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DISTORTION AND CONVOLUTIONAL THEOREMS FOR OPERATORS OF GENERALIZED FRACTIONAL CALCULUS INVOLVING WRIGHT FUNCTION

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Abstract. Using the Wright's generalized hypergeometric function, we investigate a class $W(q,s;A,B,\lambda)$ of analytic functions with negative coefficients. We derive many results for the modified Hadamard product of functions belonging to the class $W(q,s;A,B,\lambda)$. Moreover, we generalize some of the distortion theorems to the classical fractional integrals and derivatives and the Saigo (hypergeometric) operators of fractional calculus.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the unit disc $U = \{z\colon z \in C \text{ and } |z| < 1\}$.

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If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written symbolically $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ $(|w(z)| \leq |z|$ in $U)$, such that $f(z) = g(w(z))$ $(z \in U)$.

For analytic functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

by $(f * g)(z)$ we denote the Hadamard product (or convolution) of $f(z)$ and $g(z)$, defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1, B_1, \ldots, \beta_s, B_s$ $(q, s \in N = \{1, 2, \ldots\})$ be positive real parameters such that

$$1 + \sum_{k=1}^{s} B_k - \sum_{k=1}^{q} A_k \geq 0.$$

The Wright generalized hypergeometric function [23] (see also [6], [12] and [19])

$$\begin{aligned} &{}_q\Psi_s[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s); z] \\ &= {}_q\Psi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] \end{aligned}$$

is defined by

$$\begin{aligned} &{}_q\Psi_s[(\alpha_k, A_k)_{1,q}; (\beta_k, B_k)_{1,s}; z] \\ &= \sum_{k=0}^{\infty} \left\{ \prod_{n=1}^{q} \Gamma(\alpha_n + kA_n) \right\} \left\{ \prod_{n=1}^{s} \Gamma(\beta_n + kB_n) \right\}^{-1} \frac{z^k}{k!} (z \in U). \end{aligned} \quad (2)$$

If $A_n = 1$ $(n = 1, \ldots, q)$ and $B_n = 1$ $(n = 1, \ldots, s)$, we have the relationship:

$$\Omega\, {}_q\Psi_s[(\alpha_n, 1)_{1,q}; (\beta_n, 1)_{1,s}; z] = {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z), \quad (3)$$

where ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is the generalized hypergeometric function (see for details the books on special functions, as [12], [19]) and

$$\Omega = \left(\prod_{n=1}^{q} \Gamma(\alpha_n) \right)^{-1} \left(\prod_{n=1}^{s} \Gamma(\beta_n) \right). \quad (4)$$

The Wright generalized hypergeometric functions (2) have been recently involved in the geometric function theory, see [1], [2], [3], [14], [15] and [16], as well as: [7], [8], [9] and [13]. It is a special case of Fox's function (see for example [6], [12] and [19]).

Using the Wright generalized hypergeometric functions Dziok and Raina [2] defined a function ${}_q\phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z]$ by

$$ {}_q\phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] = \Omega z {}_q\Psi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] $$

and introduced the following linear operator

$$ \theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}] \colon \mathcal{A} \to \mathcal{A}, $$

defined by the convolution

$$ \theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}]f(z) = {}_q\phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] * f(z). $$

We observe that for a function $f(z)$ of the form (1) we have

$$ \theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}]f(z) = z + \sum_{k=2}^{\infty} \Omega \sigma_k a_k z^k, \tag{5} $$

where Ω is given by (4) and σ_k is defined by

$$ \sigma_k = \frac{\Gamma(\alpha_1 + A_1(k-1)) \cdots \Gamma(\alpha_q + A_q(k-1)}{\Gamma(\beta_1 + B_1(k-1)) \cdots \Gamma(\beta_s + B_s(k-1))(k-1)!}. \tag{6} $$

