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SET DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

G. N. GALANIS, T. G. BHASKAR and V. LAKSHMIKANTHAM

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Abstract. It is known that a Fréchet space $\mathbb{F}$ can be realized as a projective limit of a sequence of Banach spaces $\mathbb{E}^i$. The space $K_c(\mathbb{F})$ of all compact, convex subsets of a Fréchet space, $\mathbb{F}$, is realized as a projective limit of the semilinear metric spaces $K_c(\mathbb{E}^i)$. Using the notion of Hukuhara derivative for maps with values in $K_c(\mathbb{F})$, we prove the local and global existence theorems for an initial value problem associated with a set differential equation.

1. Introduction

One of the most convenient generalizations of differential equations is the notion of Set Differential Equations (SDEs). The main objects in this framework are multivalued functions of the form $U\colon I \to K_c(\mathbb{R}^n)$, where I is an interval of real numbers and $K_c(\mathbb{R}^n)$ the space of all nonempty compact and convex subsets of $\mathbb{R}^n$. Such functions are sought as solutions of initial value problems of the form

$$D_H U = F(t, U(t)); \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n),$$

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where $D_H U$ stands for the Hukuhara derivative of U (cf. [4], [5]).

The basic theory of SDEs in $K_c(\mathbb{R}^n)$ is developed in [5]. Natural extensions to the study of IVPs for SDEs on $K_c(\mathbb{E})$, where $\mathbb{E}$ is an infinite dimensional Banach space, is also initiated there but much work is yet to be done. On the other hand, the notion of Hukuhara derivatives, Hausdorff distances and Lipschitz continuity fail to be carried over to the setting of infinite dimensional locally convex spaces that are not Banach. Despite this, the increasing need of theoretical models for the study of several problems emerging in modern Differential Geometry and Theoretical Physics in the framework of non-Banach infinite dimensional spaces, makes it necessary to study the SDE in such a setting.

In an earlier paper [2], we proposed the generalizations of the above notions to a wide class of locally convex topological vector spaces: the Fréchet spaces. Making ample use of the fact that every Fréchet space $\mathbb{F}$ can be viewed also as a projective limit of Banach spaces, we established on $K_c(\mathbb{F})$ a separable and complete topological structure. This identification provides a framework to study SDEs in Fréchet spaces.

The main aim of this paper is to study and solve initial value problems for SDEs in Fréchet spaces. Since there are inherent difficulties even for scalar ordinary differential equations in this framework, we also indicate the application of our results obtained for SDEs to the study of ODEs in Fréchet spaces.

2. The space $K_c(\mathbb{F})$

The necessary background for the study of set (multivalued) differential equations on a Banach space $\mathbb{E}$ of finite or infinite dimension has been provided in [4] and [5] using the notion of Hukuhara differentiation. Recall that, $K_c(\mathbb{E})$ denotes the collection of all nonempty compact, convex subsets of $\mathbb{E}$. $K_c(\mathbb{E})$ is a complete metric space with the Hausdorff metric:

$$D[A,B] = \max\left[\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\right]$$

where $d(x,A) = \inf[d(x,y)\colon y \in A]$, $A, B \in K_c(\mathbb{E})$.

A set valued mapping $F\colon I \to K_c(\mathbb{E})$ is said to be *continuous* at $t_0 \in I$ if *for any* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon, t_0) > 0$ *such that*

$$D\left[F(t), F(t_0)\right] < \varepsilon, \text{ for all } t \in I \text{ with } |t - t_0| < \delta.$$

On the other hand, we call F *Lipschitz continuous*, with Lipschitz constant L, if

$$D[F(t), F(s)] \leq L \cdot |t - s|, \text{ for all } t, s \in I.$$

Moreover, F is called Hukuhara differentiable if there exists an element $D_H F(t_0) \in K_c(\mathbb{E})$, $t_0 \in I$, such that

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{F(t_0) - F(t_0 - \Delta t)}{\Delta t} = D_H F(t_0).$$

The differences in the numerators are the Hukuhara (geometric) differences of the involved sets.

