(x)\,dw + 2F_t\left(\int_0^{\alpha} F_s(x)\,ds\right) = \int_{t-\alpha}^{t+\alpha} F_w(x)\,dw + 2F_{t+u}\left(\int_0^{\alpha} F_s(x)\,ds\right).$$

Then we get by Lemma 3 substracting the term $\int_{t-\alpha+u}^{t+\alpha} F_w(x)\,dw$ from both the sides of the last equality

$$\int_{t+\alpha}^{t+\alpha+u} F_w(x)\,dw + 2F_t\left(\int_0^{\alpha} F_s(x)\,ds\right) = \int_{t-\alpha}^{t-\alpha+u} F_w(x)\,dw + 2F_{t+u}\left(\int_0^{\alpha} F_s(x)\,ds\right). \tag{13}$$

By Lemma 7 the Hukuhara difference

$$\int_{t+\alpha}^{t+\alpha+u} F_w(x)\,dw - \int_{t-\alpha}^{t-\alpha+u} F_w(x)\,dw$$

exists, which together with (13) shows that the Hukuhara difference

$$2F_{t+u}\left(\int_0^{\alpha} F_s(x)\,ds\right) - 2F_t\left(\int_0^{\alpha} F_s(x)\,ds\right)$$

also exists and

$$\frac{2F_{t+u}\left(\int_0^{\alpha} F_s(x)\,ds\right) - 2F_t\left(\int_0^{\alpha} F_s(x)\,ds\right)}{u} = \frac{1}{u}\int_{t+\alpha}^{t+\alpha+u} F_w(x)\,dw - \frac{1}{u}\int_{t-\alpha}^{t-\alpha+u} F_w(x)\,dw.$$

In virtue of Lemmas 1 and 11 there exists

$$\lim_{u\to 0^+} \frac{2F_{t+u}\left(\int_0^{\alpha} F_s(x)\,ds\right) - 2F_t\left(\int_0^{\alpha} F_s(x)\,ds\right)}{u} = F_{t+\alpha}(x) - F_{t-\alpha}(x).$$

We define

$$G_{\alpha,t}(x) := \frac{F_{t+\alpha}(x) - F_{t-\alpha}(x)}{2} = \lim_{u\to 0^+} \frac{F_{t+u}\left(\int\limits_0^\alpha F_s(x)\,ds\right) - F_t\left(\int\limits_0^\alpha F_s(x)\,ds\right)}{u}.$$

We observe that

$$G_{\alpha,t}(x) = \frac{F_{t+\alpha}(x) - F_{t-\alpha}(x)}{2} = \frac{F_{\alpha+t}(x) - F_{\alpha-t}(x)}{2} = G_{t,\alpha}(x). \tag{14}$$

Since the multifunction $s \mapsto F_s(x)$ is concave and there exist the Hukuhara differences $F_t(x) - F_s(x)$ for $t \geq s \geq 0$, so by Theorem 3.2 in [10] there exist

$$G_t^+(x) := \lim_{\alpha\to 0^+} \frac{F_{t+\alpha}(x) - F_t(x)}{\alpha},$$
$$G_t^-(x) := \lim_{\alpha\to 0^+} \frac{F_t(x) - F_{t-\alpha}(x)}{\alpha}$$

and $G_t^-(x) \subset G_t^+(x)$ for $t > 0$. Consequently there exists

$$G_t(x) := \lim_{\alpha\to 0^+} \frac{F_{t+\alpha}(x) - F_{t-\alpha}(x)}{\alpha} = G_t^+(x) + G_t^-(x) \quad \text{for } t > 0.$$

It follows from (10) that

$$\frac{F_{2t}(x) - x}{2t} = F_t\left(\frac{F_t(x) - x}{t}\right) + \frac{F_t(x) - x}{t}.$$