If, for convenience, we write

$$ \theta[\alpha_1]f(z) = \theta[(\alpha_1, A_1), \dots, (\alpha_q, A_q); (\beta_1, B_1), \dots, (\beta_s, B_s)]f(z), $$

then one can easily verify from the definition (5) that

$$ zA_1(\theta[\alpha_1]f(z))' = \alpha_1 \theta[\alpha_1 + 1]f(z) - (\alpha_1 - A_1)\theta[\alpha_1]f(z). \tag{7} $$

Using the linear operator $\theta[\alpha_1]$ Aouf and Dziok [1] defined the class $W(q, s; A, B, \lambda)$ of functions of the form

$$ f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \geq 0) \tag{8} $$

which also satisfy the following condition:

$$ \frac{1}{(1-\lambda)} \left(\alpha_1 \frac{\theta[\alpha_1 + 1]f(z)}{\theta[\alpha_1]f(z)} + A_1(1-\lambda) - \alpha_1 \right) \prec A_1 \frac{1 + Az}{1 + Bz} $$
$$ (0 \leq B \leq 1; -B \leq A < B; 0 \leq \lambda < 1). $$

In particular, for $q = s+1$ and $\alpha_{s+1} = A_{s+1} = 1$, we write $W(s; A, B, \lambda) = W(s+1, s; A, B, \lambda)$. The class $W(q, s; A, B, 0) = W(q, s; A, B)$ was studied by Dziok and Raina [2] (see also [3]).

If $A_n = 1$ $(n = 1, \dots, q)$ and $B_n = 1$ $(n = 1, \dots, s)$, then we note that $W(q, s; A, B, 0) = V_2^1(q, s; A, B)$. This class was studied by Dziok and Srivastava [4] (see also [5]). Putting moreover $\alpha_1 = n + 1$, $\alpha_2 = 1$ and $\beta_1 = 1$, we have the class $T_n(\lambda, \rho) = W(2, 1; -\rho, \rho, \lambda)$, which was studied by Patel and Acharya [11].

For the class $W(q, s; A, B, \lambda)$ we have following result.

Lemma 1 ([1]). *A function $f(z)$ of the form (8) belongs to the class $W(q,s;A,B,\lambda)$ if and only if*

$$\sum_{k=2}^{\infty} \Omega\delta_k a_k \le (B-A)(p-\lambda), \tag{9}$$

where

$$\delta_k = [(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k, \tag{10}$$

and Ω, σ_k are defined by (4) and (6), respectively.

2. Modified Hadamard product

For the functions

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \ge 0;\ j=1,2), \tag{11}$$

we denote by $(f_1 \otimes f_2)(z)$ *the modified Hadamard product* or *convolution* of the functions f_1 and f_2 defined by

$$(f_1 \otimes f_2)(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k.$$

Theorem 1. *Let the functions $f_j(z)$ $(j=1,2)$ defined by (11) be in the class $W(q,s;A,B,\lambda)$. If the sequence $\{\delta_k\}$ is nondecreasing, then $(f_1 \otimes f_2)(z) \in W(q,s;A,B,\gamma)$, where*

$$\gamma = 1 - \frac{(1+B)(B-A)(1-\lambda)^2}{\Omega[(1+B)+(B-A)(1-\lambda)]^2\sigma_2 - (B-A)^2(1-\lambda)^2}. \tag{12}$$

The result is sharp.

Proof. We need to find the largest γ such that

$$\sum_{k=2}^{\infty} \frac{\Omega[(1+B)(k-1)+(B-A)(1-\gamma)]\sigma_k}{(B-A)(1-\gamma)} a_{k,1}a_{k,2} \le 1. \tag{13}$$

By Lemma 1 and the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty} \frac{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k}{(B-A)(1-\lambda)} \sqrt{a_{k,1}a_{k,2}} \le 1. \tag{14}$$

Thus by (13) it is sufficient to show that

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{[(1+B)(k-1)+(B-A)(1-\lambda)](1-\gamma)}{[(1+B)(k-1)+(B-A)(1-\gamma)](1-\lambda)} \quad (k \ge 2).$$

