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STABILITY OF IMPULSIVE HYBRID SET-VALUED DIFFERENTIAL EQUATIONS WITH DELAY BY PERTURBING LYAPUNOV FUNCTIONS

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Abstract. We study the stability of the zero solution of an impulsive set differential system with delay by means of the perturbing Lyapunov function method. Sufficient conditions for the stability of the zero solution of impulsive set differential equations with delay are presented.

1. Introduction

The study of set differential equations has been initiated as an independent subject and several results of interest can be found in [2], [3], [9]–[11], [13]. The interesting feature of the set differential equations is that the results obtained in this new framework become the corresponding results of ordinary differential equations as the Hukuhara derivative and the integral used in formulating the set differential equations reduce to the ordinary vector derivative and integral when the set under consideration is a single

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valued mapping. Moreover, in the present setup, we have only semilinear complete metric space to work with, instead of complete normed linear space required in the study of the ordinary differential systems. Furthermore, set differential equations, that are generated by multivalued differential inclusions when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions [19]. Set differential equations can also be utilized to investigate fuzzy differential equations [10]. For recent development on the subject, see [7] and references therein.

In recent years, a number of research papers has dealt with dynamical systems with impulse effect as a class of general hybrid systems. Examples include the adequate mathematical models for numerous processes and phenomena studied in biology, applied physics, etc. Impulsive dynamical systems are characterized by the occurrence of abrupt change in the state of the system which occur at certain time instants over a period of negligible duration. The presence of impulse means that the state trajectory does not preserve the basic properties which are associated with non impulsive dynamical systems. Thus, the theory of impulsive differential equations is quite interesting and has attracted the attention of many scientists, see for instance, [1], [6], [16] and the references therein. Moreover, in certain situations, the future state of the physical problems depends not only on the present state but also on its past history. Thus, introduction of the delay in the governing equations ensures a better modelling of the processes involved [4], [15].

The stability criteria in sense of Lyapunov function is found to be quite elegant to develop the qualitative properties of the null solution of the systems of differential equations. Lakshmikantham and Leela [8] introduced the perturbing Lyapunov function method under weaker conditions to study nonuniform properties of solutions of systems of differential equations. For more details on perturbing Lyapunov function method, see [5], [12], [17], [18].

In this paper, we apply the perturbing Lyapunov function method to investigate stability of the zero solution of an impulsive set differential system with delay at fixed moments.

2. Terminology and preliminaries

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. We define the Hausdorff metric as

$$D[X,Y] = \max\left[\sup_{y\in Y} d(y,X),\ \sup_{x\in X} d(x,Y)\right],$$

where $d(y, X) = \inf[d(y, x) \colon x \in X]$ and X, Y are bounded subsets of $\mathbb{R}^n$. Notice that $K_c(\mathbb{R}^n)$ with the metric is a complete metric space. Moreover, $K_c(\mathbb{R}^n)$ equipped with the natural algebraic operations of addition and nonnegative scalar multiplication becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [14], [19]. Further, the Hausdorff metric satisfies the following properties: $\forall\ X, Y, Z \in K_c(\mathbb{R}^n)$ and $\mu \in \mathbb{R}_+$, we have

$$\begin{gathered} D[X+Z, Y+Z] = D[X, Y] \text{ and } D[X, Y] = D[Y, X], \\ D[\mu X, \mu Y] = \mu D[X, Y], \\ D[X, Y] \leq D[X, Z] + D[Z, Y]. \end{gathered}$$

Definition 2.1. The set $Z \in K_c(\mathbb{R}^n)$ satisfying $X = Y + Z$ is known as the Hukuhara difference of the sets X and Y in $K_c(\mathbb{R}^n)$ and is denoted by $X - Y$.

Definition 2.2. For any interval I in $\mathbb{R}$, the mapping $F \colon I \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if there exists an element $D_H F(t_0) \in K_c(\mathbb{R}^n)$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h},$$

exist in the topology of $K_c(\mathbb{R}^n)$ and each one is equal to $D_H F(t_0)$.

