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ON THE LIPSCHITZ CONTINUITY OF THE SOLUTIONS OF A CLASS OF ELLIPTIC FREE BOUNDARY PROBLEMS

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Abstract. In this article we prove local interior and boundary Lipschitz continuity of the solutions of a general class of elliptic free boundary problems in divergence form.

0. Introduction

The purpose of this paper is to study the regularity of the solutions of the following elliptic free boundary problem in divergence form

$$(P_0)\begin{cases}\text{Find } (u,\chi) \text{ such that:}\\ (i)\quad u\geq 0,\quad 0\leq\chi\leq 1,\quad u(1-\chi)=0\quad \text{a.e. in } \Omega\\ (ii)\ \ u=\varphi\quad \text{on } \Gamma_1\\ (iii)\,(a\nabla u+uB+\chi H)\,.\eta=0\quad \text{on } \partial\Omega\setminus\Gamma_1\\ (iv)\quad -\operatorname{div}(a\nabla u+uB+\chi H)=\chi c+C.\nabla u+du+f\end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with $\partial\Omega=\Gamma_1\cup\Gamma_2$, $a(x)=(a_{ij}(x))$ is a n-by-n matrix, $x=(x_1,\dots,x_n)$, $B(x)=(b_1(x),\dots,b_n(x))$, $C(x)=$

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$(c_1(x), \dots, c_n(x))$, $H(x) = (h_1(x), \dots, h_n(x))$ are vector functions, $c(x)$, $d(x)$ and $f(x)$ are real valued functions. η is the outward unit normal vector to $\partial\Omega$.

Note that for $B(x) = C(x) = 0$, $d(x) = c(x) = f(x) = 0$ and $H(x) = a(x)e_n$, with $e_n = (0, \dots, 0, 1)$, (P_0) corresponds to the dam problem with Dirichlet boundary conditions (see [1], [4]).

When $B(x) = C(x) = H(x) = 0$ and $d(x) = f(x) = 0$, we have the obstacle problem.

When $n = 2$, $a(x) = h^3(x)I_2$, $B(x) = C(x) = 0$, $d(x) = c(x) = f(x) = 0$ and $H(x) = h(x)e_2$, where I_2 is the 2-by-2 identity matrix and $h(x)$ is a scalar function, we have the lubrication problem.

When $n = 2$, $a(x) = k(x)I_2$, $B(x) = C(x) = 0$, $d(x) = c(x) = f(x) = 0$ and $H(x) = h(x)e_1$, with $e_1 = (1, 0)$, $k(x)$ and $h(x)$ are scalar functions, we have the aluminium electrolysis problem.

Given the jump condition along the free boundary $(\partial[u > 0]) \cap \Omega$, the optimal expected regularity for a solution u is Lipschitz continuity. This regularity result was proved in [1] for the dam problem and was extended in [2]. The objective of this paper is to consider a more general class of problems and also to establish a similar regularity up to Γ_1.

The main idea of the proof of the interior regularity is the comparison of u with a function of type $v(x) = k\left(e^{-\mu\rho^2} - e^{-\mu(r+\delta)^2}\right)$ in order to derive an estimate of the form $u(x_0) \leq Cr$ whenever $B(x_0, r)$ is a maximal open ball satisfying $B(x_0, r) \subset [u > 0]$, $\overline{B(x_0, r)} \subset \Omega$. For the regularity up to the boundary, we first reduce the problem to a flat boundary. The main idea then is the estimate $u(x) \leq C|x - x_0|$ in the half ball $B^+(x_0, R)$, where $x_0 \in \Gamma_1$. This is obtained by comparing u to the test function $\psi(|x - x_0 + Re_n| - R)$, where $\psi(t) = -\beta/\alpha t + (\beta/\alpha^2)e^{2\alpha R}(1 - e^{-\alpha t})$, and α, β are some constants.

1. Statement of the problem

Throughout this paper we assume that Ω is a Lipschitz bounded domain of $\mathbb{R}^n$ and that

$$\sum |a_{ij}(x)|^2 \leq M^2, \qquad \text{for a.e. } x \in \Omega, \quad (1.1)$$

$$a(x)\xi.\xi \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \qquad \text{for a.e. } x \in \Omega, \quad (1.2)$$

$$\lambda^{-2} \sum (|b_i(x)|^2 + |c_i(x)|^2) + \lambda^{-1}|d(x)| \leq \nu^2 \quad (1.3)$$

$$d + \operatorname{div}(B) \leq 0 \qquad \text{in } \mathcal{D}'(\Omega) \quad (1.4)$$

$$H \in \mathbb{L}^2(\Omega) \quad (1.5)$$

$$c \in L^2(\Omega) \tag{1.6}$$

$$f \in L^2(\Omega), \qquad f \geq 0 \text{ a.e. in } \Omega \tag{1.7}$$

$$\varphi \in C^{0,1}(\overline{\Omega}), \qquad \varphi \geq 0 \text{ a.e. in } \Omega \tag{1.8}$$

where λ, ν and M are positive constants. Note that assumption (1.4) and the nonnegativity of f and φ are needed in order to get nonnegative solutions.

We shall denote by $B(x,r)$ an open ball of center x and radius r. By L we denote the linear operator defined by

$$Lu = \operatorname{div}(a(x)\nabla u + uB(x)) + C(x)\nabla u + d(x)u.$$

We consider then the following weak formulation of problem (P_0)

$$(P)\begin{cases} \text{Find } (u,\chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ (i) \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0 \text{ a.e. in } \Omega \\ (ii) \quad u = \varphi \quad \text{on } \Gamma_1 \\ (iii) \displaystyle\int_\Omega (a(x)\nabla u + uB(x) + \chi H(x)).\nabla\xi \\ \qquad - (\chi c(x) + C(x).\nabla u + d(x)u)\xi dx = \displaystyle\int_\Omega f\xi dx \\ \qquad \forall \xi \in H^1(\Omega), \quad \xi = 0 \text{ on } \Gamma_1. \end{cases}$$

First we have the following proposition.