By (14) we have

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(B-A)(1-\lambda)}{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k} \quad (k \geq 2).$$

Consequently, we need only to prove that

$$\gamma \leq 1 - \frac{(k-1)(1+B)(B-A)(1-\lambda)^2}{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]^2\sigma_k - (B-A)^2(1-\lambda)^2} \quad (k \geq 2).$$

Since

$$\Phi(k) = 1 - \frac{(k-1)(1+B)(B-A)(1-\lambda)^2}{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]^2\sigma_k - (B-A)^2(1-\lambda)^2} \tag{15}$$

is an increasing function of k $(k \geq 2)$, letting $k = 2$ in (15), we obtain

$$\gamma \leq \Phi(2) = 1 - \frac{(1+B)(B-A)(1-\lambda)^2}{\Omega[(1+B)+(B-A)(1-\lambda)]^2\sigma_2 - (B-A)^2(1-\lambda)^2},$$

which proves the main assertion of Theorem 1. Finally, by taking the functions $f_j(z)(j = 1, 2)$ given by

$$f_j(z) = z - \frac{(B-A)(1-\lambda)}{\Omega[(1+B)+(B-A)(1-\lambda)]\sigma_2} z^2 \quad (j = 1, 2), \tag{16}$$

we can see that the result is sharp. □

Theorem 2. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by* (11) *be in the class $W(q, s; A, B, \lambda)$. If the sequence $\{\delta_k\}$ is nondecreasing, then the function*

$$h(z) = z - \sum_{k=2}^{\infty} (a_{k,1}^2 + a_{k,2}^2)\, z^k \tag{17}$$

belongs to the class $W(q, s; A, B, \tau)$, where

$$\tau = 1 - \frac{2(1+B)(B-A)(1-\lambda)^2}{\Omega[(1+B)+(B-A)(1-\lambda)]^2\sigma_2 - 2(B-A)^2(1-\lambda)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ defined by (16).

Proof. By Lemma 1, we obtain

$$\sum_{k=2}^{\infty} \left\{ \frac{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k}{(B-A)(1-\lambda)} \right\}^2 a_{k,1}^2$$

$$\le \left\{ \sum_{k=2}^{\infty} \frac{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k}{(B-A)(1-\lambda)} a_{k,1} \right\}^2 \le 1 \tag{18}$$

and

$$\sum_{k=2}^{\infty} \left\{ \frac{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k}{(B-A)(1-\lambda)} \right\}^2 a_{k,2}^2$$
$$\le \left\{ \sum_{k=2}^{\infty} \frac{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k}{(B-A)(1-\lambda)} a_{k,2} \right\}^2 \le 1. \tag{19}$$

It follows from (18) and (19) that

$$\sum_{k=2}^{\infty} \frac{1}{2} \left\{ \frac{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k}{(B-A)(1-\lambda)} \right\}^2 (a_{k,1}^2 + a_{k,2}^2) \le 1.$$

Therefore, we need to find the largest τ such that

$$\frac{\Omega[(1+B)(k-1)+(B-A)(1-\tau)]\sigma_k}{(B-A)(1-\tau)}$$
$$\le \frac{1}{2} \left\{ \frac{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]\sigma_k}{(B-A)(1-\lambda)} \right\}^2 \quad (k \ge 2),$$

that is,

$$\tau \le 1 - \frac{2(k-1)(1+B)(B-A)(1-\lambda)^2}{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]^2\sigma_k - 2(B-A)^2(1-\lambda)^2} \quad (k \ge 2).$$

Since

$$D(k) = 1 - \frac{2(k-1)(1+B)(B-A)(1-\lambda)^2}{\Omega[(1+B)(k-1)+(B-A)(1-\lambda)]^2\sigma_k - 2(B-A)^2(1-\lambda)^2},$$

is an increasing function of $k (k \ge 2)$, we readily have

$$\tau \le D(2) = 1 - \frac{2(1+B)(B-A)(1-\lambda)^2}{\Omega[(1+B)+(B-A)(1-\lambda)]^2\sigma_2 - 2(B-A)^2(1-\lambda)^2},$$

and Theorem 2 follows at once. □

Putting $\lambda = 0$ in Theorems 1 and 2, we obtain the following two corollaries.

