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EXISTENCE OF SOLUTIONS FOR UNILATERAL PROBLEMS IN L^1 INVOLVING LOWER ORDER TERMS IN DIVERGENCE FORM IN ORLICZ SPACES

L. AHAROUCH, E. AZROUL and M. RHOUDAF

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Abstract. This article is concerned with the existence result of the unilateral problem associated to the equations of the type

$$Au - \text{div}\phi(u) = f \in L^1(\Omega),$$

where A is a Leray-Lions operator having a growth not necessarily of polynomial type and $\phi \in C^0(\mathbb{R}, \mathbb{R}^N)$.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, and let p be a real number with $1 < p < +\infty$. Consider the following nonlinear Dirichlet problem:

$$Au - \text{div}\phi(u) = f, \tag{1.1}$$

where $Au = -\text{div}a(x, u, \nabla u)$ is a Leray-Lions operators defined from $W_0^{1,p}(\Omega)$ into its dual and ϕ lies in $C^0(\mathbb{R}, \mathbb{R}^N)$.

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Boccardo proved in [9] the existence of entropy solution for the problem (1.1). The formulation adequate in this case is the following,

$$\begin{cases} u \in W_0^{1,q}(\Omega), \quad \forall q < \dfrac{N(p-1)}{N-1} \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx \\ \forall v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \end{cases}$$

where T_k is the usual truncation defined as $T_k(s) = \max(-k, \min(k,s))$ for all $s \in \mathbb{R}$.

In this direction, Boccardo and Cirmi are studied the existence and uniqueness of solution of the following unilateral problem,

$$\begin{cases} u \in W_0^{1,q}(\Omega), \quad \forall q < \dfrac{N(p-1)}{N-1}, \quad u \geq \psi \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx \\ \forall v \in K_\psi(\Omega)\cap L^\infty(\Omega), \end{cases}$$

where

$$K_\psi = \left\{ u \in W_0^{1,p}(\Omega) \colon u \geq \psi \right\},$$

with a measurable function $\psi \colon \Omega \to \overline{\mathbb{R}}$ such that $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. In these results the function $a(\cdot)$ is supposed to satisfy a polynomial growth conditions with respect to u and ∇u.

In the case where $a(\cdot)$ satisfies a more general growth condition with respect to u and ∇u (such growth to relax the coefficients of the operator A), the adequate space in which (1.1) can be studied is the Orlicz-Sobolev spaces $W^1L_M(\Omega)$ where the N-function M is related to the actual growth of a. The solvability of (1.1) in this setting is studied by Gossez-Mustonen [14] in the variational case for $\phi = 0$. The case where f belongs to $L^1(\Omega)$ and $\phi = 0$ is treated in [7]. This last result is restricted to the N-functions which satisfy the Δ_2-condition (this condition appears in the boundedness of the term $\nabla T_k(u_n)$ in $L_M(\Omega)$, see [7, pp. 96-97]). More precisely, the authors have proved in the previous work existence and uniqueness of the following unilateral problem

$$\begin{cases} u \in W_0^1 L_Q(\Omega), \quad \forall Q \in \mathcal{A}_M \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx \\ \forall v \in K_\psi(\Omega)\cap L^\infty(\Omega), \end{cases}$$

where $\mathcal{A}_M$ equals to

$$\Big\{Q\colon Q \text{ is an } N\text{-function}, \quad \frac{Q''}{Q'} \le \frac{M''}{M'} \quad \text{and}$$
$$\int_0^1 Q\circ H^{-1}\left(\frac{1}{t^{1-1/N}}\right) dt < \infty \text{ where } H(t) = \frac{M(t)}{t}\Big\}$$

and where $K_\psi = \{u \in W_0^1 L_M(\Omega)\colon u \ge \psi\}$, with the following restrictions on the obstacle ψ

$$\psi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega), \tag{1.2}$$

$$\text{there exists } \overline{\psi} \in K_\psi \text{ such that } \psi - \overline{\psi} \text{ is continuous on } \Omega. \tag{1.3}$$

The case $\phi \neq 0$ is studied by Benkirane and Bennouna in [6] where an entropy solution for equation (1.1) is proved without assuming the Δ_2-condition.

Our purpose in this paper is to prove the existence of solutions for obstacle problem associated to (1.1) for general N-functions M.

Note that, our result (see Theorem 3.1) generalizes the analogous one in [9, 10] in Orlicz spaces and both [6, 7].

This paper is organized as follows:

1) Introduction
2) Preliminaries and some technical lemmas
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5) Proof of intermediate results.

2. Preliminaries and some technical lemmas

2.1. Let $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e., M is continuous, convex, with $M(t) > 0$ for $t > 0$,

$$\frac{M(t)}{t} \to 0 \text{ as } t \to 0 \text{ and } \frac{M(t)}{t} \to \infty \text{ as } t \to \infty.$$

Equivalently, M admits the representation: $M(t) = \int_0^t a(s)\,ds$ where $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing, right continuous function, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \bar{a}(s)\,ds$, where $\bar{a}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s\colon a(s) \le t\}$.

The N-function M is said to satisfy the Δ_2-condition if, for some k

$$M(2t) \le kM(t) \quad \forall t \ge 0. \tag{2.1}$$

It is readily seen that this will be the case if and only if for every $r > 1$ there exists a positive constant $k = k(r)$ such that for all $t > 0$

$$M(rt) \le kM(t) \quad \forall t \ge 0. \tag{2.2}$$

When (2.1) and (2.2) hold only for $t \ge t_0$ for some $t_0 > 0$, then M is said to satisfy the Δ_2-condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Moreover, we have the following Young's inequality

$$\forall s,\ t \ge 0,\ st \le M(t) + \overline{M}(s).$$

Let P and Q be two N-functions. We say that P grows essentially less rapidly than Q near infinity, denote $P \ll Q$, if for every $\varepsilon > 0$,

$$\frac{P(t)}{Q(\varepsilon t)} \to 0 \text{ as } t \to \infty.$$

This is the case if and only if

$$\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

2.2. Let M be an N-function and $\Omega \subset \mathbb{R}^N$ be an open and bounded set. The Orlicz class $\mathcal{K}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))\,dx < +\infty \quad \left(\text{resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right).$$

$L_M(\Omega)$ is a Banach space under the norm,

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0\colon \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \le 1\right\}$$

and $\mathcal{K}_M(\Omega)$ is a convex subset of $L_M(\Omega)$ but not necessarily a linear space.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The dual space of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.

Let X and Y be arbitrary Banach spaces with bilinear bicontinuous pairing $\langle\,,\rangle_{X,Y}$.

We say that a sequence $\{u_n\} \subset X$ converges to $u \in X$ with respect to the topology $\sigma(X,Y)$, denote $u_n \to u$ $(\sigma(X,Y))$ in X, if $\langle u_n, v\rangle \to \langle u, v\rangle$ for all $v \in Y$. For example, if $X = L_M(\Omega)$ and $Y = L_{\overline{M}}(\Omega)$, then the pairing is defined by $\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx$ for all $u \in X$, $v \in Y$.

2.3. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lies in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$ and $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We recall that a sequence u_n in $L_M(\Omega)$ is said to be convergent to $u \in L_M(\Omega)$ modular, denote $u_n \to u$ (mod) in $L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega M\Big(\frac{|u_n(x) - u(x)|}{\lambda}\Big)\,dx \to 0$$

as $n \to +\infty$. This implies that u_n converges to u for $\sigma\left(L_M(\Omega), L_{\overline{M}}(\Omega)\right)$. A similar definition can be given in $W^1L_M(\Omega)$ where one requires the above for u and each of its first derivatives.

If M satisfies the Δ_2-condition (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm.

We recall some lemmas introduced in [8] which will be used later.

Lemma 2.1. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, we have*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\dfrac{\partial}{\partial x_i}u & \text{a.e. in } \{x \in \Omega\colon u(x) \notin D\},\\ 0 & \text{a.e. in } \{x \in \Omega\colon u(x) \in D\}\end{cases}$$

where D is the set of discontinuity points of F'.

Lemma 2.2. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function, then the mapping $T_F\colon W^1L_M(\Omega) \to W^1L_M(\Omega)$ defined by $T_F(u) = F(u)$ is sequentially continuous with respect to the weak* topology $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$.*

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P and Q be N-functions such that $Q \ll P$, and let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$.

Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right) = \left\{u \in L_M(\Omega)\colon d(u, E_M(\Omega)) < \frac{1}{k_2}\right\} \quad \textit{into } E_Q(\Omega).$$

We introduce the functional spaces, we will need later.

For N-function M, $\mathcal{T}_0^{1,M}(\Omega)$ is defined as the set of measurable functions $u\colon\ \Omega \longrightarrow \mathbb{R}$ such that for all $k > 0$ the truncated functions $T_k(u) \in W_0^1 L_M(\Omega)$.

We give the following lemma which is a generalization of Lemma 2.1 [5] in Orlicz spaces and where its proof is a slightly modification of one in L^p case.

Lemma 2.4. *For every $u \in \mathcal{T}_0^{1,M}(\Omega)$, there exists a unique measurable function $v\colon\ \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \quad \textit{almost everywhere in } \Omega, \quad \textit{for every } k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Lemma 2.5. *Let $\lambda \in \mathbb{R}$ and let u and v be two functions which are finite almost everywhere, and which belongs to $\mathcal{T}_0^{1,M}(\Omega)$. Then,*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \ \textit{a.e. in } \Omega,$$

where ∇u, ∇v and $\nabla(u+\lambda v)$ are the gradients of u, v and $u+\lambda v$ introduced in Lemma 2.4.

The proof of this lemma is similar to the proof of Lemma 2.12 [11] in the L^p case.

Below, we will use the following technical lemma:

Lemma 2.6 ([8]). *Let $(f_n), f, \gamma \in L^1(\Omega)$ such that*

(i) $f_n \geq \gamma$ *a.e. in* Ω,

(ii) $f_n \to f$ *a.e. in* Ω,

(iii) $\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$.

Then $f_n \to f$ strongly in $L^1(\Omega)$.

Lemma 2.7 ([6]). *Let Ω be an open bounded subset of $\mathbb{R}^N$ satisfying the segment property. If $u \in W_0^1 L_M(\Omega)$, then*

$$\int_\Omega \mathrm{div} u\,dx = 0.$$

3. Statement of main result

3.1. Basic assumptions.

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property.

Given an obstacle $\psi\colon \Omega \to \overline{\mathbb{R}}$ which is *a* measurable function and consider the set

$$K_\psi = \{u \in W_0^1 L_M(\Omega);\ u \geq \psi \text{ a.e. in } \Omega\}. \tag{3.1}$$

We now state our hypotheses on the differential operator A defined by,

$$Au = -\mathrm{div}(a(x, \nabla u)). \tag{3.2}$$

(A_1) $a(x,\xi)\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.

(A_2) There exist a function $c(x)$ in $E_{\overline{M}}(\Omega)$ and a positive constants k_1, k_2 such that,

$$|a(x,\zeta)| \leq c(x) + k_1 \overline{M}^{-1} M(k_2|\zeta|),$$

for a.e. x in Ω and for all $\zeta \in \mathbb{R}^N$.

(A_3) For a.e. x in Ω and ζ, ζ' in $\mathbb{R}^N$, with $(\zeta \neq \zeta')$

$$[a(x,\zeta) - a(x,\zeta')](\zeta - \zeta') > 0.$$

(A_4) There exist $\delta(x) > 0$ in $L^1(\Omega)$ and some strictly positive constants α, ν such that, for some fixed element v_0 in $K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$, we have

$$a(x,\zeta)(\zeta - \nabla v_0) \geq \alpha M\left(\frac{|\zeta|}{\nu}\right) - \delta(x),$$

for a.e. x in Ω and all $\zeta \in \mathbb{R}^N$.

(A_5) For each $v \in K_\psi \cap L^\infty(\Omega)$ there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that,

$$v_j \to v \quad \text{for the modular convergence.}$$

Finally, we suppose that,

$$f \in L^1(\Omega) \tag{3.3}$$

and

$$\phi \in C^0(\mathbb{R}, \mathbb{R}^N). \tag{3.4}$$

Remark 3.1. Note that the hypotheses (A_5) is verified if one of the following conditions is verified:

1) There exists $\overline{\psi} \in K_\psi$ such that $\psi - \overline{\psi}$ is continuous in Ω (see [14, Proposition 9]).
2) $\psi \in W_0^1 E_M(\Omega)$ (see [14, Proposition 10]).
3) The N-function M satisfies the Δ_2-condition.
4) $\psi = -\infty$ (i.e., $K_\psi = W_0^1 L_M(\Omega)$) (see Remark 2 of [14] and Theorem 4 of [13]).

Remark 3.2. Giving some comparisons of our hypotheses and those of [6, 7]:

1) In [7], the authors have supposed the Δ_2-condition and hypotheses (1.3) which is stronger than our hypotheses (A_5) (see Remark 3.1).
2) When $\psi = -\infty$, the convex set K_ψ coincides with the space $W_0^1 L_M(\Omega)$, this implies that (A_5) is verified. For that the authors in [6] have not need to (A_5).

Remark 3.3. Remark that, if we suppose that $a(x,\xi)\xi \geq \alpha M(|\xi|)$, then the hypotheses ($A_4$) is verified for all $v_0 \in K_\psi \cap W_0^1 E_M(\Omega)$.

Indeed. Let $v_0 \in K_\psi \cap W_0^1 E_M(\Omega)$ and let $\lambda > 0$ large enough, we have

$$a(x,\xi)(\xi - \nabla v_0) = a(x,\xi)\xi - \frac{1}{\lambda} a(x,\xi)(\lambda \nabla v_0). \tag{3.5}$$

On the other hand, by using (A_3), we have

$$\begin{aligned} -\frac{1}{\lambda} a(x,\xi)(\lambda \nabla v_0) \geq & -\frac{1}{\lambda} a(x,\xi)\xi - a(x, \lambda \nabla v_0)\nabla v_0 \\ & - \frac{\alpha\left(1 - \frac{1}{\lambda}\right)}{2} \frac{|a(x,\lambda\nabla v_0)|}{\frac{\alpha(\lambda-1)}{2}} |\xi| \end{aligned} \tag{3.6}$$

using the Young's inequality, we deduce that

$$
\begin{aligned}
& -\frac{\alpha\left(1-\frac{1}{\lambda}\right)}{2}\frac{|a(x,\lambda\nabla v_0)|}{\frac{\alpha(\lambda-1)}{2}}|\xi| \\
& \geq -\frac{\alpha\left(1-\frac{1}{\lambda}\right)}{2}M(|\xi|)-\frac{\alpha\left(1-\frac{1}{\lambda}\right)}{2}\overline{M}\left(\frac{|a(x,\lambda\nabla v_0)|}{\frac{\alpha(\lambda-1)}{2}}\right). \qquad (3.7)
\end{aligned}
$$

Combining (3.5), (3.6) and (3.7), we get

$$
\begin{aligned}
a(x,\xi)(\xi-\nabla v_0) \geq & a(x,\xi)\xi-\frac{1}{\lambda}a(x,\xi)\xi \\
& -\frac{\alpha\left(1-\frac{1}{\lambda}\right)}{2}M(|\xi|)-\gamma(x), \qquad (3.8)
\end{aligned}
$$

where

$$
\gamma(x)=\frac{\alpha\left(1-\frac{1}{\lambda}\right)}{2}\overline{M}\left(\frac{|a(x,\lambda\nabla v_0)|}{\frac{\alpha(\lambda-1)}{2}}\right)+a(x,\lambda\nabla v_0)\nabla v_0.
$$

Finally, by the hypotheses, we deduce

$$
a(x,\xi)(\xi-\nabla v_0) \geq \frac{\alpha\left(1-\frac{1}{\lambda}\right)}{2}M(|\xi|)-\gamma(x).
$$

3.2. Principal result.

Our objective of this paper is to prove the following existence result:

Theorem 3.1. *Suppose that the assumptions* (A_1)–(A_5) *and* (3.3), (3.4) *are satisfied. Then the following obstacle problem,*

$$
\begin{cases}
u \in \mathcal{T}_0^{1,M}(\Omega), & u \geq \psi \ \text{a.e. in } \Omega, \\
\displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx & \\
\qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega f T_k(u-v)\,dx, & \\
\forall\ v \in K_\psi \cap L^\infty(\Omega), & \forall k > 0
\end{cases} \qquad (3.9)
$$

has at least one solution.

