Journal of Applied Analysis
Vol. 14, No. 1 (2008), pp. 53–61

UPPER AND LOWER SOLUTIONS METHOD FOR FOURTH-ORDER PERIODIC BOUNDARY VALUE PROBLEMS

S. WENG, H. GAO, D. JIANG and X. HOU

Received December 28, 2005 and, in revised form, October 19, 2007

Abstract. The purpose of this paper is to prove the existence of a solution of the following periodic boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0, 2\pi] \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi) \end{cases}$$

in the presence of an upper solution β and a lower solution α with $\beta \le \alpha$, where $f(t, u, v)$ satisfies one side Lipschitz condition.

1. Introduction and main results

In this paper, we study a fourth-order periodic boundary value problem of the form

---

2000 *Mathematics Subject Classification.* Primary: 34B15, 34B16.

Key words and phrases. Periodic boundary value problem, existence, upper and lower solutions method.

Project Supported by the National Natural Science Foundation of P. R. China (No. 10171010).

ISSN 1425-6908

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0, 2\pi] \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi), \end{cases} \tag{1.1}$$

where $f(t, u, v)$ is a Carathéodory function.

A function $f\colon [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}$ is said to be a Carathéodory function if it possesses the following three properties:

(i) For all $(u, v) \in \mathbb{R}^2$, the function $t \to f(t, u, v)$ is measurable on $[0, 2\pi]$.

(ii) For almost all $t \in [0, 2\pi]$, the function $(u, v) \to f(t, u, v)$ is continuous on $\mathbb{R}^2$.

(iii) For any given $N > 0$, there exists $g_N(t)$, a Lebesgue integrable function defined on $[0, 2\pi]$ such that

$$|f(t, u, v)| \le g_N(t) \text{ for a.e. } t \in [0, 2\pi],$$

whenever $|u|, |v| \le N$.

To develop a monotone method, we need the concepts of upper and lower solutions. We say that $\beta \in W^{4,1}[0, 2\pi]$ is an upper solution to the problem (1.1), if it satisfies

$$\begin{cases} \beta^{(4)}(t) \le f(t, \beta(t), \beta''(t)), \quad t \in [0, 2\pi] \\ \beta(0) = \beta(2\pi),\ \beta'(0) = \beta'(2\pi),\ \beta''(0) = \beta''(2\pi),\ \beta'''(0) \le \beta'''(2\pi). \end{cases} \tag{1.2}$$

Similarly, a function $\alpha \in W^{4,1}[0, 2\pi]$ is said to be a lower solution to (1.1), if it satisfies

$$\begin{cases} \alpha^{(4)}(t) \ge f(t, \alpha(t), \alpha''(t)), \quad t \in [0, 2\pi] \\ \alpha(0) = \alpha(2\pi),\ \alpha'(0) = \alpha'(2\pi),\ \alpha''(0) = \alpha''(2\pi),\ \alpha'''(0) \ge \alpha'''(2\pi). \end{cases} \tag{1.3}$$

We call a function $u \in W^{4,1}[0, 2\pi]$ a solution to the problem (1.1), if it is an upper and a lower solution to (1.1).

Recently, the equation of (1.1) with non-periodic boundary value problems has been studied by several authors, for examples, see [1], [3], [4], [6]–[8], [10], [12]–[14], [16], [18]. In [1], [6]–[8], [16], [18], all of the results are based upon the the Leray-Schauder continuation method and topological degree. In [3], [4], [10], [13], [14], the upper and lower solutions method has been studied when $f = f(t, u)$. In [12], the authors have studied the existence of the methods of lower and upper solutions and the monotone iterative technique.

Only a few have dealt with the periodic boundary value problem (1.1) (see [2], [9], [15], [17]). When $f = f(t, u)$, the authors of [2, 15], have studied the problem by the methods of lower and upper solutions and the monotone iterative technique. Wang [17] has investigated a special case of (1.1) (where $f(t, u, v) = kv + F(t, u)$) in the presence of a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ with $\beta(t) \le \alpha(t)$. Recently, Jiang, Gao and Wan [9] have dealt with (1.1) by means of a monotone iterative technique in the presence of a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ with $\beta(t) \leq \alpha(t)$. To develop a monotone method, the following hypotheses are needed in [9]:

$(A1)$ For any given $\beta, \alpha \in C[0, 2\pi]$ with $\beta(t) \leq \alpha(t)$ on $[0, 2\pi]$, there exist $0 < A \leq B$ such that

$$A(v_2 - v_1) \leq f(t, u, v_2) - f(t, u, v_1) \leq B(v_2 - v_1)$$

for a.e. $t \in [0, 2\pi]$ whenever $\beta(t) \leq u \leq \alpha(t)$, $v_1, v_2 \in \mathbb{R}$, and $v_1 \leq v_2$.