Proposition 1.1. *There exists a solution (u,χ) of (P) such that $u \leq \max_{\overline{\Gamma}_1} \varphi = U$.*

Proof. For the existence of a solution of (P), we consider for $\varepsilon > 0$ the approximated problem

$$(P_\varepsilon)\begin{cases} \text{Find } u_\varepsilon \in H^1(\Omega) \text{ such that:} \\ (i) \quad u_\varepsilon = \varphi \quad \text{on } \Gamma_1 \\ (ii) \displaystyle\int_\Omega (a(x)\nabla u_\varepsilon + u_\varepsilon B(x) + h_\varepsilon(u_\varepsilon)H(x)).\nabla\xi \\ \qquad - (h_\varepsilon(u_\varepsilon)c(x) + C(x).\nabla u_\varepsilon + d(x)u_\varepsilon)\xi dx = \displaystyle\int_\Omega f\xi dx, \\ \qquad \forall \xi \in H^1(\Omega), \quad \xi = 0 \text{ on } \Gamma_1 \end{cases}$$

where $h_\varepsilon(t) = \min(t^+/\varepsilon, 1)$.

To prove the existence of a solution of (P_ε), we consider for each $v \in L^2(\Omega)$, the following problem

$$(P_\varepsilon^v)\begin{cases}\text{Find } u_\varepsilon \in H^1(\Omega) \text{ such that:}\\ (i)\ \ u_\varepsilon = \varphi \quad \text{ on } \Gamma_1\\ (ii)\ \displaystyle\int_\Omega \left(a(x)\nabla u_\varepsilon + u_\varepsilon B(x) + h_\varepsilon(v)H(x)\right).\nabla\xi\\ \qquad - \left(h_\varepsilon(v)c(x) + C(x).\nabla u_\varepsilon + d(x)u_\varepsilon\right)\xi dx = \displaystyle\int_\Omega f\xi dx,\\ \qquad \forall \xi \in H^1(\Omega), \quad \xi = 0 \text{ on } \Gamma_1.\end{cases}$$

Under the assumptions (1.1)–(1.8), there exists at least one solution u_ε to (P_ε^v) (see [3, p. 215]). Then we apply the Schauder fixed point theorem to get a solution of (P_ε). Using u_ε^- as a test function and arguing as in [3, Theorem 8.1, p. 179], we obtain $u_\varepsilon \geq 0$ a.e. in Ω. Finally we establish an a priori estimate of $|u_\varepsilon|_{1,2}$ and use the fact that $0 \leq h_\varepsilon(u_\varepsilon) \leq 1$ a.e in Ω to pass to the limit and get a solution of the problem (P).

To establish the estimate one can adapt the proof of Theorem 8.1 in linebreak [3, p. 179]. □

Remark 1.1.

i) If for $\zeta \in \mathcal{D}(\Omega)$, one takes $\pm\zeta$ as test functions in (P) (iii), one gets $Lu = -f - \chi c(x) - \operatorname{div}(\chi H)$ in $\mathcal{D}'(\Omega)$. Then if $c, f \in L^p_{\text{loc}}(\Omega)$ and $H \in \mathbb{L}^{2p}_{\text{loc}}(\Omega)$ with $p > n/2$, and if we take into account (1.1)–(1.8), we obtain (see [3, Theorem 8.29, p. 205]) that $u \in C^{0,\alpha}_{\text{loc}}(\Omega \cup \Gamma_1)$ for some $\alpha \in (0,1)$. In particular the set $[u > 0]$ is open.

ii) Due to (P) (i), we have $Lu = -f - c(x) - \operatorname{div}(H)$ in $\mathcal{D}'([u > 0])$. So if $a, b_i \in C^{0,\alpha}_{\text{loc}}(\Omega)$ $(0 < \alpha < 1)$, $c_i, d \in L^\infty_{\text{loc}}(\Omega)$, $c, f, \operatorname{div}(H) \in L^p_{\text{loc}}(\Omega)$ with $p > n/(1-\alpha)$, we obtain (see [3, p. 212]) that $u \in C^{1,\alpha}_{\text{loc}}([u > 0])$.

2. Interior Lipschitz Continuity

In this section we assume that

$$a, b_i \in C^{0,\alpha}_{\text{loc}}(\Omega), \quad c_i, d \in L^\infty_{\text{loc}}(\Omega) \quad i = 1, \dots, n,\ \alpha \in (0,1) \tag{2.1}$$

$$\forall s > 0,\ \forall x_0 \in \Omega,\ \forall y \in B(x_0, s) \subset \Omega\colon\ \operatorname{div}(a(x)(x-y)) \leq c_0(s) \text{ in } \mathcal{D}'(B(x_0,s)) \tag{2.2}$$

$$c, f, \operatorname{div}(H) \in L^p_{\text{loc}}(\Omega), \quad p > n/(1-\alpha) \tag{2.3}$$

$$|H(x)| \leq c_0(s) \quad \text{a.e. in } B(x_0, s) \tag{2.4}$$

$$-c^- + \operatorname{div}(H) \geq -c_0(s) \quad \text{in } \mathcal{D}'(B(x_0,s)) \tag{2.5}$$

$$d + \operatorname{div}(B) \geq -c_0(s) \quad \text{in } \mathcal{D}'(B(x_0,s)) \tag{2.6}$$

where c_0 is a positive constant depending only on s. Without loss of generality, we can assume that $\lambda < 1/2$.

The main result of this section is the following theorem.

Theorem 2.1. *Let (u, χ) be a solution of (P). Then $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$.*

To prove Theorem 2.1, we need three lemmas.

Lemma 2.1. *Let $x_0 = (x_{01}, \dots, x_{0n})$ and $r, s > 0$ such that $r \le r_0 = (\sqrt{7} - 2)/(8\nu)$, $B(x_0, r) \subset [u > 0]$, $\overline{B(x_0, r)} \subset B(x_0, s) \subset \Omega$ and $\partial B(x_0, r) \cap \partial[u > 0] \ne \emptyset$. Then we have for some positive constant C depending only on λ, ν, U and s, but not on r*

$$\min_{B_{r/2}(x_0)} u \le Cr.$$

Proof. Let $m = \min_{\overline{B(x_0, r/2)}} u$ and $\delta \in (0, r)$ such that $B(x_0, r + \delta) \subset B(x_0, s)$, and let v be defined by

$$v(x) = k\left(e^{-\mu\rho^2} - e^{-\mu(r+\delta)^2}\right)$$

where

$$\rho^2 = \sum_{i=1}^{i=n} (x_i - x_{0i})^2, \quad k = \frac{m}{\left(e^{-\mu r^2/4} - e^{-\mu(r+\delta)^2}\right)}, \quad \mu = \frac{\kappa}{r^2} \quad \text{and}$$

$$\kappa = 4\nu s + \frac{2c_0 + 1}{\lambda}.$$

Then one has

$$\begin{cases} Lv \ge -\chi c - f - \operatorname{div}(H) & \text{in } D = B(x_0, r + \delta) \setminus \overline{B(x_0, r/2)} \\ v = m & \text{on } \partial B(x_0, r/2) \\ v = 0 & \text{on } \partial B(x_0, r + \delta) \\ |\nabla v| = 2k\mu\rho e^{-\mu\rho^2} & \text{decreases with respect to } \rho. \end{cases}$$