Corollary 1. *Let the functions $f_j(z)$ $(j = 1,2)$ defined by* (11) *be in the class $W(q,s;A,B)$. If the sequence $\{\delta_k\}$ is nondecreasing, then $(f_1 \otimes f_2)(z) \in W(q,s;A,B,\gamma)$, where*

$$\gamma = 1 - \frac{(1+B)(B-A)}{\Omega(1+2B-A)^2\sigma_2 - (B-A)^2}.$$

The result is sharp.

Corollary 2. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (11) be in the class $W(q, s; A, B)$. If the sequence $\{\delta_k\}$ is nondecreasing, then the function $h(z)$ defined by (17) belongs to the class $W(q, s; A, B, \tau)$, where*

$$\tau = 1 - \frac{2(1 + B)(B - A)}{\Omega(1 + 2B - A)^2 \sigma_2 - 2(B - A)^2}.$$

The result is sharp.

Taking $q = 2$, $s = 1$, $A_1 = A_2 = B_1 = 1$, $\alpha_1 = n + 1 \in N$, $\alpha_2 = \beta_1 = 1$, $B = -A = \rho$ $(0 < \rho \leq 1)$ in Theorems 1 and 2, respectively, we obtain the following consequences:

Corollary 3. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (11) be in the class $T_n(\lambda, \rho)$. Then $(f_1 \otimes f_2)(z) \in T_n(\gamma, \rho)$, where*

$$\gamma = 1 - \frac{2\rho(1 + \rho)(1 - \lambda)^2}{[1 + \rho(3 - 2\lambda)]^2(n + 1) - 4\rho^2(1 - \lambda)^2}.$$

The result is sharp.

Corollary 4. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (11) be in the class $T_n(\lambda, \rho)$. Then the function $h(z)$ defined by (17) belongs to the class $T_n(\tau, \rho)$, where*

$$\tau = 1 - \frac{4\rho(1 + \rho)(1 - \lambda)^2}{[1 + \rho(3 - 2\lambda)]^2(n + 1) - 8\rho^2(1 - \lambda)^2}.$$

The result is sharp.

3. Definitions and applications of fractional calculus

We start with some definitions of fractional calculus operators (that is fractional derivatives and fractional integrals), as adopted for use in classes of analytic functions. First we recall the definition of Saigo operator [17], [18] (see also [21]).

For real numbers γ, ζ and η, *the fractional derivative-integral operator* $I_{0,z}^{\gamma,\zeta,\eta}$ is defined by

$$I_{0,z}^{\gamma,\zeta,\eta} f(z) = \frac{z^{-\gamma-\zeta}}{\Gamma(\gamma)} \int_0^z (z - t)^{\gamma-1} \, {}_2F_1\left(\gamma + \zeta, -\eta; \gamma; 1 - \frac{t}{z}\right) f(t)dt \quad \text{for } \gamma > 0$$

$$I_{0,z}^{\gamma,\zeta,\eta} f(z) = \frac{d^n}{dz^n} I_{0,z}^{\gamma+n,\zeta-\eta,\eta-n} f(z) \quad \text{for } 0 < \gamma + n \leq 1, \; n = 1, 2, \ldots,$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}) \quad (z \to 0),$$

where $\varepsilon > \max(0, \zeta - \eta) - 1$ and the multiplicity of $(z-t)^{\gamma-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$. The function ${}_2F_1(a, b; c; z)$ is the Gaussian hypergeometric function (3) defined, in terms of the Pochhammer symbol

$$(\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}), \end{cases}$$

by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in U).$$

In particular we obtain the fractional derivative-integral operator

$$D_z^{\gamma} f(z) = I_{0,z}^{\gamma,\gamma,\eta} f(z) \tag{20}$$

introduced by Owa [10] (see also Srivastava and Owa [20]).