Remark 3.4. Remark that, in the previous result, we can not replace $K_\psi \cap L^\infty(\Omega)$ by only K_ψ, since in general the integral $\int_\Omega \phi(u)\nabla T_k(u-v)\,dx$ may not have a meaning.

Remark 3.5. The particular case $M(t) = |t|^p$ gives the corresponding existence result in the classical L^p-case (which appears a new result).

4. Proof of principal result

Without loss the generality we take $\nu = 1$ in the conditions (A_4).

Let us recall the following lemma which will be needed later:

Lemma 4.1 ([12])**.** *Let $f \in W^{-1}E_{\overline{M}}(\Omega)$ and let $K \subset W_0^1 L_M(\Omega)$ be convex, $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$ sequentially closed and such that $K \cap W_0^1 E_M(\Omega)$ is $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$ dense in K. Assume that (A_1)–(A_4) are satisfy with $v_0 \in K \cap W_0^1 E_M(\Omega)$, then the variational inequality*

$$\begin{cases} u \in \mathcal{D}(A) \cap K, \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla(u-v)\,dx \le \langle f, u-v\rangle \\ \forall v \in K, \end{cases}$$

has at least one solution.

Remark 4.1. The previous lemma can be applied if $K = W_0^1 L_M(\Omega)$ (see Remark 3.1 and Remark 2 of [14]).

Remark 4.2. Remark that the convex set K_ψ satisfies the following conditions:

1) K_ψ is $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$ sequentially closed.
2) $K_\psi \cap W_0^1 E_M(\Omega)$ is $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$ dense in K_ψ.

Indeed.

1) Let $u_n \in K_\psi$ which converges to $u \in W_0^1 L_M(\Omega)$ for $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$. Since the imbedding of $W_0^1 L_M(\Omega)$ into $E_M(\Omega)$ is compact it follows that for a subsequence $u_n \to u$ a.e. in Ω, which gives $u \in K_\psi$.
2) It suffices to apply (A_5) and the fact that $T_n(u) \to u$ (mod) in $W^1 L_M(\Omega)$ for all $u \in K_\psi$.

4.1. Approximate problem.

We consider the sequence of approximate problem,

$$\begin{cases} u_n \in K_\psi, \\ \langle Au_n, u_n - v\rangle + \int_\Omega \phi(T_n(u_n))\nabla(u_n - v)\,dx \le \int_\Omega f_n(u_n - v)\,dx \\ \forall v \in K_\psi, \end{cases} \tag{4.1}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$. Applying Remark 4.2 and Lemma 4.1, we can deduce that this approximate problem has a solution.

4.2. Some intermediate results.

Lemma 4.2. *Assume that (A_1)–(A_4) are satisfied, and let $(z_n)_n$ be a sequence in $W_0^1L_M(\Omega)$ such that*

a) $z_n \rightharpoonup z$ *in* $W_0^1L_M(\Omega)$ *for* $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$,

b) $(a(x,\nabla z_n))_n$ *is bounded in* $(L_{\overline{M}}(\Omega))^N$,

c) $\int_\Omega [a(x,\nabla z_n) - a(x, z_n, \nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx \to 0$ *as* n *and* $s \to +\infty$ (*where* χ_s *the characteristic function of* $\Omega_s = \{x \in \Omega, |\nabla z| \le s\}$).

Then

$$M(|\nabla z_n|) \to M(|\nabla z|) \text{ in } L^1(\Omega).$$

Remark 4.3. The condition b) is not necessary in the case where the N-function M satisfies the Δ_2-condition.

Indeed. The condition a) implies that the sequence $(z_n)_n$ is bounded in $W_0^1L_M(\Omega)$, hence there exists two positive constants λ, C such that

$$\int_\Omega M(\lambda|\nabla z_n|)\,dx \le C. \tag{4.2}$$

On the other hand, by the condition (2.2) there exists a constant positive $r(k_2)$ such that $M(k_2t) \le r(k_2)M(\lambda t) + c_1$, $\forall t > 0$. Let $\varepsilon > 0$. Let $\mu > 0$ large enough, we have by using (A_2)

$$\int_\Omega \overline{M}\left(\frac{|a(x,\nabla z_n)|}{\mu}\right)dx \le \frac{1}{\mu}\int_\Omega \overline{M}(c(x))\,dx + c_2 + \frac{k_1}{\mu}\int_\Omega \overline{M}(\lambda|\nabla z_n|). \tag{4.3}$$

From (4.2) and (4.3) we deduce that $(a(x,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

Remark 4.4. Note that the previous lemma holds, also in the general case where $a \equiv a(x,s,\xi)$.

Proposition 4.1. *Assume that* (A_1)–(A_5), (3.3) *and* (3.4) *hold true and let* u_n *be a solution of the approximate problem* (4.1). *Then for all* $k > 0$, *there exists a constant* $c(k)$ (*which does not depend on the* n) *such that,*

$$\|T_k(u_n)\|_{W_0^1 L_M(\Omega)} \leq c(k).$$

Proposition 4.2. *Assume that* (A_1)–(A_5), (3.3) *and* (3.4) *hold true and let* u_n *be a solution of the approximate problem* (4.1), *then there exists a measurable function* u *such that, for all* $k > 0$ *we have,*

1) $u_n \to u$ *a.e. in* Ω,
2) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^1 L_M(\Omega)$ *for* $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$,
3) $T_k(u_n) \to T_k(u)$ *strongly in* $E_M(\Omega)$ *and a.e. in* Ω.

Proposition 4.3. *Assume that* (A_1)–(A_5), (3.3) *and* (3.4) *hold true and let* u_n *be a solution of the approximate problem* (4.1). *Then for all* $k > 0$,

1) $(a(x, \nabla T_k(u_n)))_n$ *is bounded in* $(L_{\overline{M}}(\Omega))^N$,
2) $M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|)$ *in* $L^1(\Omega)$.

4.3. Proof of Theorem 3.1.

Let $v \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$. Taking $u_n - T_k(u_n - v)$ as test function in (4.1), we can write, for n large enough $(n > k + \|v\|_\infty)$,

$$\begin{aligned} &\int_\Omega a(x, \nabla u_n) \nabla T_k(u_n - v)\, dx + \int_\Omega \phi(u_n) \nabla T_k(u_n - v)\, dx \\ &\leq \int_\Omega f_n T_k(u_n - v)\, dx, \end{aligned} \tag{4.4}$$

which implies that,

$$\begin{aligned} &\int_{\{|u_n - v| \leq k\}} a(x, \nabla u_n) \nabla (u_n - v_0)\, dx \\ &+ \int_{\{|u_n - v| \leq k\}} a(x, \nabla T_{k+\|v\|_\infty}(u_n)) \nabla (v_0 - v)\, dx \\ &+ \int_\Omega \phi(u_n) \nabla T_k(u_n - v)\, dx \leq \int_\Omega f_n T_k(u_n - v)\, dx. \end{aligned} \tag{4.5}$$

Now, applying the assertion 2) of Proposition 4.3, assertions 1), 3) of Proposition 4.2 and Fatou's lemma, we have,

$$\begin{aligned} &\int_{\{|u - v| \leq k\}} a(x, \nabla u) \nabla (u - v_0)\, dx \\ &\leq \liminf_{n \to \infty} \int_{\{|u_n - v| \leq k\}} a(x, \nabla u_n) \nabla (u_n - v_0)\, dx. \end{aligned} \tag{4.6}$$

On the other hand, by Proposition 4.3 we get,

$$a(x, \nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x, \nabla T_{k+\|v\|_\infty}(u)) \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\prod L_{\overline{M}}, \prod E_M),$$

which and assertion 1) of Proposition 4.2, Lebesgue's theorem, allow to deduce

$$\int_{\{|u_n-v|\le k\}} a(x, \nabla T_{k+\|v\|_\infty}(u_n))\nabla(v_0 - v)\,dx \to \int_{\{|u-v|\le k\}} a(x, \nabla T_{k+\|v\|_\infty}(u))\nabla(v_0 - v)\,dx. \tag{4.7}$$

Moreover, thanks to assertion 1) and 2) of Proposition 4.2, we have

$$\int_\Omega \phi(u_n)\nabla T_k(u_n - v)\,dx \to \int_\Omega \phi(u)\nabla T_k(u - v)\,dx. \tag{4.8}$$