However, if we are to deal with a situation where the underlying space is not a Banach space, the above methodology collapses from the very beginning. Indeed, if the topology of the underlying space is not obtained by a single norm, then even the definition of continuity, Lipschitz continuity as well as the notion of Hukuhara derivative cannot be patterned, along the same lines.

A way out, at least partially, of these problems has been proposed in [2]. Generalizations of all the above notions have been developed in order to fit in the structure of a wide category of locally convex topological vector spaces: The Fréchet (i.e. Hausdorff, metrizable and complete) spaces. The main idea is to replace the Hausdorff metric by a family of corresponding "semi-metrics".

To be more precise, let us consider a seperable Fréchet space $\mathbb{F}$ with a topology defined by the family of seminorms $\{p_i\}_{i \in \mathbb{N}}$. Without loss of generality, we may assume that $p_1 \leq p_2 \leq \ldots \leq p_i \leq p_{i+1} \leq \ldots$. Then, $\mathbb{F}$ can viewed as a projective limit of Banach spaces. That is, $\mathbb{F} \equiv \varprojlim \{\mathbb{E}^i ; \rho^{ji}\}_{i,j \in \mathbb{N}}$, where $\mathbb{E}^i$ is the completion of the quotient $\mathbb{F}/\operatorname{Ker} p_i$ $(i \in \mathbb{N})$ and ρ^{ji} are the connecting morphisms:

$$\rho^{ji} \colon \mathbb{E}^j \to \mathbb{E}^i \colon [x + \operatorname{Ker} p_j]_j \mapsto [x + \operatorname{Ker} p_i]_i ;\ j \geq i.$$

Here the bracket $[\cdot]_i$ stands for the corresponding equivalence class (see [1], [6]).

Under this notation the space $K_c(\mathbb{F})$, of all nonempty compact and convex subsets of $\mathbb{F}$, can be realized as a projective limit space of the corresponding structures of $\mathbb{E}^i$s:

$$K_c(\mathbb{F}) \equiv \varprojlim \left\{ K_c\left(\mathbb{E}^i\right) ; \phi^{ji} \right\},$$

where the connecting mappings are defined as

$$\phi^{ji} \colon K_c\left(\mathbb{E}^j\right) \to K_c\left(\mathbb{E}^i\right) : A \mapsto \rho^{ji}(A).$$

The latter are also continuous, with respect to the topologies induced by the Hausdorff metrics $D^{\mathbb{E}^j}$, $D^{\mathbb{E}^i}$. Thus, any element A of $K_c(\mathbb{F})$ can be viewed alternatively in the form

$$A \equiv \left(\rho^i(A)\right)_{i \in \mathbb{N}} \equiv \varprojlim \rho^i(A),$$

where $\rho^i\colon \mathbb{F} \to \mathbb{E}^i$ are the canonical projections of $\mathbb{F}$ to its factors. It is worth also noting that the Minkowski addition and Hukuhara difference are compatible with projective limits.

The above presented structure of $K_c(\mathbb{F})$ makes it possible to revise the pathological (in the Fréchet framework) notion of Hausdorff distance:

Definition 2.1. We call

(i) *i-distance* from x to A:

$$d^i(x, A) = \inf \{p_i(x - a);\ a \in A\}$$

(ii) *i-separation* of A, $B \in K_c(\mathbb{F})$:

$$d^i_H(B, A) = \sup \left\{d^i(b, A);\ b \in B\right\}$$

(iii) *i-Hausdorff distance* or *i-semi-metric* between A and B:

$$D^i(B, A) = \max \left\{d^i_H(B, A), d^i_H(A, B)\right\}.$$

The following basic connection with the corresponding classical notions on the Banach factors is valid: If $A = \varprojlim A^i$, $B = \varprojlim B^i$, $A^i, B^i \in K_c(\mathbb{E}^i)$, then

$$D^i(A, B) = D^{\mathbb{E}^i}\left[A^i, B^i\right],$$

$D^{\mathbb{E}^i}$ standing for the Hausdorff metric of $\mathbb{E}^i$ $(i \in \mathbb{N})$.