Indeed we have $\nabla v = -2\mu k e^{-\mu\rho^2}(x - x_0)$ and therefore we get

$$\begin{aligned} \frac{d}{d\rho}|\nabla v| =& 2k\mu e^{-\mu\rho^2}(1 - 2\kappa\frac{\rho^2}{r^2}) \le 2k\mu e^{-\mu\rho^2}(1 - \kappa/2) \le 0 \\ & \text{since } \kappa > 1/\lambda > 2 \\ Lv =& 4\mu^2 k e^{-\mu\rho^2} a(x)(x - x_0).(x - x_0) - 2\mu k e^{-\mu\rho^2} \operatorname{div}(a(x)(x - x_0)) \\ & - 2\mu k e^{-\mu\rho^2}(B(x) + C(x))(x - x_0) + (d(x) + \operatorname{div}(B))v \\ \ge& 4\mu^2\lambda(r^2/4)k e^{-\mu\rho^2} - 2\mu k c_0 e^{-\mu\rho^2} - 4\mu s\lambda\nu k e^{-\mu\rho^2} - c_0 U \\ \ge& \mu k[\mu\lambda r^2 - 4\lambda\nu s - 2c_0]e^{-\mu\rho^2} - c_0 U \end{aligned}$$

$$
\begin{aligned}
&=\kappa\frac{m}{r^2}[\kappa\lambda-4\lambda\nu s-2c_0]\frac{e^{-\mu\rho^2}}{e^{-\mu r^2/4}-e^{-\mu(r+\delta)^2}}-c_0U\\
&=\kappa\frac{m}{r^2}\frac{e^{-\mu\rho^2}}{e^{-\mu r^2/4}-e^{-\mu(r+\delta)^2}}-c_0U=\kappa\frac{m}{r^2}\frac{e^{-\mu\rho^2}e^{\mu(r+\delta)^2}}{e^{\mu((r+\delta)^2-r^2/4)}-1}-c_0U\\
&\geq\kappa\frac{m}{r^2}\frac{1}{e^{\mu((r+\delta)^2-r^2/4)}-1}-c_0U\\
&\geq\kappa\frac{m}{r^2}\frac{1}{e^{\mu(4r^2)}-1}-c_0U \quad \text{since } 0<\delta<r\\
&=\kappa\frac{m}{r^2}\frac{1}{e^{4\kappa}-1}-c_0U. \qquad (2.7)
\end{aligned}
$$

Using (1.7) and (2.5), we get from (2.7)

$$
\begin{aligned}
Lv+f+\chi c+\operatorname{div}(H) &\geq\kappa\frac{m}{r^2}\frac{1}{e^{4\kappa}-1}-c_0U-\chi c^- +\operatorname{div}(H)\\
&\geq\kappa\frac{m}{r^2}\frac{1}{e^{4\kappa}-1}-c_0U-c_0. \qquad (2.8)
\end{aligned}
$$

Now if

$$
m\leq\frac{c_0(1+U)(e^{4\kappa}-1)}{\kappa}r^2, \quad \text{then} \quad m\leq\frac{sc_0(1+U)(e^{4\kappa}-1)}{\kappa}r
$$

and the lemma is proved. Assume that

$$
m>\frac{c_0(1+U)(e^{4\kappa}-1)}{\kappa}r^2.
$$

Then we obtain from (2.8)

$$
Lv+f+\chi c+\operatorname{div}(H)\geq 0 \quad \text{in } D. \qquad (2.9)
$$

Since $v\leq u$ on ∂D, $\zeta=(v-u)^+\in H_0^1(D)$, and $\pm\zeta$ — after being extended by zero outside D — are test functions for (P). So we have

$$
\begin{aligned}
&\int_D\left(a(x)\nabla u+uB(x)+\chi H(x)\right).\nabla(v-u)^+ \qquad (2.10)\\
&-(\chi c(x)+C(x).\nabla u+d(x)u)(v-u)^+dx=\int_D f(v-u)^+dx.
\end{aligned}
$$

By (2.9) we have

$$
\begin{aligned}
&\int_D\left(a(x)\nabla v+vB(x)+H(x)\right).\nabla(v-u)^+ \qquad (2.11)\\
&-(\chi c(x)+C(x).\nabla v+d(x)v)(v-u)^+dx\leq\int_D f(v-u)^+dx.
\end{aligned}
$$

Subtracting (2.10) from (2.11), we get

$$
\int_D a(x)\nabla(v-u).\nabla(v-u)^+dx\leq\int_D(v-u)^+(C(x)-B(x)).\nabla(v-u)^+
$$

$$+\int_D (\chi-1)H(x).\nabla(v-u)^+$$
$$+\int_D d(x)(v-u)^{+2}dx$$

which can be written by (1.2)–(1.3) and by taking into account that $\chi = 1$ a.e. in $[u>0]$

$$\int_D \lambda|\nabla(v-u)^+|^2dx \leq 2\lambda\nu\int_D (v-u)^+|\nabla(v-u)^+|dx + \int_D \lambda\nu^2(v-u)^{+2}dx$$
$$+\int_{D\cap[u=0]}(\chi-1)H(x).\nabla v dx. \tag{2.12}$$

Note that we have by Cauchy-Schwarz inequality

$$\int_D (v-u)^+|\nabla(v-u)^+|dx$$
$$\leq \left(\int_D (v-u)^{+2}dx\right)^{1/2}.\left(\int_D |\nabla(v-u)^+|^2dx\right)^{1/2}. \tag{2.13}$$