Combining (4.5)–(4.8), we get

$$\int_{\{|u-v|\le k\}} a(x, \nabla u)\nabla(u-v_0)\,dx + \int_{\{|u-v|\le k\}} a(x, \nabla T_{k+\|v\|_\infty}(u))\nabla(v_0-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx \le \int_\Omega fT_k(u-v)\,dx. \tag{4.9}$$

Hence,

$$\int_\Omega a(x, \nabla u)\nabla T_k(u - v)\,dx + \int_\Omega \phi(u)\nabla T_k(u - v)\,dx \le \int_\Omega fT_k(u - v)\,dx. \tag{4.10}$$

Now, let $v \in K_\psi \cap L^\infty(\Omega)$. By the condition (A_5) there exists $v_j \in K_\psi \cap W_0^1E_M(\Omega) \cap L^\infty(\Omega)$ such that v_j converges to v in the modular sense. Let $h \ge \max(\|v_0\|_\infty, \|v\|_\infty)$ and taking $v = T_h(v_j)$ in (4.10), we have

$$\int_\Omega a(x, \nabla u)\nabla T_k(u - T_h(v_j))\,dx + \int_\Omega \phi(u)\nabla T_k(u - T_h(v_j))\,dx \le \int_\Omega fT_k(u - T_h(v_j))\,dx. \tag{4.11}$$

We can easily pass to the limit as $j \to +\infty$ and get,

$$\int_\Omega a(x, \nabla u)\nabla T_k(u - T_h(v))\,dx + \int_\Omega \phi(u)\nabla T_k(u - T_h(v))\,dx \le \int_\Omega fT_k(u - T_h(v))\,dx \quad \forall\, v \in K_\psi \cap L^\infty(\Omega). \tag{4.12}$$

Finally, since $h \geq \max(\|v_0\|_\infty, \|v\|_\infty)$, we get

$$\begin{aligned}&\int_\Omega a(x,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx\\&\leq \int_\Omega fT_k(u-v)\,dx \quad \forall v \in K_\psi \cap L^\infty(\Omega),\ \forall k>0,\end{aligned} \tag{4.13}$$

this, completes the proof of Theorem 3.1.

5. Proof of intermediates results

5.1. Proof of Lemma 4.2.

Fix $r>0$ and let $s>r$, since $\Omega_r \subset \Omega_s$ we have,

$$\begin{aligned}0 &\leq \int_{\Omega_r} [a(x,\nabla z_n) - a(x,\nabla z)][\nabla z_n - \nabla z]\,dx\\&\leq \int_{\Omega_s} [a(x,\nabla z_n) - a(x,\nabla z)][\nabla z_n - \nabla z]\,dx\\&= \int_{\Omega_s} [a(x,\nabla z_n) - a(x,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx\\&\leq \int_{\Omega} [a(x,\nabla z_n) - a(x,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx.\end{aligned} \tag{5.1}$$

Which with c) imply that,

$$\lim_{n\to\infty} \int_{\Omega_r} [a(x,\nabla z_n) - a(x,\nabla z)][\nabla z_n - \nabla z]\,dx = 0. \tag{5.2}$$

So, (as in [12])

$$\nabla z_n \to \nabla z \quad \text{a.e. in } \Omega. \tag{5.3}$$

On the one side, we have

$$\begin{aligned}\int_\Omega a(x,\nabla z_n)\nabla z_n\,dx =& \int_\Omega [a(x,\nabla z_n) - a(x,\nabla z\chi_s)] \times [\nabla z_n - \nabla z\chi_s]\,dx\\&+ \int_\Omega a(x,\nabla z\chi_s)(\nabla z_n - \nabla z\chi_s)\,dx\\&+ \int_\Omega a(x,\nabla z_n)\nabla z\chi_s\,dx.\end{aligned} \tag{5.4}$$

Since $(a(x,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, from (5.3), we obtain

$$a(x,\nabla z_n) \rightharpoonup a(x,\nabla z) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma\left(\prod L_{\overline{M}}, \prod E_M\right). \tag{5.5}$$

Consequently,

$$\int_\Omega a(x,\nabla z_n)\nabla z\chi_s\,dx \to \int_\Omega a(x,\nabla z)\nabla z\chi_s\,dx \tag{5.6}$$

as $n \to \infty$.

Letting also $s \to \infty$, we obtain,

$$\int_\Omega a(x, \nabla z)\nabla z \chi_s \, dx \to \int_\Omega a(x, \nabla z)\nabla z \, dx. \tag{5.7}$$

On the other hand, it is easy to see that the second term of the right hand side of (5.4) tends to 0 as $n \to \infty$ and $s \to \infty$.

Moreover, from c), (5.6) and (5.7) we have,

$$\lim_{n\to\infty} \int_\Omega a(x, \nabla z_n)\nabla z_n \, dx = \int_\Omega a(x, \nabla z)\nabla z \, dx, \tag{5.8}$$

hence

$$\lim_{n\to\infty} \int_\Omega a(x, \nabla z_n)(\nabla z_n - \nabla v_0) \, dx = \int_\Omega a(x, \nabla z)\nabla(z - \nabla v_0) \, dx.$$

Finally, using (A_4) one obtain by Lemma 2.6 and Vitali's theorem,

$$M(|\nabla z_n|) \longrightarrow M(|\nabla z|) \text{ in } L^1(\Omega).$$

5.2. Proof of Proposition 4.1.

Let $k > 0$. Taking $u_n - T_k(u_n - v_0)$ as test function in (4.1), we obtain for n large enough

$$\int_\Omega a(x, \nabla u_n)\nabla T_k(u_n - v_0) \, dx + \int_\Omega \phi(u_n)\nabla T_k(u_n - v_0) \, dx \\ \leq \int_\Omega f_n T_k(u_n - v_0) \, dx.$$

Since, $\nabla T_k(u_n - v_0)$ is identically zero on the set where $|u_n(x) - v_0(x)| > k$, hence we can write

$$\int_\Omega a(x, \nabla u_n)\nabla T_k(u_n - v_0) \, dx \leq \int_{\{|u_n - v_0| \leq k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla u_n| \, dx \\ + \int_{\{|u_n - v_0| \leq k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla v_0| \, dx \\ + \int_\Omega f_n T_k(u_n - v_0) \, dx,$$

which gives, by using (3.4) and Young's inequality,

$$\int_{\{|u_n - v_0| \leq k\}} a(x, \nabla u_n)\nabla(u_n - v_0) \, dx \\ \leq \frac{\alpha}{2} \int_{\{|u_n - v_0| \leq k\}} M(|\nabla u_n|) \, dx + c_1(k), \tag{5.9}$$

where $c_1(k)$ is a constant which depends of k, which with (A_4) yields

$$\int_{\{|u_n - v_0| \le k\}} M(|\nabla u_n|)\,dx \le c_2(k). \tag{5.10}$$

Since k is arbitrary and

$$\{|u_n| \le k\} \subset \{|u_n - v_0| \le k + \|v_0\|_\infty\},$$

we deduce that,

$$\int_\Omega M(|\nabla T_k(u_n)|)\,dx \;\le\; \int_{\{|u_n - v_0| \le k + \|v_0\|_\infty\}} M(|\nabla u_n|)\,dx \;\le\; c_3(k), \tag{5.11}$$

from which, we get

$$\|T_k(u_n)\|_{W_0^1 L_M(\Omega)} \le c(k). \tag{5.12}$$

5.3. Proof of Proposition 4.2.

Step 1. We claim that: for $k > h > \|v_0\|_\infty$

$$\int_\Omega M(|\nabla T_k(u_n - T_h(u_n))|)\,dx \le kC \tag{5.13}$$

where C is a constant does not depends of n, k and h.

Using the Proposition 4.1, there exists some $v_k \in W_0^1 L_M(\Omega)$ such that,

$$T_k(u_n) \rightharpoonup v_k \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma\left(\prod L_M, \prod E_{\overline{M}}\right),$$
$$T_k(u_n) \to v_k \text{ strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \tag{5.14}$$

On the other hand, let $k > h \ge \|v_0\|_\infty$. By using $v = u_n - T_k(u_n - T_h(u_n))$ as test function in (4.1) we obtain,

$$\int_\Omega a(x, \nabla u_n)\nabla T_k(u_n - T_h(u_n))\,dx + \int_\Omega \phi(T_n(u_n))\nabla T_k(u_n - T_h(u_n))\,dx$$
$$\le \int_\Omega f_n T_k(u_n - T_h(u_n))\,dx.$$

The second term of the left hand side of the last inequality vanishes for n large enough. Indeed, we have by virtue of Lemma 2.7,

$$\begin{aligned}\int_\Omega \phi(T_n(u_n))\nabla T_k(u_n - T_h(u_n))\,dx &= \int_\Omega \phi(u_n)\nabla T_k(u_n - T_h(u_n))\,dx\\ &= \int_\Omega \operatorname{div}\left[\int_0^{u_n} \phi(s)\chi_{\{h \le |s| \le k+h\}}\,ds\right]dx\\ &= 0,\end{aligned}$$

(this is due to $\int_0^{u_n} \phi(s)\chi_{\{h \le |s| \le k+h\}}\,ds$ lies in $W_0^1 L_M(\Omega)$).