This fact allows us to endow the space $K_c(\mathbb{F})$ with a separable and complete topological structure. It is worth noticing that the notion of a *Cauchy sequence* is defined here by means of *all* semi-metrics $(D^i)_{i\in\mathbb{N}}$:

$$(A_n)_{n\in\mathbb{N}} \subset K_c(\mathbb{F}) \text{ is } \textit{Cauchy} \text{ if and only if}$$
$$\lim_{n,m\to+\infty} D^i\left(A_n, A_m\right) = 0, \text{ for every } i \in \mathbb{N}.$$

In view of the above, we may proceed to the, crucial for our study, notions of continuity, Lipschitz continuity and Hukuhara differentiability for set valued mappings within the framework of a Fréchet space $\mathbb{F}$.

Let $F\colon I \subset \mathbb{R} \to K_c(\mathbb{F})$ be any $\mathbb{F}$-set valued mapping. Then it can be realized as a projective limit of the corresponding $\mathbb{E}^i$-mappings:

$$F = \varprojlim F^i,$$

where $F^i := \phi^i \circ F\colon I \to K_c(\mathbb{E}^i)$, if $\phi^i\colon K_c(\mathbb{F}) \to K_c(\mathbb{E}^i)\colon A \mapsto \rho^i(A)$ denote the canonical projections of the limit $K_c(\mathbb{F}) \equiv \varprojlim K_c(\mathbb{E}^i)$. This realization, gives the opportunity to translate the study of F onto its factors:

(1) F is continuous if and only if each F^i is continuous, for every index $i \in \mathbb{N}$.

(2) F is called *Lipschitz continuous* if, for any $i \in \mathbb{N}$, there exists a positive constant L^i such that

$$D^i\left(F(t), F\left(t'\right)\right) \le L^i \cdot \left|t - t'\right|,$$

for all $t, t' \in I$. This property is also characterized by the corresponding behavior onto the factors: F is Lipschitz continuous if and only if every factor F^i $(i \in \mathbb{N})$ is Lipschitz continuous.

Since the projective limits are compatible with continuity and Hukuhara differences, we may extend the classical definition of Hukuhara derivative in the Fréchet framework:

Definition 2.2. Let I be an interval of $\mathbb{R}$ and $F\colon I \to K_c(\mathbb{F})$ a set valued function. F is called *Hukuhara differentiable* at $t_0 \in I$ if there exists a $D \in K_c(\mathbb{F})$ such that

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{F(t_0) - F(t_0 - \Delta t)}{\Delta t} = D.$$

Such a D is called the *Hukuhara derivative* of F at t_0 and is denoted by $D_H F(t_0)$.

Again, if we adopt the realization of $F\colon I \to K_c(\mathbb{F})$ as a projective limit $F = \varprojlim F^i$; $F^i\colon I \to K_c(\mathbb{E}^i)$, then the Hukuhara differentiability at t_0 reduces to the corresponding notion on the factors:

Proposition 2.3. *$F = \varprojlim F^i\colon I \subseteq \mathbb{R} \to K_c(\mathbb{F})$ is Hukuhara differentiable at t_0 if and only if each F^i $(i \in \mathbb{N})$ is Hukuhara differentiable at t_0 and*

$$D_H F(t_0) = \varprojlim D_H F^i(t_0).$$

For the relevant proof and the full details of this approach we refer to [2].

3. Set differential equations in Banach spaces

Let $\mathbb{E}$ be a real Banach space with norm $\|\cdot\|$ and the metric generated by it be denoted by d. It is known that if the space $K_c(\mathbb{E})$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then it becomes a semilinear metric space that can be embedded as a complete cone in a corresponding Banach space (see [7]). For a set valued mapping $F\colon I \to E$, by $(A)\int_{I_0} F(s)\,ds$, we shall denote the integral in the sense of Aumann, on a measurable set $I_0 \subset I$, as

$$(A)\int_{I_0} F(s)\,ds = \left\{ \int_{I_0} f(s)\,ds \colon f \text{ is a Bochner integrable selector of } F \right\}.$$