In order to prove our result for the operator $I_{0,z}^{\gamma,\zeta,\eta}$, we recall here the following lemma.

Lemma 2 ([22])**.** *If* $n > \max\{0, \zeta - \eta\} - 1$, *then*

$$I_{0,z}^{\gamma,\zeta,\eta} z^n = \frac{\Gamma(n+1)\Gamma(n-\zeta+\eta+1)}{\Gamma(n-\zeta+1)\Gamma(n+\gamma+\eta+1)} z^{n-\zeta}.$$

We can prove easy the following theorem.

Theorem 3. *Let* $l \in N \cup \{0\}$, $-\gamma \le \zeta \le l$, $\zeta < 2 + \eta$, *and let the sequence* $\{\delta_k/(k)_l\}$ *be nondecreasing. If the function* $f(z)$ *is in the class* $W(q, s; A, B, \lambda)$ *and* $0 < |z| < 1$, *then*

$$\left| I_{0,z}^{\gamma,\zeta,\eta} f(z) \right| \ge \frac{\Gamma(2-\zeta+\eta)|z|^{1-\zeta}}{\Gamma(2-\zeta)\Gamma(2+\gamma+\eta)} \left\{ 1 - \frac{2(B-A)(1-\lambda)(2-\zeta+\eta)}{\Omega\delta_2(2-\zeta)(2+\gamma+\eta)} |z| \right\},$$

$$\left| I_{0,z}^{\gamma,\zeta,\eta} f(z) \right| \le \frac{\Gamma(2-\zeta+\eta)|z|^{1-\zeta}}{\Gamma(2-\zeta)\Gamma(2+\gamma+\eta)} \left\{ 1 + \frac{2(B-A)(1-\lambda)(2-\zeta+\eta)}{\Omega\delta_2(2-\zeta)(2+\gamma+\eta)} |z| \right\}.$$

The result is sharp for extremal function f *of the form*

$$f(z) = z - \frac{(B-A)(1-\lambda)}{\Omega\delta_2} z^2. \tag{21}$$

Putting $\zeta = \gamma$ in Theorem 3 and using (20) we obtain the following corollary.

Corollary 5. *Let $l \in N \cup \{0\}$, $\gamma \geq -l$ and let the sequence $\{\delta_k/(k)_l\}$ be nondecreasing. If the function $f(z)$ is in the class $W(q,s;A,B,\lambda)$ and $0<|z|<1$, then*

$$|D_z^\gamma f(z)| \geq \frac{|z|^{1-\gamma}}{\Gamma(2-\gamma)}\left\{1-\frac{2(B-A)(1-\lambda)}{\Omega\delta_2(2-\gamma)}|z|\right\},$$

$$|D_z^\gamma f(z)| \leq \frac{|z|^{1-\gamma}}{\Gamma(2-\gamma)}\left\{1+\frac{2(B-A)(1-\lambda)}{\Omega\delta_2(2-\gamma)}|z|\right\}.$$

The result is sharp for extremal function f of the form (21).

Corollary 6. *Let $l \in N \cup \{0\}$, $\gamma \geq -l$ and let the sequence $\{\delta_k/(k)_l\}$ be nondecreasing. If the function $f(z)$ is in the class $W(q,s;A,B,\lambda)$, then $D_z^\gamma f(z)$ is included in a disc with its center at the origin and radius r given by*

$$r = \frac{1}{\Gamma(2-\gamma)}\left\{1+\frac{2(B-A)(1-\lambda)}{\Omega\delta_2(2-\gamma)}\right\}.$$

Remark 1. Taking $\lambda = 0$ in Theorem 3 and Corollaries 5 and 6, respectively, we obtain the results for the class $W(q,s;A,B,0) = W(q,s;A,B)$ studied by Dziok and Raina [2]. Corollaries 1 and 2 are improvements of the results obtained by Patel and Acharya [11].

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