Thus,

$$\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n-T_h(u_n))\,dx \le \int_\Omega f_n T_k(u_n-T_h(u_n))\,dx$$

which implies that,

$$\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n-T_h(u_n))\,dx \le kc_3, \tag{5.15}$$

where c_3 is a nonnegative constant independent of n, k and h.

Now, let a constant c such that $0<c<1$ and satisfies

$$\frac{\alpha(1-c)}{2c} > \lambda > 1+k_1.$$

(Such constant c is well existed since $\lim\limits_{c\to 0^+} \dfrac{\alpha(1-c)}{2c} = +\infty$.)

From (5.15) we have

$$\begin{aligned}
&\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))[\nabla T_k(u_n-T_h(u_n))-(1-c)\nabla v_0]\,dx\\
&\le c_3k+\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))(c-1)\nabla v_0\,dx\\
&= c_3k+c\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))\left(\frac{c-1}{c}\nabla v_0\right)\,dx
\end{aligned}$$

and from the monotonicity condition (A_3) we get,

$$\begin{aligned}
&\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))[\nabla T_k(u_n-T_h(u_n))-(1-c)\nabla v_0]\,dx\\
&\le c_3k+c\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))\nabla T_k(u_n-T_h(u_n))\,dx\\
&-c\int_\Omega a(x,\frac{c-1}{c}\nabla v_0)[\nabla T_k(u_n-T_h(u_n))-\frac{c-1}{c}\nabla v_0]\,dx.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(1-c)\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))[\nabla T_k(u_n-T_h(u_n))-\nabla v_0]\,dx\\
&\le c_3k+c_4+c\int_\Omega \left|a\left(x,\frac{c-1}{c}\nabla v_0\right)\right| |\nabla T_k(u_n-T_h(u_n))|\,dx\\
&= c_3k+c_4+\frac{\alpha(1-c)}{2}.\frac{2c}{\alpha(1-c)}\int_\Omega \left|a\left(x,\frac{c-1}{c}\nabla v_0\right)\right| |\nabla T_k(u_n-T_h(u_n))|\,dx\\
&= c_3k+c_4+\frac{\alpha(1-c)}{2}\int_\Omega \left|\frac{a\left(x,\frac{c-1}{c}\nabla v_0\right)}{\frac{\alpha(1-c)}{2c}}\right| |\nabla T_k(u_n-T_h(u_n))|\,dx.
\end{aligned}$$

Thanks to Young's inequality, we can deduce that

$$\begin{aligned}
&(1-c)\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))[\nabla T_k(u_n-T_h(u_n))-\nabla v_0]\,dx\\
&\le c_3k+c_4+\frac{\alpha(1-c)}{2}\int_\Omega \overline{M}\left(\frac{\left|a\left(x,\dfrac{c-1}{c}\nabla v_0\right)\right|}{\lambda}\right)dx\\
&+\frac{\alpha(1-c)}{2}\int_\Omega M(|\nabla T_k(u_n-T_h(u_n))|)\,dx
\end{aligned}$$

from which we can deduce (5.13) after using (A_4).

STEP 2. Convergence in measure of u_n. In this step, we prove that u_n converges to some function u in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). We shall show that u_n is a Cauchy sequence in measure.

Let $k>h>\|v_0\|_\infty$ large enough. Thanks to Lemma 5.7 of [12] and (5.13), there exist two positive constants c_7 and c_8 independent of k and h such that,

$$\begin{aligned}
&\int_\Omega M(c_7|T_k(u_n-T_h(u_n))|)\,dx\\
&\le c_8\int_\Omega M(|\nabla T_k(u_n-T_h(u_n))|)\,dx\le c_9k. \qquad (5.16)
\end{aligned}$$

This yields, using (5.16),

$$\begin{aligned}
&M(c_7k)\text{meas}\{|u_n-T_h(u_n)|>k\}\\
&=\int_{\{|u_n-T_h(u_n)|>k\}} M(c_7|T_k(u_n-T_h(u_n))|)\,dx\\
&\le c_8\int_\Omega M(|\nabla T_k(u_n-T_h(u_n))|)\,dx\\
&\le kc_9.
\end{aligned}$$

So,

$$\text{meas}(\{|u_n-T_h(u_n)|>k\})\le\frac{kc_9}{M(kc_7)} \quad \text{for all } n \text{ and for all } k>h>\|v_0\|_\infty. \qquad (5.17)$$

Hence,

$$\text{meas}(\{|u_n|>k\})\le\text{meas}(\{|u_n-T_h(u_n)|>k-h\})\le\frac{(k-h)c_9}{M((k-h)c_7)} \text{ for all } n.$$

Therefore, as k tends to infinity, using

$$\frac{t}{M(t)} \to 0 \text{ as } t \to \infty,$$

we obtain

$$\text{meas}(\{|u_n| > k\}) \to 0 \text{ as } k \text{ tends to infinity uniformly in } n. \tag{5.18}$$

Now, let $\lambda > 0$, we have

$$\begin{aligned}\text{meas}(\{|u_n - u_m| > \lambda\}) \leq \text{meas}(\{|u_n| > k\}) + \text{meas}(\{|u_m| > k\}) \\ + \text{meas}(\{|T_k(u_n) - T_k(u_m)| > \lambda\}).\end{aligned}$$

From (5.14), we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$, by (5.18), (5.19) and the fact that $T_k(u_n)$ is a Cauchy sequence in measure, there exists some $k(\varepsilon) > 0$ such that $\text{meas}(\{|u_n - u_m| > \lambda\}) < \varepsilon$ for all n, $m \geq n_0(k(\varepsilon), \lambda)$. This proves that $(u_n)_n$ is a Cauchy sequence in measure in Ω, thus converges almost everywhere to some measurable function u. Then we deduce the result of Proposition 4.2.

5.4. Proof of Proposition 4.3.

1) Boundedness of $(a(x, \nabla T_k(u_n))_n$ in $(L_{\overline{M}}(\Omega))^N$.
Let $w \in (E_M(\Omega))^N$ be arbitrary. By condition (A_3) we have,

$$(a(x, \nabla u_n) - a(x, w))(\nabla u_n - w) \geq 0$$

which implies that,

$$a(x, \nabla u_n)(w - \nabla v_0) \leq a(x, \nabla u_n)(\nabla u_n - \nabla v_0) - a(x, w)(\nabla u_n - w).$$

Consequently,

$$\begin{aligned}&\int_{\{|u_n - v_0| \leq k\}} a(x, \nabla u_n)(w - \nabla v_0)\, dx \\ &\leq \int_{\{|u_n - v_0| \leq k\}} a(x, \nabla u_n)(\nabla u_n - \nabla v_0)\, dx \\ &+ \int_{\{|u_n - v_0| \leq k\}} a(x, w)(w - \nabla u_n)\, dx.\end{aligned} \tag{5.20}$$

Combining (5.9) and (5.10), we get

$$\int_{\{|u_n - v_0| \leq k\}} a(x, \nabla u_n)(\nabla u_n - \nabla v_0)\, dx \leq C_{11}, \tag{5.21}$$

with C_{11} is a positive constant.

On the other hand, we have by (A_2)

$$|a(x, w)| \leq c(x) + k_1 \overline{M}^{-1} M(k_2 |w|).$$

Therefore,

$$\int_\Omega \overline{M}(\frac{a(x,w)}{\lambda})\,dx \;\leq\; \int_\Omega \overline{M}(\frac{c(x)}{\lambda}) \;+\; \int_\Omega \frac{k_1}{\lambda} M(k_2|w|) \;\leq\; C_{12} \qquad (5.22)$$

when $\lambda > 0$ is large enough.