Now using Poincaré's inequality and majoring the constant, we obtain

$$\int_D (v-u)^{+2}dx \leq 16r^2\int_D |\nabla(v-u)^+|^2dx. \tag{2.14}$$

Therefore we deduce from (2.12)–(2.14) that

$$\int_D \lambda|\nabla(v-u)^+|^2dx \leq 8\lambda\nu r\int_D |\nabla(v-u)^+|^2dx + 16\lambda\nu^2r^2\int_D |\nabla(v-u)^+|^2dx$$
$$+\int_{D\cap[u=0]} c_0|\nabla v|dx$$

which can be written

$$\lambda\left(1-8\nu r-16\nu^2r^2\right)\int_D |\nabla(v-u)^+|^2dx \leq \int_{D\cap[u=0]} c_0|\nabla v|dx.$$

This leads for $r \leq r_0 = (\sqrt{7}-2)/(8\nu)$ to

$$\frac{\lambda}{4}\int_D |\nabla(v-u)^+|^2dx \leq \int_{D\cap[u=0]} c_0|\nabla v|dx$$

or

$$\int_{D\cap[u>0]} |\nabla(v-u)^+|^2dx \leq \int_{D\cap[u=0]} |\nabla v|\left(\frac{4c_0}{\lambda}-|\nabla v|\right)dx. \tag{2.15}$$

Now we claim that

$$\int_{D\cap[u>0]} |\nabla(v-u)^+|^2dx > 0. \tag{2.16}$$

Indeed assume that

$$\int_{D\cap[u>0]} |\nabla(v-u)^+|^2 dx = 0.$$

In particular we have

$$\int_{B(x_0,r)\setminus\overline{B(x_0,r/2)}} |\nabla(v-u)^+|^2 dx = 0$$

and then $\nabla(v-u)^+ = 0$ in $B(x_0,r)\setminus\overline{B(x_0,r/2)}$. Since $v \le u$ on $\partial B(x_0,r/2)$, we get $v \le u$ in $B(x_0,r)\setminus\overline{B_{r/2}}(x_0)$. This leads to a contradiction with $\partial B(x_0,r)\cap\partial[u>0]\neq\emptyset$ and $v>0$ in D. Hence (2.16) is true and we deduce from (2.15) that

$$\int_{D\cap[u=0]} |\nabla v|\left(\frac{4c_0}{\lambda}-|\nabla v|\right)dx > 0. \tag{2.17}$$

Since $|\nabla v|$ is non-increasing with respect to ρ, we infer from (2.17) that

$$|\nabla v|_{|_{\partial B(x_0,r+\delta)}} = 2k\mu(r+\delta)e^{-\mu(r+\delta)^2} < \frac{4c_0}{\lambda}.$$

Letting $\delta\to 0$, we get

$$m \le \frac{2c_0}{\kappa\lambda}(e^{3\kappa/4}-1)r = C\,r.$$

□

Lemma 2.2. *Assume that $u>0$ in $B(x_0,\mu r)\subset B(x_0,s)\subset\Omega$. Then*

$$u_r(y) = \frac{u(x_0+ry)}{r}$$

is defined in $B(O,\mu)$, where $B(O,\mu)$ is the open ball of center $O=(0,\dots,0)$ and radius μ. Moreover we have

i) $\text{div}(a_r(y)\nabla u_r + u_r B_r(y)) - (rC_r(y)\nabla u_r + r^2 d_r(y)u_r) = rc_r(y) + rf_r(y) + \text{div}(H_r)$ *in* B_μ,
where $g_r(y) = g(x_0+ry)$.

ii) $\max_{B(x_0,\mu r)} u \le C\left(\min_{B(x_0,\mu r/2)} u + r\right)$
where C is a positive constant depending only on μ, λ, ν, M, U, n, p and s.

iii) $\sup_{B(x_0,\mu r/2)} |\nabla u| \le C\left(\frac{\sup_{B(x_0,\mu r)} u}{r}+1\right)$
where C is a positive constant depending only on μ, λ, ν, M, U, n, p, $|a_{ij}|_{0,\alpha,B(x_0,s)}$, $|b_i|_{0,\alpha,B(x_0,s)}$, $|c_i|_{0,B(x_0,s)}$, $|d|_{0,B(x_0,s)}$ and s.

Proof. i) Straightforward calculation.

ii) Since r is bounded by s/μ, we can assume that a_r, B_r, rC_r, rd_r, satisfy the assumptions (1.1)–(1.3) with the constants independent on r. Therefore we can apply Theorems 8.17 and 8.18 in [3, p. 194] (Harnack inequality) to the equation in i). We get for a positive constant C_1 depending only on μ, λ, ν, M, U, n, p and s

$$\max_{B(O,\mu/2)} u_r \le C_1 \left(\min_{B(O,\mu/2)} u_r + |r(c_r + f_r)|_{L^p(B_\mu)} + |H_r|_{L^{2p}(B_\mu)} \right).$$

Since $p > n/(1-\alpha) > n$ and due to (2.3), we have for some constant $C_2(s)$

$$\begin{aligned}
|r(c_r + f_r)|_{L^p(B_\mu)} &= \left(\int_{B_\mu} r^p |c+f|^p (x_0 + ry) dy \right)^{1/p} \\
&= \left(\int_{B(x_0,\mu r)} \frac{r^p}{r^n} |c+f|^p (x) dx \right)^{1/p} \\
&= r^{1-n/p} |c+f|_{L^p(B(x_0,\mu r))} \le C_2 r^{1-n/p} \\
|H_r|_{L^{2p}(B(O,\mu))} &= \left(\int_{B(O,\mu)} |H|^{2p} (x_0 + ry) dy \right)^{1/2p} \le c_0(s) |B(O,\mu)|^{1/2p}.
\end{aligned}$$

Hence we obtain for a constant C_3 independent of r

$$\max_{B(O,\mu/2)} u_r \le C_3 \left(\min_{B(O,\mu/2)} u_r + r^{1-n/p} + 1 \right).$$

Now using the definition of u_r, one can verify that

$$\max_{B(O,\mu/2)} u_r = \frac{1}{r} \max_{B(x_0,\mu r/2)} u \quad \text{and} \quad \min_{B(O,\mu/2)} u_r = \frac{1}{r} \min_{B(x_0,\mu r/2)} u.$$