Given any $\tau > 0$, we define $\mathrm{C} = C\,[[-\tau, 0], K_c(\mathbb{R}^n)]$. Let $U \in [J_0, K_c(\mathbb{R}^n)]$ for any $t \in J_0 = [t_0 - \tau, t_0 + a]$, $a > 0$ and let U_t denote a translation of the restriction of U to the interval $[t - \tau, t]$, that is, $U_t \in \mathrm{C}$ be defined by $U_t(s) = U(t + s)$, $-\tau \leq s \leq 0$. For any $\Theta, \Psi \in \mathrm{C}$, we define a metric $D_0[\Theta, \Psi] = \max_{-\tau \leq s \leq 0} D\,[\Theta(s), \Psi(s)]$.

Consider the impulsive set differential equation with delay

$$\begin{cases} D_H U(t) = F(t, U_t), & t \neq t_k, \\ U_{t_k^+} = I_k(U_{t_k}), & t = t_k, \\ U_{t_0} = \Theta_0 \in \mathrm{C}, \end{cases} \tag{2.1}$$

where $F \in PC\,[\mathbb{R}_+ \times \mathrm{C}, K_c(\mathbb{R}^n)]$ is piecewise continuous and in particular $F \colon (t_{k-1}, t_k] \times \mathrm{C} \to K_c(\mathbb{R}^n)$ is continuous, $I_k \colon \mathrm{C} \to \mathrm{C}$ is continuous for each k and $\{t_k\}$ is a sequence of points such that $0 \leq t_0 < t_1 < \ldots t_k < \ldots$ with $\lim_{k \to \infty} t_k = \infty$.

By a solution of (2.1), we mean a piecewise continuous function $U(t) = U(t_0, \Theta_0)(t)$ on $[t_0, \infty)$ which is left continuous on $(t_k, t_{k+1}]$ and is defined by

$$U(t_0,\Theta_0)(t) = \begin{cases} \Theta_0, & t_0-\tau \le t \le t_0, \\ U_0(t_0,\Theta_0)(t), & t_0 \le t \le t_1, \\ U_1(t_1,\Theta_1)(t), & t_1 < t \le t_2, \\ \vdots & \vdots \\ U_k(t_k,\Theta_k)(t), & t_k < t \le t_{k+1}, \\ \vdots & \vdots \end{cases}$$

where $U_k(t_k,\Theta_k)(t)$ is a solution of the set differential equation with delay

$$D_H U(t) = F(t,U_t), \quad U_{t_k^+} = \Theta_k,\ k = 0,1,2,\dots .$$

Definition 2.3. Let $V\colon \mathbb{R}_+ \times K_c(\mathbb{R}^n) \times \mathrm{C} \to \mathbb{R}_+$. Then V is said to belong to class V_0 if

(a$_1$) $V(t,U,\Theta)$ is continuous in $(t_{k-1},t_k] \times K_c(\mathbb{R}^n) \times \mathrm{C}$ and for each $U \in K_c(\mathbb{R}^n)$, $\Theta \in \mathrm{C}$, $k = 1,2,\dots$, $\lim_{(t,Y,\Theta)\to(t_k^+,U,\Theta)} V(t,Y,\Theta) = V(t_k^+,U,\Theta)$ exists;

(a$_2$) $V(t,U,\Theta)$ is Lipschitzian in U.

For $(t,U,\Theta) \in (t_{k-1},t_k] \times K_c(\mathbb{R}^n) \times \mathrm{C}$, let us define

$$D^+V(t,U,\Theta) = \lim_{h\to 0^+} \frac{1}{h}\left[V(t+h,U+hF(t,U_t),U_{t+h}) - V(t,U,U_t)\right].$$

Definition 2.4. The zero solution $U(t) \equiv \phi$ of (2.1) is said to be

(b$_1$) stable if for $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0,\varepsilon) > 0$ such that $D_0[\Theta_0,\phi] < \delta$ implies that $D[U(t),\phi] < \varepsilon$, $t \ge t_0$;