$(A2)$ Inequality

$$f(t, u_2, v) - f(t, u_1, v) \geq -\frac{A^2}{4}(u_2 - u_1)$$

holds for a.e. $t \in [0, 2\pi]$, whenever $\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)$, $v \in \mathbb{R}$.

The purpose of this paper is to give the existence result of solution of (1.1) under the assumption that there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1.1) with $\beta(t) \leq \alpha(t)$ and $f(t, u, v)$ only satisfies one side Lipschitz condition. We develop the upper and lower solutions method and prove that the solution $u(t)$ of (1.1) satisfies $\beta(t) \leq u(t) \leq \alpha(t)$. Our result extends and complements those in [2], [5], [9], [15], [17].

To develop upper and lower solutions method, we need the following hypotheses:

$(H1)$ For any given $\beta, \alpha \in C[0, 2\pi]$ with $\beta(t) \leq \alpha(t)$ on $[0, 2\pi]$, there exist $A > 0$ and $B > 0$ such that $B^2 \geq 4A$ and

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -A(u_2 - u_1) + B(v_2 - v_1) \tag{1.4}$$

for a.e. $t \in [0, 2\pi]$ whenever $\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)$, $v_1, v_2 \in \mathbb{R}$, and $v_1 \leq v_2$.

Let $m < 0$ and $M < 0$ are two roots to the equation $x^2 + Bx + A = 0$, then

$$m + M = -B, \quad mM = A.$$

Let

$$A(t) := \alpha''(t) + m\alpha(t), \quad B(t) := \beta''(t) + m\beta(t). \tag{1.5}$$

The main result of this paper is stated as follows.

Theorem 1. *Suppose that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of* (1.1) *such that $\beta(t) \leq \alpha(t)$ on $[0, 2\pi]$, and $f(t, u, v)$ is a Carathéodory function satisfying the hypotheses $(H1)$: there exist $a > 0$ and $b > 0$ such that $b^2 \geq 4a$ and*

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -a(u_2 - u_1) + b(v_2 - v_1).$$

Then $A(t) \leq B(t)$ *on* $[0, 2\pi]$ *and* (1.1) *has one solution* $u \in W^{4,1}[0, 2\pi]$ *such that*

$$\beta(t) \leq u(t) \leq \alpha(t), \quad A(t) \leq u''(t) + mu(t) \leq B(t).$$

2. Proof of Theorem 1

Lemma 1 (Maximum principle)**.** *Let* $y \in W^{2,1}[0, 2\pi]$*, and satisfies*

$$\begin{cases} y''(t) - ky(t) \geq 0 \quad \text{for a.e. } t \in [0, 2\pi], \\ y(0) = y(2\pi), \quad y'(0) \geq y'(2\pi), \end{cases}$$

where $k > 0$*. Then* $y(t) \leq 0$ *on* $[0, 2\pi]$*.*

Lemma 2. *Suppose that there exists a lower solution* $\alpha(t)$ *and an upper solution* $\beta(t)$ *of* (1.1) *such that* $\beta(t) \leq \alpha(t)$ *on* $[0, 2\pi]$*, and* $f(t, u, v)$ *is a Carathéodory function satisfying the hypotheses* $(H1)$*. Then* $A(t) \leq B(t)$ *on* $[0, 2\pi]$*.*

Proof. It follows from (1.2) and (1.3) that

$$\begin{cases} A''(t) + MA(t) \geq f(t, \alpha(t), A(t) - m\alpha(t)) + (m + M)A(t) - m^2\alpha(t), \\ \qquad\qquad t \in [0, 2\pi] \\ A(0) = A(2\pi), \ A'(0) \geq A'(2\pi), \end{cases}$$

and

$$\begin{cases} B''(t) + MB(t) \leq f(t, \beta(t), B(t) - m\beta(t)) + (m + M)B(t) - m^2\beta(t), \\ \qquad\qquad t \in [0, 2\pi] \\ B(0) = B(2\pi), \ B'(0) \leq B'(2\pi). \end{cases}$$

Let $y(t) = A(t) - B(t)$, then $y(0) = y(2\pi)$, $y'(0) \geq y'(2\pi)$.