Letting $t \to 0^+$ we get

$$\lim_{t\to 0^+} F_t\left(\frac{F_t(x) - x}{t}\right) = \{0\}$$

and since

$$0 \in \frac{F_t(x) - x}{t} \subset F_t\left(\frac{F_t(x) - x}{t}\right)$$

we have

$$G_0^+(x) := \lim_{t\to 0^+} \frac{F_t(x) - x}{t} = \{0\}.$$

Our next claim is that the multifunction $t \mapsto G_t^+(x)$ is concave. Replacing in (10) t by $t+u$, $u > 0$ and substract $F_{t+s}(x) + F_{t-s}(x)$ from both the sides of this equality we get

$$F_{t+s+u}(x) - F_{t+s}(x) + F_{t-s+u}(x) - F_{t-s}(x) = 2F_{t+u}(F_s(x)) - 2F_t(F_s(x)).$$

Dividing the last equality by u we get

$$\frac{F_{t+s+u}(x) - F_{t+s}(x)}{u} + \frac{F_{t-s+u}(x) - F_{t-s}(x)}{u} = 2F_s\left(\frac{F_{t+u}(x) - F_t(x)}{u}\right)$$

and letting $u \to 0^+$ we obtain

$$G^+_{t+s}(x) + G^+_{t-s}(x) = 2F_s(G^+_t(x)). \tag{15}$$

Setting $t = (v+u)/2$, $s = (v-u)/2$, $u, v \in \mathbb{R}$ in (15) we have

$$G^+_v(x) + G^+_u(x) = 2F_{(v-u)/2}(G^+_{(v+u)/2}(x)).$$

By assumption $x \in F_t(x)$ we get

$$G^+_{(v+u)/2}(x) \subset \frac{G^+_v(x) + G^+_u(x)}{2}.$$

Fix an interval $[a, b] \subset [0, \infty)$ and let $t \in [a, b]$. Since $G^+_t(x) \subset F_{t+1}(x) - F_t(x)$ we have

$$G^+_t(x) + x \subset G^+_t(x) + F_t(x) \subset F_{t+1}(x) \subset F_{b+1}(x).$$

Therefore the multifunction $t \mapsto G^+_t(x)$ is bounded on $[a, b]$. By Theorem 4.4 in [6] the multifunction $t \mapsto G_t(x)$ is continuous in $(0, \infty)$ and concave by Theorem 4.1 in [6]. Let $\lambda \in (0, 1)$. Then $G^+_{\lambda t}(x) \subset \lambda G^+_t(x)$ and

$$\frac{G^+_{\lambda t}(x)}{\lambda t} \subset \frac{G^+_t(x)}{t}$$

it follows that there exists

$$\lim_{t \to 0^+} \frac{G^+_t(x)}{t} =: H(x).$$

Since $x \mapsto F_t(x)$ for every $t \in \mathbb{R}$ are linear continuous we see H is linear and $\|H(x)\| \le \|G_1(x)\| \le \|G_1\|\|x\|$, hence H is continuous, too.

By uniform convergence of $\lim_{u \to 0^+} (F_{\alpha+u}(x) - F_\alpha(x))/u$ on each compact subset of K (see Theorem 1 in [8]), Lemma 15 and equality (14) we get

$$\begin{aligned}
&2G_t^-(x)\\
&\subset G_t(x) = \lim_{\alpha\to 0^+} \frac{2}{\alpha} G_{\alpha,t}(x) = \lim_{\alpha\to 0^+} \frac{2}{\alpha} G_{t,\alpha}(x)\\
&= \lim_{\alpha\to 0^+} \frac{2}{\alpha} \lim_{u\to 0^+} \frac{F_{\alpha+u}\left(\int_0^t F_s(x)\,ds\right) - F_\alpha\left(\int_0^t F_s(x)\,ds\right)}{u}\\
&\subset \lim_{\alpha\to 0^+} \frac{2}{\alpha} \lim_{u\to 0^+} \frac{(F_{\alpha+u} - F_\alpha)\left(\int_0^t F_s(x)\,ds\right)}{u}\\
&= \lim_{\alpha\to 0^+} \frac{2}{\alpha} G_\alpha^+\left(\int_0^t F_s(x)\,ds\right) = 2H\left(\int_0^t F_s(x)\,ds\right).
\end{aligned} \tag{16}$$