(b$_2$) uniformly stable if δ in (b$_1$) is independent of t_0;

(b$_3$) asymptotically stable if (b$_1$) holds and given $\varepsilon > 0$, $t_0 \in \mathbb{R}_+$, there exist $\delta_0 = \delta_0(t_0) > 0$ and $T(t_0,\varepsilon) > 0$ such that $D_0[\Theta_0,\phi] < \delta_0$ implies that $D[U(t),\phi] < \varepsilon$, $t \ge t_0 + T$;

(b$_4$) uniformly asymptotically stable if (b$_2$) holds and δ_0, T in (b$_3$) are independent of t_0.

Remark. For the stability criteria of the null solution of (2.1), one can employ the measure $\|U(t)\| = \operatorname{diam}[U(t)]$, $t \ge t_0$. But the $\operatorname{diam}[U(t)]$ is nondecreasing in t once the Hukuhara differences are assumed to exist. This problem can be overcome by utilizing the existence of Hukuhara difference in the initial conditions also, which in fact makes it possible to match the behavior of the solution of set differential equations with the corresponding solutions of ordinary differential equations . In order to do so, we suppose that the Hukuhara difference exists for any given initial values $\Phi_0, \Psi_0 \in \mathrm{C}$

so that we set $\Phi_0 - \Psi_0 = \Theta_0$ and consider the stability of the solution $U(t, t_0, \Phi_0 - \Psi_0) = U(t, t_0, \Theta_0)$ of (2.1).

We now define the following spaces:

$$\begin{aligned}\mathcal{K} &= [\nu \colon \nu \in C[\mathbb{R}_+, \mathbb{R}_+] \text{ is strictly increasing and } \nu(0) = 0]\,,\\ S(\rho) &= [U \in K_c(\mathbb{R}^n) \colon D[U(t), \phi] < \rho]\,,\\ S_1(\rho) &= [\Theta \in \mathrm{C} \colon D[\Theta, \phi] < \rho]\,.\end{aligned}$$

The following comparison result [15] is needed to investigate the stability of the impulsive set-valued differential equations with delay (2.1) by means of perturbing Lyapunov functions.

Theorem 2.1. *Suppose that*

(i) $V \colon \mathbb{R}_+ \times K_c(\mathbb{R}^n) \times \mathrm{C} \to \mathbb{R}_+$ *and* $V \in V_0$*;*

(ii) $D^+V(t, U, \Theta) \le g\,(t, V(t, U, \Theta))$, $t \neq t_k$ *where* $g \in PC[\mathbb{R}^2_+, \mathbb{R}_+]$*;*

(iii) $V\left(t_k^+, U(t_0, \Theta_0)(t_k^+), U_{t_k^+}(t_0, \Theta_0)\right)$
$\le J_k\,(t_k, V\,(t_k, U(t_0, \Theta_0)(t_k), U_{t_k}(t_0, \Theta_0)))$,
where $J_k(t, w)$ *is nondecreasing in* w*.*

Then $V\,(t, U(t_0, \Theta_0)(t), U_t(t_0, \Theta_0)) \le r(t)$, $t \ge t_0$, *where* $U(t_0, \Theta_0)(t)$ *is any solution of the impulsive set differential equation with delay* (2.1) *and* $r(t)$ *is the maximal solution of the scalar impulsive differential equation*

$$\begin{cases} w' = g(t, w), & t \neq t_k,\\ w(t_k^+) = J_k\,(t_k, w(t_k))\,, & t = t_k,\ k = 1, 2, \dots,\\ w(t_0) = w_0 \ge 0. \end{cases}$$

3. Stability by the method of perturbing Lyapunov functions

In this section, we discuss the stability and asymptotic stability of the zero solution of impulsive set-valued differential equations with delay (2.1) by means of perturbing Lyapunov functions.