Assume that $y(t) > 0$ for some $t \in [0, 2\pi]$. Since $y(t)$ is a continuous function defined on a closes interval $[0, 2\pi]$, it can attain its maximum value on $[0, 2\pi]$.

If $\max y(t) = y(t_0) > 0$ where $t_0 \in (0, 2\pi)$, then there is an interval $[c, d] \subseteq (0, 2\pi)$ such that $y(c) = y(d)$ and $y(t) > 0$ in $[c, d]$, it follows from (H_1) that

$$\begin{aligned} y'' + My(t) \geq & f(t, \alpha(t), A(t) - m\alpha(t)) - f(t, \beta(t), B(t) - m\beta(t)) \\ & + (m + M)y(t) - m^2(\alpha(t) - \beta(t)) \\ \geq & -(A + Bm + m^2)(\alpha(t) - \beta(t)) + (B + m + M)y(t) \\ = & 0, \quad t \in [c, d], \end{aligned}$$

then by Lemma 1, we have $y(t) \leq 0$ on $[c, d]$, which is a contradiction.

If $\max y(t) = y(0) = y(2\pi) > 0$, and hence there exists $s \in (0, 2\pi)$ with $y(s) > 0$, then there would be $0 \le a < s < b \le 2\pi$ such that $y(t) > 0$ in (a, b) with $y(a) = y(b) = 0$. By (1.2) and (1.3), we have

$$y''(t) + My(t) \ge 0, \quad t \in [a, b], \qquad y(a) = y(b) = 0.$$

This leads to $y(t) \le 0$ on $[a, b]$, which is again a contradiction.

If $y(0) > 0$ (then $y(2\pi) = y(0) > 0$), then there would be $a < b$ in $(0, 2\pi)$ such that $y(t) > 0$ on $[0, a) \cup (b, 2\pi]$ with $y(a) = y(b) = 0$, $y'(a) \le 0$, $y'(b) \ge 0$. So we have $y''(t) + My(t) \ge 0$ on $[0, a) \cup (b, 2\pi]$, hence $y'(t)$ is strictly increasing in $[0, a) \cup (b, 2\pi]$, which implies that $y'(0) < y'(a) \le 0 \le y'(b) < y'(2\pi)$, this is a contradiction with the boundary conditions.

The proof of Lemma 2 is completed. $\square$

Lemma 3. *If $m < 0$, then for any $q(t) \in L^1[0, 2\pi]$, the problem*

$$\begin{cases} u'' + mu(t) = q(t), & \text{for a.e. } t \in [0, 2\pi] \\ u(0) = u(2\pi), & u'(0) = u'(2\pi), \end{cases}$$

has a unique solution $u \in W^{2,1}[0, 2\pi]$, and

$$u(t) = L^{-1}q(t) = \int_0^{2\pi} G_m(t, s)q(s)ds,$$

where $\rho = \sqrt{-m}$ and

$$G_m(t, s) := \begin{cases} -\dfrac{e^{\rho(t-s)} + e^{\rho(2\pi - t + s)}}{2\rho(e^{2\rho\pi} - 1)}, & 0 \le s \le t \le 2\pi, \\ -\dfrac{e^{\rho(s-t)} + e^{\rho(2\pi - s + t)}}{2\rho(e^{2\rho\pi} - 1)}, & 0 \le t \le s \le 2\pi. \end{cases}$$

Proof. Let

$$p(t, x) = \begin{cases} A(t), & x < A(t), \\ x, & A(t) \le x \le B(t), \\ B(t), & x > B(t). \end{cases}$$

By Lemma 1, we have

$$\alpha(t) = L^{-1}A(t), \quad \beta(t) = L^{-1}B(t), \quad \beta(t) \le L^{-1}p(t, x) \le \alpha(t).$$

Now we consider the following modified problem

$$\begin{cases} x'' + Mx(t) = f(t, L^{-1}p(t, x(t)), (I - mL^{-1})p(t, x(t))) \\ \qquad\qquad + (m + M)p(t, x(t)) - m^2 L^{-1}p(t, x(t)), \\ x(0) = x(2\pi), \quad x'(0) = x'(2\pi). \end{cases} \tag{2.1}$$