For a continuous map $F\colon I \to K_c(\mathbb{E})$ the integral $\int_{I_0} F(s)\,ds$ can be introduced in a natural way in the sense of Bochner. Due to the theorem on differentiation of the Bochner integral (see [7]) it can be shown for a map $\Phi\colon I \to K_c(\mathbb{E})$ that if

$$\Phi(t) = U_0 + \int_{t_0}^{t} F(s)\,ds, \quad U_0 \in K_c(\mathbb{E}),\ t_0 \in I, \tag{3.1}$$

then the Hukuhara derivative $D_H\Phi(t)$ exists a.e. on I and

$$D_H\Phi(t) = F(t).$$

In view of the above, we first state below a local existence theorem for SDEs in the framework of $K_c(\mathbb{E})$ where $\mathbb{E}$ is a Banach space:

$$D_H U(t) = F(t, U(t)); \quad U(t_0) = U_0 \in K_c(\mathbb{E}), \tag{3.2}$$

where $F\colon I \times K_c(\mathbb{E}) \to K_c(\mathbb{E})$.

Theorem 3.1. *Let* $R_0 = J \times B(U_0, b)$ *where* $J = [t_0, t_0 + a]$, $B(U_0, b) = \{U \in K_c(\mathbb{E})\colon D[U, U_0] \le b\}$. *Assume that:*

(1) $F \in C[R_0, K_c(\mathbb{E})]$ *and* $D[F(t, U), 0] \le M_0$ *on* R_0.
(2) $g \in C[J \times [0, 2b], \mathbb{R}_+]$, $g(t, w) \le M_1$ *on* $J \times [0, 2b]$, $g(t, 0) \equiv 0$, $g(t, w)$ *is nondecreasing in* w, *for each* $t \in J$, *and* $w(t) \equiv 0$ *is the only solution of*

$$w' = g(t, w), w(t_0) = 0, \ \text{on } J.$$

(3) $D[F(t, U), F(t, V)] \le g(t, D[U, V])$ *on* R_0.

Then, the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} F(s, U_n(s))\,ds, \quad n = 0, 1, 2, \dots,$$

exist as continuous functions on $J_0 = [t_0, t + \eta)$, *where* $\eta = \min\{a, b/M\}$, $M = \max\{M_0, M_1\}$ *and converge uniformly to the unique solution of the IVP on* J_0.

Assuming the local existence, we can prove the following global existence result.

Theorem 3.2. *Let* $F \in C[\mathbb{R}_+ \times K_c(\mathbb{E}), K_c(\mathbb{E})]$ *and*

$$D[F(t, U), 0] \le g(t, D[U, 0]), \quad (t, U) \in \mathbb{R}_+ \times K_c(\mathbb{E}),$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t, w)$ *is nondecreasing in* w, *for each* $t \in \mathbb{R}_+$, *and the maximal solution* $r(t, t_0, w_0)$ *of*

$$w' = g(t, w) \quad w(t_0) = w_0 \ge 0,$$

exists on $[t_0, \infty)$. *Suppose further that* F *is smooth enough to guarantee local existence of solutions of* (3.2) *for any* $(t_0, U_0) \in \mathbb{R}_+ \times K_c(\mathbb{E})$. *Then the largest interval of existence of any solution* $U(t, t_0, U_0)$ *of* (3.2) *such that* $D[U_0, 0] \leq w_0$, *is* $[t_0, \infty]$.

The proofs of the above stated theorems can be modelled along the lines of the proofs of Theorem 2.3.1 and Theorem 2.6.1 in [5]. Alternatively, these results can also be obtained as special cases of Theorem 3.6 (p. 29) and Theorem 4.1 (p. 31) of [7].

4. Set differential equations in Fréchet spaces

As already mentioned in the Introduction, the local structure of a Fréchet space $\mathbb{F}$ prevents the classical mechanism of set differential equations from being patterned here. However, the new approach of the space of compact convex subsets $K_c(\mathbb{F})$ presented in Section 2 and mainly its realization as a projective limit, gives a way out. In this section we propose a mechanism of solving multivalued initial problems within the framework of Fréchet spaces based on this new structure of $K_c(\mathbb{F})$.