Which implies that the second term on the right in (5.20) is also bounded. By the theorem of Banach-Steinhaus, the sequence $(a(x,\nabla u_n)\chi_{\{|u_n - v_0| \leq k\}})_n$ remains bounded in $(L_{\overline{M}}(\Omega))^N$. Since k is arbitrary, we deduce that $(a(x,\nabla T_k(u_n)))_n$ is also bounded in $(L_{\overline{M}}(\Omega))^N$. Which implies that, for all $k > 0$, there exists a function $\rho_k \in (L_{\overline{M}}(\Omega))^N$ such that,

$$a(x,\nabla T_k(u_n)) \rightharpoonup \rho_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N$$
$$\text{for } \sigma\left(\prod L_{\overline{M}}(\Omega), \prod E_M(\Omega)\right). \qquad (5.23)$$

2) We claim that $M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|)$ in $L^1(\Omega)$.
We fix $k > 0$ and let $\Omega_r = \{x \in \Omega, |\nabla T_k(u(x))| \leq r\}$ and denote by χ_r the characteristic function of Ω_r. Clearly, $\Omega_r \subset \Omega_{r+1}$ and $\text{meas}(\Omega \backslash \Omega_r) \longrightarrow 0$ as $r \longrightarrow \infty$.

By using (A_5), there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1 L_M(\Omega)$.

We will introduce the following function of one real variable s, which is defined as

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \leq m \\ -|s| + m + 1 & \text{if } m \leq |s| \leq m+1 \\ 0 & \text{if } |s| \geq m+1. \end{cases}$$

The choose of the $u_n - h_m(u_n - v_0)(T_k(u_n) - T_k(v_j))$ as test function in (4.1), we gives (using the fact that the derivative of $h_m(s)$ is different from zero only where $m < |s| < m+1$),

$$\begin{aligned} &\int_\Omega a(x,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j))h_m(u_n - v_0)\,dx \\ &+ \int_{\{m<|u_n - v_0|<m+1\}} a(x,\nabla u_n)\nabla(u_n - v_0)(T_k(u_n) - T_k(v_j))h'_m(u_n - v_0)\,dx \\ &+ \int_{\{m<|u_n - v_0|<m+1\}} \phi(u_n)\nabla(u_n - v_0)(T_k(u_n) - T_k(v_j))h'_m(u_n - v_0)\,dx \\ &+ \int_\Omega \phi(u_n)(\nabla T_k(u_n) - \nabla T_k(v_j))h_m(u_n - v_0)\,dx \\ &\leq \int_\Omega f_n h_m(u_n - v_0)(T_k(u_n) - T_k(v_j))\,dx. \end{aligned} \qquad (5.24)$$

In the sequel and throughout the paper, we will denote $\varepsilon(n,j,m,s)$ all quantities (possibly different) such that

$$\lim_{s\to+\infty}\lim_{m\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\varepsilon(n,j,m,s)=0$$

and this will be the other in which the parameters we use will tend to infinity, that is, first n, then j, m and finally s. Similarly, we write only $\varepsilon(n)$, or $\varepsilon(n,j)$, ... to mean that the limits are made only on the specified parameters.

We will deal with each term of (5.24). First of all, observe that

$$\int_\Omega f_n h_m(u_n-v_0)(T_k(u_n)-T_k(v_j))\,dx=\varepsilon(n,j). \tag{5.25}$$

Indeed. In view of assertion 1) of Proposition 4.2, we have

$$h_m(u_n-v_0)(T_k(u_n)-T_k(v_j))\to h_m(u-v_0)(T_k(u)-T_k(v_j))$$
$$\text{weakly}^* \text{ as } n\to+\infty \text{ in } L^\infty(\Omega),$$

and then,

$$\int_\Omega f_n h_m(u_n-v_0)(T_k(u_n)-T_k(v_j))\,dx$$
$$\to\int_\Omega f h_m(u-v_0)(T_k(u)-T_k(v_j))\,dx \text{ as } n\to+\infty.$$

Since

$$h_m(u-v_0)(T_k(u)-T_k(v_j))\to 0 \text{ weak}^* \text{ in } L^\infty(\Omega) \text{ as } j\to+\infty,$$

we get

$$\int_\Omega f h_m(u-v_0)(T_k(u)-T_k(v_j))\,dx\to 0 \text{ as } j\to+\infty.$$

For what concerns the third term of the left hand side of (5.24), we have by letting $n\to\infty$

$$\int_{\{m<|u_n-v_0|<m+1\}}\phi(u_n)\nabla(u_n-v_0)(T_k(u_n)-T_k(v_j))h'_m(u_n-v_0)\,dx$$
$$=\int_{\{m<|u-v_0|<m+1\}}\phi(u)\nabla(u-v_0)(T_k(u)-T_k(v_j))h'_m(u-v_0)\,dx+\varepsilon(n)$$

since

$$\phi(u_n)\chi_{\{m<|u_n-v_0|<m+1\}}(T_k(u_n)-T_k(v_j))$$
$$\to\phi(u)\chi_{\{m<|u-v_0|<m+1\}}(T_k(u)-T_k(v_j)),$$

strongly in $(E_{\overline{M}}(\Omega))^N$ by assertion 1) of Proposition 4.2 and Lebesgue theorem while $\nabla T_{m+1}(u_n)\rightharpoonup\nabla T_{m+1}(u_n)$ weakly in $(L_M(\Omega))^N$ by assertion 2) of Proposition 4.2.

Letting $j \to \infty$ in the right term of the above equality, one has, by using the modular convergence of $(v_j)_j$

$$\int_{\{m<|u-v_0|<m+1\}} \phi(u)\nabla(u-v_0)(T_k(u)-T_k(v_j))h'_m(u-v_0)\,dx = \varepsilon(j)$$

and so

$$\int_{\{m<|u_n-v_0|<m+1\}} \phi(u_n)\nabla(u_n-v_0)(T_k(u_n)-T_k(v_j))h'_m(u_n-v_0)\,dx$$
$$= \varepsilon(n,j). \tag{5.26}$$

Similarly, we have

$$\int_\Omega \phi(u_n)(\nabla T_k(u_n)-\nabla T_k(v_j))h_m(u_n-v_0)\,dx = \varepsilon(n,j). \tag{5.27}$$

Starting with the second term of the left hand side of (5.24), we have

$$\left|\int_{\{m<|u_n-v_0|<m+1\}} a(x,\nabla u_n)\nabla(u_n-v_0)(T_k(u_n)-T_k(v_j))h'_m(u_n-v_0)\,dx\right|$$
$$\le 2k\left|\int_{\{m<|u_n-v_0|<m+1\}} a(x,\nabla u_n)\nabla(u_n-v_0)+\delta(x)\,dx\right|$$
$$+ 2k\int_{\{m<|u_n-v_0|<m+1\}} \delta(x)\,dx. \tag{5.28}$$

Moreover, since $\{m < |u_n - v_0| < m+1\} \subset \{l < |u_n| < l+s\}$ where $l = m - \|v_0\|_\infty, s = 2\|v_0\|_\infty + 1$, we get

$$2k\left|\int_{\{m<|u_n-v_0|<m+1\}} (a(x,\nabla u_n)\nabla(u_n-v_0)+\delta(x))\,dx\right|$$
$$\le 2k\int_{\{l<|u_n|<l+s\}} (a(x,\nabla u_n)\nabla(u_n-v_0)+\delta(x))\,dx$$
$$= 2k\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla u_n\,dx - 2k\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla v_0\,dx$$
$$+ 4k\int_{\{l<|u_n|<l+s\}} \delta(x)\,dx. \tag{5.29}$$

Now, we take $u_n - T_s(u_n - T_l(u_n))$ as test function in (4.1), we get

$$\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla u_n\,dx + \int_\Omega \operatorname{div}\left[\int_0^{u_n} \phi(t)\chi_{\{l\le|t|\le l+s\}}\,dt\right]dx$$
$$\le \int_\Omega f_n T_s(u_n - T_l(u_n))\,dx \le s\int_{\{|u_n|>l\}} |f_n|\,dx$$

and using the fact that

$$\int_0^{u_n} \phi(t)\chi_{\{l\leq|t|\leq l+s\}}\,dt \in W_0^1 L_M(\Omega)$$

and Lemma 2.7 one has,

$$\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla u_n\,dx \leq \int_\Omega f_n T_s(u_n - T_l(u_n))\,dx$$
$$\leq s\int_{\{|u_n|>l\}} |f_n|\,dx. \tag{5.30}$$

On the other side, the Hölder's inequality gives

$$\left|-2k\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla v_0\,dx\right|$$
$$\leq 4k\|a(x,\nabla T_s(u_n - T_l(u_n)))\|_{\overline{M}}\|\nabla v_0\chi_{\{|u_n|>l\}}\|_M. \tag{5.31}$$

Furthermore, by the same argument as in the proof of the Proposition 4.3 (step 1), we get

$$\|a(x,\nabla T_s(u_n - T_l(u_n)))\|_{\overline{M}} \leq C_{14},$$

where C_{14} is a positive constant independent of n and m.