It follows that

$$\max_{B(x_0,\mu r/2)} u \le C_3 \left(\min_{B(x_0,\mu r/2)} u + r^{2-n/p} + r \right).$$

Finally since $p > n$ and r is bounded, we have $r^{2-n/p} \le C_4 r$, which leads finally to

$$\max_{B(x_0,\mu r/2)} u \le C \left(\min_{B(x_0,\mu r/2)} u + r \right).$$

iii) From the equation in i), we know (see [3, p. 212], Corollary 8.36 and the Remark after it) that $u_r \in C^{1,\alpha}(B(O,\mu))$ and that

$$\begin{aligned}
&|u_r|_{1,\alpha,\overline{B(O,\mu/2)}} \\
&\le C \left(|u_r|_{0,B_\mu} + |rc_r|_{p,B(O,\mu)} + |rf_r|_{p,B(O,\mu)} + |\operatorname{div}(H_r)|_{p,B(O,\mu)} \right),
\end{aligned}$$

where C depends only on $\operatorname{dist}(\overline{B(O,\mu/2)}, \partial B_\mu)$, λ, ν, M, U, n, p, $|a_{ij}|_{0,\alpha,B(x_0,s)}$, $|b_i|_{0,\alpha,B(x_0,s)}$, $|c_i|_{0,B(x_0,s)}$, $|d|_{0,B(x_0,s)}$ and s. In particular, we obtain $\sup_{B(O,\mu/2)} |\nabla u_r| \leq C\Big(\sup_{B(O,\mu)} u_r + 1\Big)$ or equivalently

$$\sup_{B(x_0,\mu r/2)} |\nabla u| \leq C\Big(\frac{\sup_{B(x_0,\mu r)} u}{r} + 1\Big).$$

□

Lemma 2.3. *Under the assumptions of Lemma* 2.1*, we have for a constant* $C > 0$ *depending only on* λ, ν, M, U, c_0, n, p *and* s*, but not on* r

$$u(x_0) \leq Cr.$$

Proof. Applying the result ii) of Lemma 2.2 for $\mu = 1$, we get

$$\max_{B(x_0,r/2)} u \leq C\left(\min_{B(x_0,r/2)} u + r\right),$$

where C is a positive constant depending only on λ, ν, M, U, n, p and s. Using Lemma 2.1, we obtain

$$u(x_0) \leq \max_{B(x_0,r/2)} u \leq C\left(\min_{B(x_0,r/2)} u + r\right) \leq C(Cr + r) = C_1 r.$$

□

Remark 2.1. If in (2.3) $L^p_{\text{loc}}(\Omega)$ is replaced by $L^p(\Omega)$, and if the constant c_0 in (2.4)–(2.6) does not depend on s, then clearly the constants in Lemmas 2.1–2.3 will not depend on s.

Proof of Theorem 2.1. Let $\Omega_\varepsilon = \{x \in \Omega / d(x, \partial\Omega) > \varepsilon\}$ for $\varepsilon \in (0, r_0/4)$, where r_0 is as in Lemma 2.1. We shall prove that ∇u is bounded in $\Omega_{4\varepsilon}$ by a constant depending only on λ, ν, M, n, p, U, $|a_{ij}|_{0,\alpha,\Omega_{3\varepsilon}}$, $|b_i|_{0,\alpha,\Omega_{3\varepsilon}}$, $|c_i|_{0,\Omega_{3\varepsilon}}$, $|d|_{0,\Omega_{3\varepsilon}}$ and ε.

Let $x_0 \in \Omega_{4\varepsilon}$. We distinguish two cases:

i) $B(x_0, 2\varepsilon) \subset [u > 0]$

Applying Lemma 2.2 iii) with $\mu = 2$ and $r = \varepsilon$, we get

$$\sup_{B(x_0,\varepsilon)} |\nabla u| \leq C\Big(\frac{\sup_{B(x_0,2\varepsilon)} u}{\varepsilon} + 1\Big) \leq C\left(\frac{U}{\varepsilon} + 1\right).$$

where C is a positive constant independent of ε.

ii) $B(x_0, 2\varepsilon) \cap [u = 0] \neq \emptyset$

Let $x \in B_\varepsilon(x_0)$ such that $u(x) > 0$ and let $r(x) = \operatorname{dist}(x, [u = 0])$ be the distance between x and the set $[u = 0]$. Clearly we have $B(x, r(x)) \subset [u > 0]$. Moreover we claim that $r(x) < 3\varepsilon$ and $\overline{B(x, r(x))} \subset B(x_0, 4\varepsilon)$.

Indeed we know that there exists $x' \in B(x_0, 2\varepsilon) \cap [u = 0]$. So we have

$$r(x) \leq |x - x'| \leq |x - x_0| + |x_0 - x'| < \varepsilon + 2\varepsilon = 3\varepsilon.$$

Now if $y \in \overline{B(x, r(x))}$, we have

$$|x_0 - y| \leq |x_0 - x| + |x - y| \leq \varepsilon + r(x) < \varepsilon + 3\varepsilon = 4\varepsilon.$$

Since $u > 0$ in $B(x, r(x))$, $\overline{B(x, r(x))} \subset B(x_0, 4\varepsilon)$, $r(x) < 4\varepsilon < r_0$ and $\partial B(x, r(x)) \cap \partial[u > 0] \neq \emptyset$, we deduce from Lemma 2.3 that $u(x) \leq Cr(x)$ for some constant independent of $r(x)$. Applying Lemma 2.2 ii) and iii) with $\mu = 1/2$ and $r = r(x)$, we get

$$\sup_{B(x,r(x)/4)} |\nabla u| \leq C \left(\frac{\sup_{B(x,r(x)/2)} u}{r(x)} + 1 \right) \leq C$$

where C is a positive constant depending only on ε and the data. In particular $|\nabla u(x)| \leq C$.