For each $x \in C[0,2\pi]$,we define the mapping $\Phi\colon C[0,2\pi] \to C[0,2\pi]$

$$(\Phi x)(t) = \int_0^{2\pi} G_M(t,s)(Fx)(s)ds, \tag{2.2}$$

where

$$\begin{aligned}(Fx)(t) :=& f(t, L^{-1}p(t,x(t)), (I - mL^{-1})p(t,x(t))) \\ &+ (m+M)p(t,x(t)) - m^2L^{-1}p(t,x(t)).\end{aligned}$$

Since $p(t,x(t))$ and $L^{-1}p(t,x(t))$ are bounded and $f(t,u,v)$ is a Carathéodory function, there exists $g(t)$, a Lebesgue integrable function defined on $[0,2\pi]$ such that

$$|(Fx)(t)| \le g(t) \text{ for a. e. } t \in [0,2\pi].$$

Thus $(\Phi x)(t)$ is also bounded.

We can easily prove that $\Phi\colon C[0,2\pi] \to C[0,2\pi]$ is completely continuous. Then Leray-Schauder fixed point Theorem assures that Φ has a fixed point $x \in C[0,2\pi]$ and

$$x(t) = \int_0^{2\pi} G_M(t,s)(Fx)(s)ds, \tag{2.3}$$

thus the modified problem (2.1) has one solution $x \in W^{2,1}[0,2\pi]$. The proof of Lemma 3 is completed. $\square$

Lemma 4. *Suppose that* $(H1)$ *holds. Assume that* $\alpha(t)$ *and* $\beta(t)$ *are lower and upper solutions to* (1.1) *and* $\beta(t) \le \alpha(t)$ *on* $[0,2\pi]$. *Let* $x \in W^{2,1}[0,2\pi]$ *be a solution to* (2.1), *then* $A(t) \le x(t) \le B(t)$ *on* $[0,2\pi]$.

Remark 1. Lemma 4 implies $u(t) = L^{-1}x(t) = \int_0^{2\pi} G_m(t,s)x(s)ds$ is a solution to (1.1), since $u'' + mu(t) = x(t), u(0) = u(2\pi), u'(0) = u'(2\pi)$ and $A(t) \le x(t) \le B(t)$.

Proof of Lemma 4. Since $\alpha(t) = L^{-1}A(t)$, $\beta(t) = L^{-1}B(t)$,

$$\begin{cases} B''(t) + MB(t) \le f(t, L^{-1}B(t), (I - mL^{-1})B(t)) - m^2L^{-1}B(t) \\ \qquad\qquad\qquad + (m+M)B(t), \\ B(0) = B(2\pi), \quad B'(0) \le B'(2\pi), \end{cases}$$

and

$$\begin{cases} A''(t) + MA(t) \ge f(t, L^{-1}A(t), (I - mL^{-1})A(t)) - m^2L^{-1}A(t) \\ \qquad\qquad\qquad + (m+M)A(t), \\ A(0) = A(2\pi), \quad A'(0) \ge A'(2\pi). \end{cases}$$

Let $y(t) = x(t) - B(t)$ and $z(t) = A(t) - x(t)$, then

$$y(0) = y(2\pi), \quad y'(0) \geq y'(2\pi)$$

and

$$z(0) = z(2\pi), \quad z'(0) \geq z'(2\pi).$$

Applying an analogous approach used in the proof of Lemma 2, we can show that $y(t) \leq 0$ and $g(t) \leq 0$, and that is, $A(t) \leq x(t) \leq B(t)$.

The proof of Lemma 4 is completed. $\square$

By Remark 1, we have the results of Theorem 1.

3. Example

In this section, we consider the periodic boundary value problem:

$$\begin{cases} u^{(4)}(t) - ku''(t) = F(t,u), \quad t \in [0,2\pi] \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi), \end{cases} \tag{3.1}$$

where $F(t,u)$ is a Carathéodory function, $k > 0$.

To develop upper and lower solutions method, we also need the following hypothesis:

(H) For any given $\beta, \alpha \in C[0,2\pi]$ with $\beta(t) \leq \alpha(t)$ on $[0,2\pi]$, inequality

$$F(t,u_2) - F(t,u_1) \geq -\frac{k^2}{4}(u_2 - u_1)$$

holds for a.e. $t \in [0,2\pi]$, whenever $\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)$.