Combining (5.29), (5.30) and (5.31), we deduce

$$\left|2k\int_{\{m<|u_n-v_0|<m+1\}} \left(a(x,\nabla u_n)\nabla(u_n - v_0) + \delta(x)\right)\,dx\right|$$
$$\leq C_{15}\int_{\{|u_n|>l\}} (\delta(x) + |f_n|)\,dx + C_{16}\|\nabla v_0\chi_{\{|u_n|>l\}}\|_M. \tag{5.32}$$

Letting successively first n, then m $\;(l = m - \|v_0\|_\infty)$ go to infinity, we find, by using the fact that $\delta \in L^1(\Omega), v_0 \in W_0^1 E_M(\Omega)$ and the strong convergence of f_n

$$\left|\int_{\{m<|u_n-v_0|<m+1\}} \left(a(x,\nabla u_n)\nabla(u_n - v_0) + \delta(x)\right)\,dx\right|$$
$$= \varepsilon(n,m). \tag{5.33}$$

Finally, we have

$$\left|\int_{\{m<|u_n-v_0|<m+1\}} a(x,\nabla u_n)\nabla(u_n - v_0)(T_k(u_n) - T_k(v_j))h'_m(u_n - v_0)\,dx\right|$$
$$= \varepsilon_j(n,m). \tag{5.34}$$

By means of (5.24)–(5.27), (5.34), we obtain

$$\int_\Omega a(x,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j))h_m(u_n - v_0)\,dx$$

$$\leq \varepsilon(n,m) + \varepsilon(n,j). \tag{5.35}$$

Splitting the integral on the left hand side of (5.35) where $|u_n| \leq k$ and $|u_n| > k$, we can write,

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j))h_m(u_n - v_0)\,dx \\
&= \int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(v_j)]h_m(u_n - v_0)\,dx \\
&+ \int_{\{|u_n|>k\}} a(x,0)\nabla T_k(v_j)h_m(u_n - v_0)\,dx \\
&- \int_{\{|u_n|>k\}} a(x,\nabla u_n)\nabla T_k(v_j)h_m(u_n - v_0)\,dx \qquad (5.36)\\
&\geq \int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(v_j)]h_m(u_n - v_0)\,dx \\
&- \int_{\{|u_n|>k\}} \big|a(x,0) + a(x,\nabla T_{m+\|v_0\|_\infty+1}(u_n))||\nabla v_j\big|\ dx.
\end{aligned}$$

Since $(|a(x,0) + a(x,\nabla T_{m+\|v_0\|_\infty+1}(u_n))|)_n$ is bounded in $L_{\overline{M}}(\Omega)$, we get, for a subsequence still denoted u_n

$$|a(x,0) + a(x,\nabla T_{m-\|v_0\|_\infty+1}(u_n))| \rightharpoonup l_m \text{ weakly in } L_{\overline{M}}(\Omega) \text{ for } \sigma(L_{\overline{M}}, E_M),$$

and since, $|\nabla v_j|\chi_{\{|u_n|>k\}}$ converges strongly to $|\nabla v_j|\chi_{\{|u|>k\}}$ in $E_M(\Omega)$, we have by letting $n \to \infty$

$$-\int_{\{|u_n|>k\}} |a(x,0) + a(x,\nabla T_{m+\|v_0\|_\infty+1}(u_n))|\nabla v_j|\,dx \to -\int_{\{|u|>k\}} l_m|\nabla v_j|\,dx$$

as n tends to infinity.

Using now, the modular convergence of $(v_j)_j$, we get

$$-\int_{\{|u|>k\}} l_m|\nabla v_j|\,dx \to -\int_{\{|u|>k\}} l_m|\nabla T_k(u)|\,dx$$

as j tends to infinity.

Since $\nabla T_k(u) = 0$ in $\{|u| > k\}$ we deduce that,

$$-\int_{\{|u_n|>k\}} |a(x,0) + a(x,\nabla T_{m+\|v_0\|_\infty+1}(u_n))|\nabla v_j|\,dx = \varepsilon(n,j). \tag{5.37}$$

We then have by (5.36),

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j))h_m(u_n - v_0)\,dx \qquad (5.38)\\
&\geq \int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(v_j)]h_m(u - v_0)\,dx + \varepsilon(n,j).
\end{aligned}$$

It is easily to see that,

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)(\nabla T_k(u_n)-\nabla T_k(v_j))h_m(u_n-v_0)\,dx\\
&\geq \int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(v_j)\chi_s^j)]\\
&\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx \qquad (5.39)\\
&+\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx\\
&-\int_{\Omega\setminus\Omega_s^j}|a(x,\nabla T_k(u_n))||\nabla v_j|\,dx+\varepsilon(n,j),
\end{aligned}$$

where χ_s^j denotes the characteristic function of the subset $\Omega_s^j = \{x\in\Omega\colon |\nabla T_k(v_j)|\leq s\}$, and as above we have

$$\begin{aligned}
&-\int_{\Omega\setminus\Omega_s^j}|a(x,\nabla T_k(u_n))||\nabla v_j|\,dx\\
&=-\int_{\Omega\setminus\Omega_s}\rho_k|\nabla T_k(u)|\,dx+\varepsilon(n,j). \qquad (5.40)
\end{aligned}$$

where ρ_k is some function in $L_{\overline{M}}(\Omega)$ such that

$$|a(x,\nabla T_k(u_n))|\rightharpoonup\rho_k \text{ weakly in } L_{\overline{M}}(\Omega) \text{ for } \sigma(L_{\overline{M}},E_M).$$

For what concerns the second term of the right hand side of (5.39) we can write,

$$\begin{aligned}
&\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx\\
&=\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)h_m(T_k(u_n)-v_0)\,dx \qquad (5.41)\\
&-\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j h_m(u_n-v_0)\,dx.
\end{aligned}$$

Starting of the second term of the last equality, we have

$$\begin{aligned}
&\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)h_m(u_n-v_0)\,dx\\
&=\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(u)h_m(u-v_0)\,dx+\varepsilon(n)
\end{aligned}$$

since

$$a(x,\nabla T_k(v_j)\chi_s^j)h_m(T_k(u_n)-v_0)\to a(x,\nabla T_k(v_j)\chi_s^j)h_m(T_k(u)-v_0)$$

strongly in $(E_{\overline{M}}(\Omega))^N$ by Lemma 2.3 while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ for $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$. Letting again $j \to \infty$, one has, since

$$a(x, \nabla T_k(v_j)\chi_s^j) h_m(T_k(u) - v_0) \to a(x, \nabla T_k(u)\chi_s) h_m(T_k(u) - v_0)$$

strongly in $(E_{\overline{M}}(\Omega))^N$ by using the modular convergence of v_j and Lebesgue theorem

$$\begin{aligned}&\int_\Omega a(x, \nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n) h_m(u_n - v_0)\, dx\\ &= \int_\Omega a(x, \nabla T_k(u)\chi_s)\nabla T_k(u) h_m(u - v_0)\, dx + \varepsilon(n, j).\end{aligned}$$

In the same way, we have

$$\begin{aligned}&-\int_\Omega a(x, \nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j h_m(u_n - v_0)\, dx\\ &= \int_{\Omega\setminus\Omega_s} a(x, \nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s h_m(u - v_0)\, dx + \varepsilon(n, j).\end{aligned}$$

Adding the two equalities we conclude

$$\begin{aligned}&\int_\Omega a(x, \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] h_m(u_n - v_0)\, dx\\ &= \int_{\Omega\setminus\Omega_s} a(x, 0)\nabla T_k(u) h_m(u - v_0)\, dx + \varepsilon(n, j).\end{aligned}$$