Since $\nabla u(x) = 0$ a.e. in $B_\varepsilon(x_0) \cap [u = 0]$, it follows that ∇u is uniformly bounded in $B(x_0, \varepsilon)$. □

3. Boundary Lipschitz continuity

In this section we assume that $\varphi = 0$ on a nonempty $C^{1,1}$ portion T of Γ_1 and prove that u is locally Lipschitz continuous up to T. In fact by the same arguments one can prove the same result for any $C^{1,1}$ portion of Γ_1 on which φ is of class C^2. We need the following assumptions on the data.

$$\varphi = 0 \quad \text{on a nonempty } C^{1,1} \text{ portion } T \text{ of } \Gamma_1 \tag{3.1}$$

$$\forall x \in T\ \exists s > 0\colon a, b_i \in C^{0,1}(\overline{B(x, s) \cap \Omega}),\ c_i, d \in L^\infty(B(x, s) \cap \Omega),$$
$$i = 1, \dots, n \tag{3.2}$$

$$c, f, \operatorname{div}(H) \in L^p\left(B(x, s) \cap \Omega\right),\ p > \frac{n}{1 - \alpha}, \quad 0 < \alpha < 1 \tag{3.3}$$

$$|H(x)| \leq c_0 \quad \text{a.e. in } B(x, s) \cap \Omega \tag{3.4}$$

$$c + f + \operatorname{div}(H) \leq c_0 \quad \text{in } \mathcal{D}'\left(B(x, s) \cap \Omega\right) \tag{3.5}$$

where c_0 is a positive constant depending only on s.

The main result of this section is the following.

Theorem 3.1. *Let (u, χ) be a solution of (P). Then we have $u \in C^{0,1}_{\mathrm{loc}}(\Omega \cup T)$.*

We shall first transform the problem locally. Indeed let $x_0 \in T$. Then there exists a neighborhood V of x_0 in $\mathbb{R}^n$ and a $C^{1,1}$ bijection $\Phi\colon Q \to V$ such that

$$\Phi \in C^{1,1}(\overline{Q}),\ \Psi = \Phi^{-1} \in C^{1,1}(\overline{V}),\ \Phi(Q_+) = V \cap \Omega \text{ and } \Phi(Q_0) = V \cap T,$$

where

$$\begin{aligned} Q =& \{x = (x', x_n) \in \mathbb{R}^n\colon |x'| < 1 \text{ and } |x_n| < 1\} \\ Q_+ =& Q \cap \mathbb{R}^n_+,\ \mathbb{R}^n_+ = \mathbb{R}^n \cap [x_n > 0] \\ Q_0 =& Q \cap [x_n = 0]. \end{aligned}$$

Now we set for $y \in Q_+$

$$\begin{aligned} v(y) =& uo\Phi(y), \quad \gamma(y) = \chi o\Phi(y) \\ b(y) =& |\operatorname{Jac}\Phi(y)|^t D\Psi(y) ao\Phi(y) D\Psi(y) \\ \widetilde{G}(y) =& |\operatorname{Jac}\Phi(y)|^t D\Psi(y) Go\Phi(y) \\ \widetilde{g}(y) =& |\operatorname{Jac}\Phi(y)| go\Phi(y). \end{aligned}$$

Then one can verify that (v, γ) satisfies

$$(\widetilde{P}) \begin{cases} (v,\gamma) \in H^1(Q_+) \times L^\infty(Q_+) \text{ such that:} \\ (i)\ \ v \geq 0, \quad 0 \leq \gamma \leq 1, \quad v(1-\gamma) = 0 \text{ a.e. in } Q_+ \\ (ii)\ v = 0 \quad \text{on } Q_0 \\ (iii) \displaystyle\int_{Q_+} \Big(b(x)\nabla v + v\widetilde{B}(x) + \gamma \widetilde{H}(y)\Big).\nabla\zeta \\ \qquad - (\gamma\widetilde{c}(y) + \widetilde{G}(y).\nabla v + \widetilde{d}(y)v)\zeta dy = \int_{Q_+} \widetilde{f}\zeta dy \\ \qquad \forall \zeta \in H^1_0(Q_+). \end{cases}$$

It is then obvious that it suffices to prove Theorem 3.1 for a solution v of $(\widetilde{P})$. Therefore we will perform the proof assuming that $\Omega = Q_+$ and $\Gamma_1 = Q_0$. The proof is based on the following lemma.

Lemma 3.1. *Let $x_0 = (x_{01}, \ldots, x_{0n}) \in Q_0$ and $R > 0$ such that $B^+(x_0, R) = B(x_0, R) \cap \mathbb{R}^n_+ \subset Q^+$. Then we have for some positive constant C depending only on λ, ν, n, M, U and R,*

$$u(x) \leq C\,|x - x_0| \quad \forall x \in B^+(x_0, R).$$

Proof. Let $x_1 = x_0 - Re_n$ and let $\Omega_R = B_{2R}(x_1) \cap \Omega$. We consider the function $v(x) = \psi(d_1(x))$ defined for $x \in \Omega_R$, where $d_1(x) = |x - x_1| - R$ and

$$\psi(t) = -\frac{\beta}{\alpha}t + \frac{\beta}{\alpha^2}e^{2\alpha R}(1 - e^{-\alpha t}),$$

$$\alpha = \frac{(nM-\lambda)}{\lambda R} + \frac{K}{\lambda} + 2\nu,$$
$$K = n\sqrt{n} \sup_{i,j\in\{1,\dots,n\}, x\in\overline{\Omega_R}} \left|\frac{\partial a_{ij}}{\partial x_i}\right|, \quad \beta = \frac{c_0}{\lambda},$$

and $c_0 = c_0(x_0, R)$ is a constant from (3.5).

We claim that

$$Lv + c + f + \mathrm{div}(H) \le \lambda(\psi''(d_1) + \alpha\psi'(d_1) + \beta) = 0 \quad \text{in } \mathcal{D}'(\Omega_R). \tag{3.6}$$

Indeed we first have

$$\frac{\partial v}{\partial x_i} = \psi'(d_1)\frac{\partial d_1}{\partial x_i} = \psi'(d_1)\frac{x_i - x_{1i}}{|x-x_1|} \tag{3.7}$$

$$\begin{aligned}\frac{\partial^2 v}{\partial x_i \partial x_j} &= \psi''(d_1)\frac{\partial d_1}{\partial x_i}\frac{\partial d_1}{\partial x_j} + \psi'(d_1)\frac{\partial^2 d_1}{\partial x_i \partial x_j} \\ &= \psi''(d_1)\frac{(x_i - x_{1i})(x_j - x_{1j})}{|x-x_1|^2} \\ &\quad + \psi'(d_1)\left(\frac{\delta_{ij}}{|x-x_1|} - \frac{(x_i - x_{1i})(x_j - x_{1j})}{|x-x_1|^3}\right).\end{aligned} \tag{3.8}$$