To be more specific, let $\mathbb{F}$ be an arbitrarily chosen seperable Fréchet space and its realization as a projective limit of Banach spaces:

$$\mathbb{F} = \varprojlim \{\mathbb{E}^i; \rho^{ji}\}_{i,j \in \mathbb{N}}.$$

Let also

$$D_H U(t) = F(t, U(t)); \quad U(t_0) = U_0 \in K_c(\mathbb{F}),\ t_0 \geq 0, \tag{4.1}$$

be an initial value problem for a SDE where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{F}), K_c(\mathbb{F})]$ and $D_H F$ the Hukuhara derivative of F. Using the required notations and results from Sections 2 and 3, we can now establish local existence result for the initial value problem (4.1):

Theorem 4.1. *Let* $R_0 = J \times B(U_0, b)$ *where* $J = [t_0, t_0 + a]$, $B(U_0, b) = \{U \in K_c(\mathbb{F})\colon D^i[U, U_0] \leq b,\ i \in \mathbb{N}\}$. *Assume that:*

(1) *The function* F *can be realized as a projective limit* $F = \varprojlim F^i$, $F^i \in C[\mathbb{R}_+ \times K_c(\mathbb{E}^i), K_c(\mathbb{E}^i)]$, *and is bounded with respect to every semi metric of* $K_c(\mathbb{F})$, *that is there exists a positive* $M_0 > 0$ *with* $D^i(F(t, A), 0) \leq M_0$, *for every* $i \in \mathbb{N}$, *if* $t \in J = [t_0, t_0 + a]$ *and* $A \in K_c(\mathbb{F})$ *with* $D^i(A, U_0) \leq b$.
(2) $g \in C[J \times [0, 2b], \mathbb{R}_+]$, $g(t, w) \leq M_1$ *on* $J \times [0, 2b]$, $g(t, 0) \equiv 0$, $g(t, w)$ *is nondecreasing in* w, *for each* $t \in J$, *and* $w(t) \equiv 0$ *is the only solution of* $w' = g(t, w)$, $w(t_0) = 0$, *on* J.
(3) $D^i(F(t, A), F(t, B)) \leq g(t, D^i(A, B))$, *for every* $i \in \mathbb{N}$, *on* R_0.

Then, the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} F\left(s, U_n(s)\right) ds, \quad n = 0, 1, 2, \ldots,$$

exist on $J_0 = [t_0, t+\eta)$*, where* $\eta = \min\{a, b/M\}$*,* $M = \max\{M_0, M_1\}$ *as continuous functions and converge uniformly to the unique solution of the IVP on* J_0*.*

Proof. Let $U_0^i := \rho^i(U_0)$. Then, a sequence of SDEs on the spaces $\mathbb{E}^i$ is defined:

$$D_H^i U^i(t) = F^i\left(t, U^i(t)\right), \quad U^i(t_0) = U_0^i \in K_c\left(\mathbb{E}^i\right),\ t_0 \geq 0. \tag{4.2}$$

The fact that the semi metrics of $K_c(\mathbb{F})$ are, in a way, an upload of the metrics of $K_c(\mathbb{E}^i)$, ensures that the set valued mapping F^i is bounded by M_0 on $J \times \{A \in K_c(\mathbb{E}^i)$ with $D^{\mathbb{E}^i}[A, U_0^i] \leq b\}$. On the other hand, it satisfies the condition: For $A, B \in K_c(\mathbb{E}^i)$,

$$D^{\mathbb{E}^i}\left[F^i(t,A), F^i(t,B)\right] \leq g\left(t, D^{\mathbb{E}^i}[A,B]\right),\ i \in \mathbb{N},\ t \in J.$$

We now appeal to the Theorem 3.1, to obtain the unique solution

$$U^i\colon\ [t_0, t_0+\eta) \to K_c\left(\mathbb{E}^i\right)$$

of the IVP (4.2). The sequence of the obtained solutions are related via the connecting morphisms $\phi^{ji}\colon K_c(\mathbb{E}^j) \to K_c(\mathbb{E}^i)\colon A \mapsto \rho^{ji}(A)$. As a matter of fact, taking into account that the multifunctions $(F^i)_{i\in\mathbb{N}}$ form a projective limit, we see that for any choice of indices $i, j \in \mathbb{N}$ with $j \geq i$, next relations hold:

$$\begin{aligned}
D_H^i\left(\phi^{ji} \circ U^j\right)(t) &= \lim_{\Delta t \to 0^+} \frac{\phi^{ji}(U^j(t+\Delta t)) - \phi^{ji}(U^j(t))}{\Delta t} \\
&= \lim_{\Delta t \to 0^+} \phi^{ji}\left(\frac{U^j(t+\Delta t) - U^j(t)}{\Delta t}\right) \\
&= \phi^{ji}\left(\lim_{\Delta t \to 0^+} \frac{U^j(t+\Delta t) - U^j(t)}{\Delta t}\right) = \phi^{ji}\left(D_H^j U^j(t)\right) \\
&= \left(\phi^{ji} \circ F^j\right)\left(t, U^j(t)\right) = \left(F^i \circ \left(id_{\mathbb{R}} \times \phi^{ji}\right)\right)\left(t, U^j(t)\right) \\
&= F^i\left(t, \left(\phi^{ji} \circ U^j\right)(t)\right).
\end{aligned}$$

Therefore, the mapping $\phi^{ji} \circ U^j$ is a solution of (4.2) satisfying also the initial condition $(\phi^{ji} \circ U^j)(t_0) = \phi^{ji}(U_0^j) = U_0^i$. As a result, it coincides with U^i, for every $j \geq i$, and the Hukuhara differentiable limit $U = \varprojlim U^i\colon [t_0, t_0+\eta) \to K_c(\mathbb{F})$ exists. Moreover,

$$D_H U(t) = \varprojlim\left(D_H^i U^i(t)\right) = \varprojlim\left(F^i\left(t, U^i(t)\right)\right) = F(t, U(t)),$$

$$U(t_0) = \left(U^i(t_0)\right)_{i\in\mathbb{N}} = \left(U_0^i\right)_{i\in\mathbb{N}} = U_0,$$

and $U(t)$ is the desired solution of the initial value problem (4.1).

On the other hand, the unique solution U^i of the IVP (4.2) is the limit of the sequence of successive approximations, $\{U_n^i\}$, defined for $n = 0, 1, 2, \dots,$ by

$$U_{n+1}^i(t) = U_0^i + \int_{t_0}^t F^i\left(s, U_n^i(s)\right)\, ds.$$

These successive approximations are also related via the connecting morphisms ϕ^{ji}s. Indeed, by the linearity of the connecting morphisms, and the fact that each $U_0^i \in K_c(\mathbb{E}^i)$ has been obtained as a projection of $U_0 \in K_c(\mathbb{F})$, we have,

$$\phi^{ji} \circ U_{n+1}^j = U_0^i + \phi^{ji}\left(\int_{t_0}^t F^j\left(s, U_n^j(s)\right)\, ds\right).$$

We have already shown that $D_H^i \circ \phi^{ji} = \phi^{ji} \circ D_H^j$, $j \geq i$. Thus,

$$\begin{aligned} D_H^i\left(\phi^{ji}\left(\int_{t_0}^t F^j\left(s, U_n^j(s)\right)\, ds\right)\right) &= \phi^{ji}\left(D_H^j\left(\int_{t_0}^t F^j\left(s, U_n^j(s)\right)\, ds\right)\right) \\ &= \phi^{ji}\left(F^j\left(t, U_n^j(t)\right)\right) \\ &= F^i\left(t, U_n^i(t)\right). \end{aligned}$$

Thus,

$$\phi^{ji}\left(\int_{t_0}^t F^j\left(s, U_n^j(s)\right)\, ds\right) = \int_{t_0}^t F^i\left(s, U_n^i(s)\right)\, ds,$$

and $U_{n+1}^i = \phi^{ji} \circ U_{n+1}^j$. As a result it follows that the projective limits of U_n^i exist and the successive approximations

$$U_n(t) = \varprojlim U_n^i(t), \quad n = 0, 1, 2, \dots,$$

defined on $J_0 = [t_0, t + \eta)$, are given by

$$U_{n+1}(t) = U_0 + \int_{t_0}^t F(s, U_n(s))\, ds, \quad n = 0, 1, 2, \dots.$$

It can be easily shown that these successive approximations uniformly converge to the solution $U(t)$ of the IVP (4.1). $\square$

As a special case, we state the following theorem where the set valued map F is assumed to satisfy a Lipschitz type condition with respect to all semi metrics defined on the space $K_c(\mathbb{F})$.