According to Theorem 4.3 in [10] we have

$$F_t(x) = x + \int_0^t G_s^-(x)\,ds.$$

Hence, by (16), Lemmas 5, 12 and 14 we obtain

$$\begin{aligned}
F_t(x) &\subset x + \int_0^t H\left(\int_0^s F_u(x)\,du\right)\,ds\\
&= x + H\left(\int_0^t \left(\int_0^s F_u(x)\,du\right)\,ds\right) = x + H\left(\int_0^t (t-u)F_u(x)\,du\right).
\end{aligned}$$

Using Lemma 13 instead of Lemma 12 we get succesively

$$\begin{aligned}
F_t(x) &\subset x + H\left[\int_0^t (t-u)\left(x + H\left(\int_0^u (u-s)F_s(x)\,ds\right)\right)\,du\right]\\
&= x + H\left[\int_0^t (t-u)x\,du + \int_0^t (t-u)H\left(\int_0^u (u-s)F_s(x)\,ds\right)\,du\right]\\
&= x + H\left(\frac{1}{2}t^2x\right) + H^2\left[\int_0^t \left((t-u)\int_0^u (u-s)F_s(x)\,ds\right)\,du\right]
\end{aligned}$$

$$= x + \frac{t^2}{2} H(x) + H^2 \left(\int_0^t \frac{(t-s)^3}{3!} F_s(x)\, ds \right).$$

Repeating the same steps we have

$$F_t(x) \subset x + \frac{t^2}{2!} H(x) + \ldots + \frac{t^{2n}}{(2n)!} H^n(x) + H^{n+1} \left(\int_0^t \frac{(t-s)^{2n+1}}{(2n+1)!} F_s(x)\, ds \right).$$

It remains to prove that

$$H^{n+1} \left(\int_0^t \frac{(t-s)^{2n+1}}{(2n+1)!} F_s(x)\, ds \right) \to \{0\}.$$

For $s \in [0,t]$, $F_s(x) \subset F_t(x)$, so $\|F_s(x)\| \leq \|F_t(x)\|$. Let $m = \|F_t(x)\|$. By Lemma 9 we have

$$\begin{aligned} &\left\| H^{n+1} \left(\int_0^t \frac{(t-s)^{2n+1}}{(2n+1)!} F_s(x)\, ds \right) \right\| \\ &\leq \|H\|^{n+1} \left\| \int_0^t \frac{(t-s)^{2n+1}}{(2n+1)!} F_s(x)\, ds \right\| \\ &\leq \|H\|^{n+1} \int_0^t \frac{(t-s)^{2n+1}}{(2n+1)!} \|F_s(x)\|\, ds \\ &\leq \|H\|^{n+1} m \int_0^t \frac{(t-s)^{2n+1}}{(2n+1)!}\, ds \\ &\leq \|H\|^{n+1} m \frac{t^{2n+2}}{(n+1)!} = \frac{(\|H\| t^2)^{n+1}}{(n+1)!} m. \end{aligned}$$

Since

$$\frac{(\|H\| t^2)^{n+1}}{(n+1)!} m \to 0$$

we have

$$H^{n+1} \left(\int_0^t \frac{(t-s)^{2n+1}}{(2n+1)!} F_s(x)\, ds \right) \to \{0\}.$$

Therefore

$$F_t(x) \subset \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} H^n(x) \quad \text{for } t \in \mathbb{R},\ x \in K.$$

□

S. Kurepa in [4] proved that every single-valued function of the cosine family is in the form of the series. Theorem 3 is not generalization of Theorems for single-valued case.

Example 1. A family $\{F_t \colon t \in \mathbb{R}\}$ of multifunctions $F_t \colon [0,\infty) \to cc([0,\infty))$ such that $F_t(x) = x[1, \cosh t]$ is a regular cosine family. Our wanted multifunction is $H(x) = [0, x]$ and

$$F_t(x) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} H^n(x) = x + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} [0, x].$$

Example 2. A family $\{F_t \colon t \in \mathbb{R}\}$ of multifunctions $F_t \colon [0,\infty)^2 \to cc([0,\infty)^2)$ defined by $F_t((x,y)) = [x, x\cosh t] \times [y, y\cosh t]$ is a regular cosine family. Our wanted multifunction is $H((x,y)) = [0,x] \times [0,y]$ and

$$F_t((x,y)) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} H^n((x,y)) = (x,y) + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} [0,x] \times [0,y].$$

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