Then we deduce from (3.7)–(3.8) that

$$\begin{aligned}\mathrm{div}(a(x)\nabla v) &= \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \frac{\partial a_{ij}}{\partial x_i}\frac{\partial v}{\partial x_j} \\ &= \frac{\psi''(d_1)}{|x-x_1|^2}\left(\sum_{i,j=1}^n a_{ij}(x)(x_i - x_{1i})(x_j - x_{1j})\right) \\ &+ \frac{\psi'(d_1)}{|x-x_1|}\left(\sum_{i,j=1}^n a_{ij}\delta_{ij}\right) \\ &- \frac{\psi'(d_1)}{|x-x_1|^3}\left(\sum_{i,j=1}^n a_{ij}(x)(x_i - x_{1i})(x_j - x_{1j})\right) \\ &+ \frac{\psi'(d_1)}{|x-x_1|}\left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_i}(x_j - x_{1j})\right) \\ &\le \lambda\psi''(d_1) + (nM-\lambda)\frac{\psi'(d_1)}{|x-x_1|} + K\psi'(d_1) \\ &\le \lambda\psi''(d_1) + \left(\frac{(nM-\lambda)}{R} + K\right)\psi'(d_1).\end{aligned} \tag{3.9}$$

Moreover we have by (1.3)–(1.4) and (3.7)

$$\begin{aligned}\operatorname{div}(vB(x)) + C(x).\nabla v + d(x)v &= (B(x)+C(x)).\nabla v + (d+\operatorname{div}(B))v\\ &\le |B(x)+C(x)|.|\nabla v| \le 2\lambda\nu\psi'(d_1). \quad (3.10)\end{aligned}$$

Using (3.5), (3.9) and (3.10), we get

$$\begin{aligned}Lv + c + f + \operatorname{div}(H) \le& \lambda\psi''(d_1) + \left(\frac{(nM-\lambda)}{R} + K + 2\lambda\nu\right)\psi'(d_1) + c_0\\ =&\lambda(\psi''(d_1) + \alpha\psi'(d_1) + \beta) = 0\end{aligned}$$

which is (3.6).

Now we have

$$u = 0 \le v \quad \text{on} \quad \overline{\Omega_R}\cap\Gamma_1 \tag{3.11}$$

and

$$v = \psi(d_1(x)) = \psi(2R-R) = \psi(R) \quad \text{on} \quad \Gamma_R = \partial\Omega_R\backslash\Gamma_1. \tag{3.12}$$

If $\psi(R) \ge \max_{\overline{\Omega_R}} u$, then by (3.12), we have

$$v \ge u \quad \text{on} \quad \Gamma_R. \tag{3.13}$$

If $\psi(R) < \max_{\overline{\Omega_R}} u$, we take $v = k\psi(d_1)$, where $k = \max_{\overline{\Omega_R}} u/\psi(R) > 1$.

Since $k\psi''(d_1) + \alpha k\psi'(d_1) + \beta = \beta(1-k) < 0$, v satisfies (3.6) and also $v \ge u$ on $\partial\Omega_R$.

Now let $\zeta \in \mathcal{D}(\Omega_R)$, $\zeta \ge 0$ and $\varepsilon > 0$. Using (3.6) and (P) iii) with the test function

$$\xi = \min\left(\zeta, \frac{(u-v)^+}{\varepsilon}\right),$$

we obtain by taking into account that $\chi = 1$ a.e. in $[u > 0]$

$$\begin{aligned}&\int_{\Omega_R} (a(x)\nabla u + uB(x) + H(x)).\nabla\xi - (c(x) + C(x).\nabla u + d(x)u)\xi dx\\ &= \int_{\Omega_R} f\xi dx \qquad (3.14)\\ &-\int_{\Omega_R} (a(x)\nabla v + vB(x) + H(x)).\nabla\xi - (c(x) + C(x).\nabla v + d(x)v)\xi dx\\ &\le -\int_{\Omega_R} f\xi dx. \qquad (3.15)\end{aligned}$$

Adding (3.14) and (3.15), we obtain

$$\begin{aligned}&\int_{\Omega_R} (a(x)\nabla(u-v) + (u-v)B(x)).\nabla\xi\\ &-(C(x).\nabla(u-v) + d(x)(u-v))\xi dx \le 0\end{aligned}$$

which can be written

$$\begin{aligned}
&\int_{\Omega_R\cap[\varepsilon\zeta\leq(u-v)^+]} \left(a(x)\nabla(u-v)^+ + (u-v)^+B(x)\right).\nabla\zeta \\
&- (C(x).\nabla(u-v)^+ + d(x)(u-v)^+)\zeta dx \\
&\leq -\frac{1}{\varepsilon}\int_{\Omega_R\cap[\varepsilon\zeta>(u-v)^+]} \left(a(x)\nabla(u-v)^+ + (u-v)^+B(x)\right).\nabla(u-v)^+ \\
&+ \frac{1}{\varepsilon}\int_{\Omega_R\cap[\varepsilon\zeta>(u-v)^+]} (C(x).\nabla(u-v)^+ + d(x)(u-v)^+)(u-v)^+dx \\
&\leq -\frac{1}{\varepsilon}\int_{\Omega_R\cap[\varepsilon\zeta>(u-v)^+]} (u-v)^+\left(B(x).\nabla(u-v)^+ - C(x).\nabla(u-v)^+\right. \\
&\left.-d(x)(u-v)^+\right)dx \\
&\leq \int_{\Omega_R\cap[\varepsilon\zeta>(u-v)^+]} \zeta\left(|B(x)-C(x)|.|\nabla(u-v)^+|\right. \\
&\left.+|d(x)|(u-v)^+\right)dx. \qquad (3.16)
\end{aligned}$$

Letting $\varepsilon \to 0$ in (3.16), we get $L(u-v)^+ \geq 0$ in $\mathcal{D}'(\Omega_R)$. Taking into account (3.11)–(3.13), we get (see [3, Theorem 8.1, page 179]) $(u-v)^+ \leq 0$ in Ω_R. This leads to $u \leq v$ in Ω_R.