Since $1 - h_m(u - v_0) = 0$ in $\{|u(x) - v_0(x)| \le m\}$ and since $\{|u(x)| \le k\} \subset \{|u(x) - v_0(x)| \le m\}$ for m large enough, we deduce

$$\begin{aligned}&\int_\Omega a(x, \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] h_m(u_n - v_0)\, dx\\ &= \int_{\Omega\setminus\Omega_s} a(x, 0)\nabla T_k(u)\, dx + \varepsilon(n, j). \qquad (5.42)\end{aligned}$$

Combining (5.39), (5.40) and (5.42), we get

$$\begin{aligned}&\int_\Omega a(x, \nabla u_n)[\nabla T_k(u_n) - \nabla T_k(v_j)] h_m(u_n - v_0)\, dx\\ &\ge \int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(v_j)\chi_s^j)] \qquad (5.43)\\ &\times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j)] h_m(u_n - v_0)\, dx\\ &- \int_{\Omega\setminus\Omega_s} \rho_k |\nabla T_k(u)|\, dx + \int_{\Omega\setminus\Omega_s} a(x, 0)\nabla T_k(u)\, dx + \varepsilon(n, j).\end{aligned}$$

This and (5.35) yield

$$\begin{aligned}&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(v_j)\chi_s^j)]\\&\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx\\&\le \int_{\Omega\setminus\Omega_s}\rho_k|\nabla T_k(u)|\,dx+\int_{\Omega\setminus\Omega_s}a(x,0)\nabla T_k(u)\,dx\\&+\varepsilon(n,j)+\varepsilon(n,m).\end{aligned}\tag{5.44}$$

On the other hand, we have

$$\begin{aligned}&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx\\&-\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(v_j)\chi_s^j)]\\&\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx\\&=\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx\\&-\int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx\\&+\int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(v_j)\chi_s^j-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx,\end{aligned}\tag{5.45}$$

an, as it can be easily seen that the term of the right-hand side is the form $\varepsilon(n,j)$ implying that

$$\begin{aligned}&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)]\\&\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx\\&=\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(v_j)\chi_s^j)]\\&\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx+\varepsilon(n,j).\end{aligned}\tag{5.46}$$

Furthermore, using (5.45) and (5.47), we have

$$\begin{aligned}&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)]\\&\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx\\&\le \int_{\Omega\setminus\Omega_s}\rho_k|\nabla T_k(u)|\,dx+\int_{\Omega\setminus\Omega_s}a(x,0)\nabla T_k(u)\,dx\\&+\varepsilon(n,j)+\varepsilon(n,m).\end{aligned}\tag{5.47}$$

Now, we remark that

$$\begin{aligned}
&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\\
&= \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)]\\
&\times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n - v_0)\,dx\\
&+ \int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_m(u_n - v_0))\,dx\\
&- \int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_m(u_n - v_0))\,dx + \varepsilon(n,j)\\
&+ \varepsilon(n,m).
\end{aligned} \tag{5.48}$$

Since $1 - h_m(u_n - v_0) = 0$ in $\{|u_n(x) - v_0(x)| \le m\}$ and since $\{|u_n(x)| \le k\} \subset \{|u_n(x) - v_0(x)| \le m\}$ for m large enough, we deduce from (5.48)

$$\begin{aligned}
&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\\
&= \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)]\\
&\times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n - v_0)\,dx\\
&- \int_{\{|u_n(x)|>k\}} a(x,0)\nabla T_k(u)\chi_s(1 - h_m(u_n - v_0))\,dx \qquad (5.49)\\
&+ \int_{\{|u_n(x)|>k\}} a(x,\nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s(1 - h_m(u_n - v_0))\,dx.
\end{aligned}$$

It is easy to see that, the two last terms of the last inequality tends to zero as $n \to \infty$, this implies that,

$$\begin{aligned}
&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\\
&= \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)]\\
&\times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n - v_0)\,dx\\
&+ \varepsilon(n,j) + \varepsilon(n,m).
\end{aligned} \tag{5.50}$$

Combining (5.35), (5.45), (5.47) and (5.50), we have

$$\begin{aligned}
&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\\
&\le \int_{\Omega\setminus\Omega_s} \rho_k \nabla T_k(u)\,dx + \int_{\Omega\setminus\Omega_s} a(x,0)\nabla T_k(u)\,dx + \varepsilon(n,j,m).
\end{aligned} \tag{5.51}$$

By passing to the lim sup over n, and letting j, m, s tend to infinity, we obtain

$$\limsup_{s\to+\infty}\limsup_{n\to+\infty}\int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx = 0.$$

Thus, by the Lemma 4.2, we get

$$M(|\nabla T_k(u_n)|)\to M(|\nabla T_k(u)|)\ \text{ in }\ L^1(\Omega). \tag{5.52}$$

Remark 5.1. If we assume that $\mathcal{A}_M\neq\emptyset$, then any solution of (3.9) belongs to $W_0^1L_Q(\Omega)$ for each $Q\in\mathcal{A}_M$.

Indeed. Let $t\geq\|v_0\|_\infty$ and take $v=T_t(u)$ in (3.9), we get

$$\int_\Omega a(x,\nabla u)\nabla T_h(u-T_t(u))\,dx+\int_\Omega \phi(u)\nabla T_h(u-T_t(u))\,dx \leq \int_\Omega fT_h(u-T_t(u))\,dx.$$

Hence,

$$\frac{1}{h}\int_\Omega a(x,\nabla u)\nabla T_h(u-T_t(u))\,dx\leq c.$$

Reasoning as above and letting $h\to 0$, we get

$$\lim_{h\to 0}\frac{1}{h}\int_{\{t\leq|u(x)|\leq t+h\}} M(|\nabla u|)\,dx\leq c.$$

Thus,

$$-\frac{d}{dt}\int_{\{|u(x)|>t\}} M(|\nabla u|)\,dx\leq c.$$

Following the same method used in the work of Benkirane and Bennouna [7] (see Step 2, pp. 93–97) one proves easily that $u\in W_0^1L_Q(\Omega)\ \forall\ Q\in\mathcal{A}_M$.

In the case where $\psi=-\infty$ (i.e. $K_\psi=W_0^1L_M(\Omega)$) it is possible to state:

Corollary 5.1. *Assume that* (A_1)–(A_4) *and* (3.3), (3.4) *are satisfied. Then there exists at least one solution of the following problem*

$$\begin{cases} u\in\mathcal{T}_0^{1,M}(\Omega),\\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx\\ \qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega fT_k(u-v)\,dx,\\ \forall v\in K_\psi\cap L^\infty(\Omega),\ \forall k>0.\end{cases} \tag{5.53}$$

Remark 5.2. Observe that the hypotheses (A_5) is not used in the previous corollary, this is due obviously to the density of $\mathcal{D}(\Omega)$ in $W_0^1L_M(\Omega)\cap L^\infty(\Omega)$ in the modular sense (see [13]).

Remark 5.3. In the same particular case as above (i.e. $\psi = -\infty$), the element v_0 introduced in (A_4) lies in $W_0^1E_M(\Omega)\cap L^\infty(\Omega)$, then if we assume that $\delta = v_0 = 0$ and $\mathcal{A}_M \neq \emptyset$, then any solution of (5.53) belongs to $W_0^1L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$, which gives the result of [6].

The proof is similar to that given in Remark 5.1.

Remark 5.4. Let $M(t) = |t|^p$ and $Q(t) = |t|^q$. Then the condition $Q \in \mathcal{A}_M$ is equivalent to the following conditions:

1) $2 - \dfrac{1}{N} < p < N$,
2) $q < \overline{q} = \dfrac{N(p-1)}{N-1}$.

Remark 5.5. In the case where $M(t) = |t|^p$. The Corollary 5.1 gives the result of Boccardo [9] (i.e. $u \in W_0^{1,q}(\Omega)$, $\forall q < \dfrac{N(p-1)}{N-1}$).

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L. Aharouch
University Ibn Zohr
Faculté Polydisciplinaire
Quarzazate
B.P. 638 Quarzazate
Maroc
e-mail: l_aharouch@yahoo.fr

E. Azroul
Département de Mathématiques
et Informatique
Faculté des Sciences Dhar-Mahraz
B.P. 1796 Atlas Fès
Maroc
e-mail: azroul_elhoussine@yahoo.fr

M. Rhoudaf
Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P 1796 Atlas Fès
Maroc
e-mail: rhoudaf_mohamed@yahoo.fr