Let $A = k^2/4$, $B = k$, then $(H1)$ holds. Hence the conclusions of Theorem 1 hold. Then

$$\alpha''(t) - \frac{k}{2}\alpha(t) \leq \beta''(t) - \frac{k}{2}\beta(t)$$

and problem (3.1) has one solution $u \in W^{4,1}[0,2\pi]$ such that

$$\beta(t) \leq u(t) \leq \alpha(t), \quad \alpha''(t) - \frac{k}{2}\alpha(t) \leq u''(t) - \frac{k}{2}u(t) \leq \beta''(t) - \frac{k}{2}\beta(t).$$

In [17], Wang studied the problem (4.1) when F is continuous on $[0,2\pi] \times \mathbb{R}$. In [9, 17], the authors obtained one solution $u \in W^{4,1}[0,2\pi]$ of (3.1) such that $\beta(t) \leq u(t) \leq \alpha(t)$.

We have improved the results of [9, 17].

References

[1] Aftabizadeh, A. R., *Existence and uniqueness theorems for fourth-order boundary value problems*, J. Math. Anal. Appl. **116** (1986), 415–426.

[2] Cabada, A., *The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems*, J. Math. Anal. Appl. **185** (1994), 302–320.

[3] De Coster, C., Sanchez, L., *Upper and lower solutions, Ambrosetti-Prodi problems and positive solutions for fourth order O.D.E.*, Riv. Mat. Pura Appl. **14** (1994), 1129–1138.

[4] Dunninger, D., *Existence of positive solutions for fourth-order nonlinear problems*, Boll. Un. Mat. Ital. B (7) **1**(4) (1987), 1129–1138.

[5] Gao, H., Weng, S., Jiang, D., Hou, X., *On second order periodic boundary value problems with upper and lower solutions in the reversed order*, Electron. J. Differential Equations **2006**(25) (2006), 8 pp.

[6] Gupta, C. P., *Existence and uniqueness results for a bending of an elastic beam equation at resonance*, J. Math. Anal. Appl. **135** (1988), 208–225.

[7] Gupta, C. P., *Existence and uniqueness theorem for a bending of an elastic beam equation*, Appl. Anal. **26** (1988), 289–304.

[8] Gupta, C. P., *Existence and uniqueness results for some fourth order fully quasilinear boundary value problems*, Appl. Anal. **36** (1990), 169–175.

[9] Jiang, D. Q., Gao, W. J., Wan, A. Y., *A monotone method for constructing extremal solutions to fourth-order periodic boundary value problems*, Appl. Math. Comput. **132** (2002), 411–421.

[10] Korman, P., *A maximum principle for fourth-order ordinary differential equations*, Appl. Anal. **33** (1989), 267–273.

[11] Ladde, G. S., Lakshmikantham, V., Vatsala, A. S., *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[12] Ma, R. Y., Zhang, J. H., Fu, S. M., *The method of lower and upper solutions for fourth-order two-point boundary value problems*, J. Math. Anal. Appl. **215** (1997), 415–422.

[13] Sadyrabaev, F., *A two-point boundary value problem for a fourth-order equation* (Russian), Latv. Univ. Zinātn Raksti **553** (1990), 84–91.

[14] Schroder, J., *Fourth-order two-point boundary value problems; estimates by two side bounds*, Nonlinear Anal. **8** (1984), 107–114.

[15] Seda, V., Nieto, J. J., Gera, M., *Periodic boundary value problems for nonlinear higher order ordinary differential equations*, Appl. Math. Comput. **48** (1992), 71–82.

[16] Usmani, R. A., *A uniqueness theorem for a boundary value problem*, Proc. Amer. Math. Soc. **77** (1979), 327–335.

[17] Wang, H.-Z., *Periodic solutions of four-order differential equations* (Chinese), Acta Sci. Natur. Univ. Jilin. **4** (1993), 415-422.

[18] Yang, Y., *Fourth-order two-point boundary value problem*, Proc. Amer. Math. Soc. **104** (1988), 175–180.

Shiyou Weng
General Course Department
Suzhou Vocational University
Suzhou 215104, P. R. China
e-mail: wengshiyou2001@yahoo.com.cn

Haiyin Gao
College of Applied Science
Changchun University
Changchun, Jilin 130022
P. R. China
e-mail: gaohaiyinhealthy@yahoo.com

Daqing Jiang
School of Mathematics
and Statistics
Northeast Normal University
Changchun 130024, P. R. China
e-mail: daqingjiang@vip.163.com

Xuezhang Hou
Mathematics Department
Towson University
Baltimore
Maryland 21252, USA
e-mail: xhou@towson.edu