Theorem 3.1. *Assume that*

(A$_1$) $V_1 \in PC(\mathbb{R}_+ \times S(\rho) \times S_1(\rho), \mathbb{R}_+)$ *be such that* $V_1(t, U, \Theta) \in V_0$, $V_1(t, 0, \Theta) = 0$ *and there exists* $\rho_0 > 0$ *such that* $U_{t_k} \in S_1(\rho_0)$ *implies* $I_k(U_{t_k}) \in S_1(\rho)$ *for all* k *and*

$$\begin{cases} D^+V_1(t, U, \Theta) \le g_1\,(t, V_1(t, U, \Theta))\,, & t \neq t_k,\\ V_1\left(t_k^+, U(t_0, \Theta_0)(t_k^+), U_{t_k^+}(t_0, \Theta_0)\right) & \\ \le J_k\,(t_k, V_1\,(t_k, U(t_0, \Theta_0)(t_k), U_{t_k}(t_0, \Theta_0)))\,, & k = 1, 2, \dots\,, \end{cases} \tag{3.1}$$

where $g_1 \in PC(\mathbb{R}^2_+, \mathbb{R}_+)$ *is such that* $g_1(t,0) = 0$ *and* $J_k \in C(\mathbb{R}^2_+, \mathbb{R}_+)$ *is continuous.*

(A$_2$) *For* $\zeta > 0$*, there exists* $V_{2,\zeta} \in PC\left(\mathbb{R}_+ \times (S(\rho) \cap S^c(\zeta)) \times S_1(\rho), \mathbb{R}_+\right)$ *such that* $V_{2,\zeta}(t,0,\Theta) = 0$ *and* $V_{2,\zeta}(t,U,\Theta) \in V_0$*, where* $S^c(\zeta)$ *is complement of* $S(\zeta)$*. Further*

$$b_1\left(D[U(t),\phi]\right) \le V_{2,\zeta}(t,U,\Theta) \le a_1\left(D[U(t),\phi]\right), \quad a_1, b_1 \in \mathcal{K}, \tag{3.2}$$

and

$$\begin{cases} D^+\left(V_1(t,U,\Theta) + V_{2,\zeta}(t,U,\Theta)\right) & \\ \le g_2\left(t, V_1(t,U,\Theta) + V_{2,\zeta}(t,U,\Theta)\right), & t \ne t_k \\ V_1\left(t_k^+, U(t_0,\Theta_0)(t_k^+), U_{t_k^+}(t_0,\Theta_0)\right) & \\ +V_{2,\zeta}\left(t_k^+, U(t_0,\Theta_0)(t_k^+), U_{t_k^+}(t_0,\Theta_0)\right) & \\ \le F_k\left(t_k, V_1\left(t_k, U(t_0,\Theta_0)(t_k), U_{t_k}(t_0,\Theta_0)\right)\right. & \\ \left.+\, V_{2,\zeta}\left(t_k, U(t_0,\Theta_0)(t_k), U_{t_k}(t_0,\Theta_0)\right)\right), & k = 1,2,\dots, \end{cases} \tag{3.3}$$

where $g_2 \in PC(\mathbb{R}^2_+, \mathbb{R}_+)$ *with* $g_2(t,0) = 0$ *and* $F_k \in C(\mathbb{R}^2_+, \mathbb{R}_+)$ *is continuous.*

(A$_3$) *The zero solution of the problem*

$$\begin{cases} w' = g_1(t,w), & t \ne t_k, \\ w(t_k^+) = J_k\left(t_k, w(t_k)\right), & t = t_k,\ k = 1,2,\dots, \\ w(t_0) = w_0 \ge 0, & \end{cases} \tag{3.4}$$

is stable and the zero solution of

$$\begin{cases} v' = g_2(t,v), & t \ne t_k, \\ v(t_k^+) = F_k\left(t_k, v(t_k)\right), & t = t_k,\ k = 1,2,\dots, \\ v(t_0) = v_0 \ge 0, & \end{cases} \tag{3.5}$$

is uniformly stable.