We deduce that for all $x \in \Omega_R$

$$\begin{aligned}
u(x) \leq& v(x) = |v(x) - v(x_0)| \\
&\leq \sup_{0\leq t\leq R} \psi'(t)|x-x_0| = \psi'(0)|x-x_0| = C(R)|x-x_0|.
\end{aligned}$$

Since $B^+(x_0,R) \subset \Omega_R$, the lemma is proved. □

Proof of Theorem 3.1. Let $x_0 \in Q_0$ and $R > 0$ such that $B^+(x_0,3R) \subset \Omega$. We shall prove that ∇u is bounded in $B^+(x_0,R)$ by a constant C depending only on λ, ν, M, U and R.

We distinguish two cases:

i) $B^+(x_0,2R) \subset [u>0]$

Since u satisfies

$$\begin{cases} L(u) = c + f + \mathrm{div}(H) \quad \text{in} \quad B^+(x_0,2R) \\ 0 \leq u \leq U \quad \text{in} \quad B^+(x_0,2R), \quad u=0 \quad \text{on} \quad B_{2R}(x_0)\cap Q_0, \end{cases}$$

we deduce that $u \in C^{1,\alpha}\left(B^+(x_0,2R)\cup(B_{2R}(x_0)\cap Q_0)\right)$ (see [3, p. 212], Corollary 8.36 and the Remark after it). In particular we obtain $|\nabla u(x)| \leq C$ for all $x \in B^+(x_0,R)$.

ii) $\exists x_f \in B^+(x_0,2R)\cap[u=0]$

Let $x \in B^+(x_0, R)$ such that $u(x) > 0$ and let $r = \text{dist}(x, [u = 0])$ be the distance between x and the set $[u = 0]$. Remark that we have $r \leq |x - x_f| < 2R$. Moreover $B_r(x) \cap \Omega \subset [u > 0]$ and $\overline{B_r}(x) \cap \Omega \subset B^+_{3R}(x_0) \cap \Omega$.

Indeed, if $y \in \overline{B_r}(x) \cap \Omega$, we have $|x_0 - y| \leq |x_0 - x| + |x - y| < R + r < R + 2R = 3R$.

Again we distinguish two cases:

a) $\overline{B(x,r)} \cap Q_0 = \emptyset$

In this case, we have by Lemma 2.3 $u(x) \leq Cr$, where C depends only on λ, ν, M, U, n, p and R. Then by arguing exactly as in the proof of Theorem 2.1, we can prove that ∇u is uniformly bounded in $B_{r/2}(x)$ by a constant depending only on λ, ν, M, U, n, p and R.

b) $\exists y_1 \in \overline{B(x,r)} \cap Q_0$

Again we have two cases:

α) $B(x, r/4) \subset \Omega$

By Lemma 3.1, we have for all $y \in B_{r/4}(x)$

$$u(y) \leq C|y - y_1| \leq C(|y - x| + |x - y_1|) \leq C(r/4 + r) = \frac{5}{4}Cr$$

where C depends only on λ, ν, M, U and R. Then by arguing exactly as in the proof of Theorem 2.1, we can prove that ∇u is uniformly bounded in $B_{r/8}(x)$ by a constant depending only on λ, ν, n, p, M, U, $|a_{ij}|_{0,1,B^+(x_0,2R)}$, $|b_i|_{0,1,B^+(x_0,2R)}$, $|c_i|_{0,B^+(x_0,2R)}$, $|d|_{0,B^+(x_0,2R)}$ and R.

β) $\exists y_2 \in B(x, r/4) \cap Q_0$

So we have $x \in B^+(y_2, r/4)$. Note that $B^+(y_2, r/2) \subset B^+(x_0, 3R)$. Indeed for $y \in B^+(y_2, r/2)$, one has

$$|y - x_0| \leq |y - y_2| + |y_2 - x| + |x - x_0| \leq r/2 + r/4 + R < 2R/2 + 2R/4 + R < 3R.$$

Moreover by Lemma 3.1, we have

$$u(y) \leq C|y - y_2| \leq \frac{C}{2}r \quad \forall y \in B^+(y_2, r/2) \tag{3.17}$$

where C depends only on λ, ν, n, M, U and R.

We consider the function u_r defined by

$$u_r(z) = \frac{u(y_2 + rz)}{r}, \quad z \in B^+(O, 1/2).$$

By (3.17), we see that u_r is uniformly bounded in $B^+(O, 1/2)$, i.e. $u_r(z) \leq C/2 \ \forall z \in B^+(O, 1/2)$. Moreover, u_r satisfies

$$\begin{cases} \text{div}(a_r(z)\nabla u_r + u_r B_r(z)) - (rC_r(z)\nabla u_r + r^2 d_r u_r) = rc_r + rf_r + \text{div}(H_r) \\ \hfill \text{in } B^+(O, 1/2) \\ u_r \in C^{1,\alpha}(B(O,1/2)^+ \cup (B(O,1/2) \cap Q_0)) \text{ (see [3, p. 212])}. \end{cases}$$

Applying Corollary 8.36 of [3, p. 212] and taking into account the Remark after it, we get

$$|u_r|_{1,\alpha,\overline{B^+(O,1/4)}} \leq C \left(|u_r|_{0,B^+(O,1/2)} + |rc_r|_{p,B^+(O,1/2)} + |rf_r|_{p,B^+(O,1/2)} + |\operatorname{div}(H_r)|_{p,B^+(O,1/2)}\right),$$

where C depends only on dist $\left(\overline{B(O,1/4)}, \partial B(O,1/2)\right)$, $|a_{ij}|_{0,1,B^+(X_0,3R)}$, $|b_i|_{0,1,B^+(X_0,3R)}$, $|c_i|_{0,B^+(X_0,3R)}$, $|d|_{0,B^+(X_0,3R)}$, λ, ν, M, U, n, p, and R. In particular, $|\nabla u_r|_{0,\overline{B^+(O,1/4)}}$ is uniformly bounded. Hence $|\nabla u(x)| \leq C$ since $x \in B^+(y_2, r/4)$.

Finally because we have $\nabla u(x) = 0$ a.e. in $B^+(x_0, R) \cap [u = 0]$, it follows that ∇u is uniformly bounded in $B^+(x_0, R)$. □

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