Theorem 4.2. *Problem* (4.1) *can be uniquely solved under the following conditions:*

The function F can be realized as a projective limit $F = \varprojlim F^i$, $F^i \in C[\mathbb{R}_+ \times K_c(\mathbb{E}^i), K_c(\mathbb{E}^i)]$, and is bounded with respect to every semi metric of $K_c(\mathbb{F})$, that is there exists a positive $M > 0$ with $D^i(F(t,A), 0) \leq M$, for every $i \in \mathbb{N}$, if $t \in J = [t_0, t_0 + a]$ and $A \in K_c(\mathbb{F})$ with $D^i(A, U_0) \leq b$. $D^i(F(t,A), F(t,B)) \leq kD^i(A,B)$, for every index $i \in \mathbb{N}$, $t \in J$, if k is a positive constant. If this is the case, then the unique solution of (4.1) *can be defined on $[t_0, t_0 + \eta)$, where $\eta = \min\{a, b/\max\{M, 2kb\}\}$.*

Theorems 4.1 and 4.2, apart from answering at least partly, to the problem of solving set differential equations within the framework of Fréchet spaces, can be also applied to provide generalized solvability methods of ordinary differential equations in Fréchet spaces. Indeed, if we restrict our study to single (instead of multi) valued functions, then Theorem 4.2 reduces to:

Corollary 4.3. *Let*

$$u'(t) = F(t, u(t)), \quad u(t_0) = u_0 \in \mathbb{F}, \tag{4.3}$$

be an initial value problem on the Fréchet space $\mathbb{F}$, where

(i) *The mapping $F\colon J \times \mathbb{F} \to \mathbb{F}$, $J \subset \mathbb{R}$, is a projective limit $F = \varprojlim F^i$ and bounded on $\mathbb{F}$.*
(ii) *F satisfies the generalized Lipschitz condition*

$$p_i(F(t,u), F(t,v)) \leq k \cdot p_i(u,v),$$

for every index $i \in \mathbb{N}$ and $t \in J$, where (p_i) are the seminorms of $\mathbb{F}$ and k a positive constant.

Then, problem (4.3) *admits a unique solution.*

This is exactly the main theorem of [3] as well as a generalization of Theorem 2.2 in [1] for the case of linear differential equations. Therefore, one obtains, as corollary of the proposed methodology, techniques that lead to the study of a wide class of differential equations in Fréchet spaces.

Finally, we state a global existence theorem for (4.1), that can be established along the lines of the proof of Theorem 4.1, analogously.

Theorem 4.4. *Assume that*

(1) *$F \in C[\mathbb{R}_+ \times K_c(\mathbb{F}), K_c(\mathbb{F})]$ can be realized as a projective limit $F = \varprojlim F^i$, $F^i \in C[\mathbb{R}_+ \times K_c(\mathbb{E}^i), K_c(\mathbb{E}^i)]$.*

(2) $D^i[F(t,U),0] \leq g(t, D^i[U,0])$, $(t,U) \in \mathbb{R}_+ \times K_c(\mathbb{F})$, *for every* $i \in \mathbb{N}$, *where* $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in \mathbb{R}_+$ *and the maximal solution* $r(t,t_0,w_0)$ *of*

$$w' = g(t,w) = 0, \quad w(t_0) = w_0 \geq 0,$$

exists on $[t_0, \infty)$.

(3) F *is smooth enough to guarantee local existence of solutions of* (4.1) *for any* $(t_0, U_0) \in \mathbb{R}_+ \times K_c(\mathbb{F})$.

Then, the largest interval of existence of any solution $U(t,t_0,U_0)$ *of* (4.1) *such that* $D[U_0, 0] \leq w_0$ *is* $[t_0, \infty]$.

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G. N. Galanis
Section of Mathematics
Naval Academy of Greece
Xatzikyriakion, Piraeus 185 39
Greece
e-mail: ggalanis@snd.edu.gr

T. Gnana Bhaskar
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA
e-mail: gtenali@fit.edu

V. Lakshmikantham
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA
e-mail: lakshmik@fit.edu