Then the zero solution of (2.1) *is stable.*

Proof. Let $0 < \varepsilon < \min(\rho, \rho_0)$, $t_0 \in \mathbb{R}_+$ and $b_1(\varepsilon) > 0$. Since the zero solution of (3.5) is uniformly stable, there is a $\delta' = \delta'(\varepsilon) > 0$ such that $v_0 < \delta'$ implies that $v(t;t_0,v_0) < b_1(\varepsilon)$, $t \ge t_0$, where $v(t;t_0,v_0)$ is any solution of (3.5). Now, we choose $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$a_1(\delta_2) < \frac{\delta'}{2}. \tag{3.6}$$

Now, in view of the fact that the zero solution of (3.4) is stable; for $\delta'/2 > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_3 = \delta_3(t_0,\varepsilon) > 0$ such that

$$w_0 < \delta_3 \quad \text{implies that} \quad w(t;t_0,w_0) < \frac{\delta'}{2}, \quad t \ge t_0, \tag{3.7}$$

where $w(t; t_0, w_0)$ is any solution of (3.4). Fix $w_0 = V_1(t_0, \Theta_0, \Theta)$ and choose some $\delta_1 > 0$ such that $D_0[\Theta_0, \phi] < \delta_1$ and $V_1(t_0, \Theta_0, \Theta) < \delta_3$.

Let us set $\delta = \min(\delta_1, \delta_2)$ so that $D_0[\Theta_0, \phi] < \delta$ implies that $D[U(t_0, \Theta_0)(t), \phi] < \varepsilon$, $t \geq t_0$, where $U(t_0, \Theta_0)(t)$ is any solution of (2.1). If this is not true, then there would exist a solution $U(t_0, \Theta_0)(t)$ of (2.1) with $D_0[\Theta_0, \phi] < \delta$ and t_1, t_2 satisfying $t_k < t_1 < t_2 \leq t_{k+1}$ for some k such that

$$\begin{cases} D[U(t_0, \Theta_0)(t), \phi] < \varepsilon, & t_k \geq t \geq t_0, \quad \text{and} \\ D[U(t_0, \Theta_0)(t_2), \phi] \geq \varepsilon, & D[U(t_0, \Theta_0)(t_1), \phi] = \delta_2, \end{cases} \tag{3.8}$$

and $U(t_0, \Theta_0)(t) \in \overline{S(\varepsilon)}$ on $[t_1, t_2]$. Now, we take $\delta_2 = \zeta$ by requiring that $V_{2,\zeta}$ satisfies (A$_2$). For $t \in [t_1, t_2]$, we set

$$\begin{aligned} m(t) =& V_1\left(t, U(t_0, \Theta_0)(t), U_t(t_0, \Theta_0)\right) \\ &+ V_{2,\zeta}\left(t, U(t_0, \Theta_0)(t), U_t(t_0, \Theta_0)\right), \end{aligned} \tag{3.9}$$

satisfying

$$m(t_1) \leq r_2(t_1; t_0, v_0), \tag{3.10}$$

where $r_2(t; t_0, v_0)$ is the maximal solution of (3.5). In view of (3.3) together with (3.9) and (3.10), it follows that

$$m(t) \leq r_2(t; t_0, v_0), t \in [t_1, t_2].$$

Also, by Theorem 2.1, we have

$$V_1\left(t_1, U(t_0, \Theta_0)(t_1), U_{t_1}(t_0, \Theta_0)\right) \leq r_1(t_1; t_0, w_0),$$

which, in view of (3.7), yields

$$V_1\left(t_1, U(t_0, \Theta_0)(t_1), U_{t_1}(t_0, \Theta_0)\right) \leq \frac{\delta'}{2},$$

where $r_1(t; t_0, w_0)$ is the maximal solution of (3.4). Using (A$_2$), (3.6) and (3.8), we get

$$V_{2,\zeta}\left(t_1, U(t_0, \Theta_0)(t_1), U_{t_1}(t_0, \Theta_0)\right) \leq a_1\left(D[U(t_0, \Theta_0)(t_1), \phi]\right) = a_1(\delta_2) < \frac{\delta'}{2}.$$

Hence, by virtue of (A$_2$) and the fact that $V_1 \in V_0$, we have

$$\begin{aligned} b_1(\varepsilon) &\leq b_1\left(D[U(t_0, \Theta_0)(t_2), \phi]\right) \\ &\leq V_{2,\zeta}\left(t_2, U(t_0, \Theta_0)(t_2), U_{t_2}(t_0, \Theta_0)\right) \\ &\leq r_2(t_2; t_0, v_0) < b_1(\varepsilon), \end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 3.2. *Assume that*

(B$_1$) *The assumptions* (A_2)–(A_3) *of Theorem* 3.1 *hold with the exception that the zero solution of* (3.4) *is stable;*

(B$_2$) *The zero solution of* (3.4) *is uniformly stable;*

(B$_3$) $V_1(t,U,\Theta)\in V_0$, $V_1(t,0,\Theta)=0$ *and*

$$\begin{cases} D^+V_1(t,U,\Theta)+P(t,U,\Theta)\le g_1\left(t,V_1(t,U,\Theta)\right), \quad t\ne t_k,\\ V_1\left(t_k^+,U(t_0,\Theta_0)(t_k^+),U_{t_k^+}(t_0,\Theta_0)\right)+\displaystyle\int_{t_0}^{t_k}P(s,U,\Theta_0)ds\\ \le J_k\left(t_k,V_1\left(t_k,U(t_0,\Theta_0)(t_k),U_{t_k}(t_0,\Theta_0)\right)\right), \quad k=1,2,\dots, \end{cases} \tag{3.11}$$

where $g_1\colon\mathbb{R}_+^2\to\mathbb{R}_+$ *is continuous and* $g_1(t,w)$ *is nondecreasing in* w *with* $g_1(t,0)=0$ *and* $J_k\colon\mathbb{R}_+^2\to\mathbb{R}_+$ *is continuous and* $J_k(t,w)$ *is nondecreasing in* w, $P\colon\mathbb{R}_+\times S(\rho)\times S_1(\rho)\to\mathbb{R}_+$ *is continuous, integrable and locally Lipschitzian in* U *and* $P(t,U,\Theta)\ge b_2(U)$, $b_2\in\mathcal{K}$.

Then the zero solution of (2.1) *is asymptotically stable.*

Proof. In view of the assumptions (B$_1$)–(B$_2$), we let $0<\varepsilon=\sigma$ such that $D_0[\Theta_0,\phi]<\delta(t_0,\sigma)$ implies that $D[U(t_0,\Theta_0)(t),\phi]<\sigma$, $t\ge t_0$, where $U(t_0,\Theta_0)(t)$ is any solution of (2.1). Select $T=T(\varepsilon)>(\delta'/2)b_2\left(\delta(\varepsilon)\right)$ such that $t_0+T\ne t_k$, $k=1,2,\dots$. We claim that there is a $t^*\in[t_0,t_0+T]$ such that $P(t^*,U,\Theta_0)<b_2\left(\delta(\varepsilon)\right)$ for any solution of (2.1) provided $D_0[\Theta_0,\phi]<\delta(t_0,\sigma)$. If it is not true, then $\forall\ t\in[t_0,t_0+T]$, $P(t,U,\Theta)\ge b_2\left(\delta(\varepsilon)\right)$.

For $t\in[t_0,t_1]$, by Theorem 2.1, we have

$$\int_{t_0}^{t}P(s,U,\Theta)ds+V_1(t,U,\Theta)\le r_1\left(t;t_0,V_1(t_0,\Theta_0,\Theta)\right)=r_1(t;t_0,w_0),$$

where $r_1(t;t_0,w_0)$ is the maximal solution of (3.4). Now, for $t\in(t_{k-1},t_k]$,

$$\begin{aligned} &V_1\left(t_k^+,U(t_0,\Theta_0)(t_k^+),U_{t_k^+}(t_0,\Theta_0)\right)+\int_{t_0}^{t_k}P(s,U,\Theta_0)ds\\ &\le J_k\left(t_k,V_1\left(t_k,U(t_0,\Theta_0)(t_k),U_{t_k}(t_0,\Theta_0)\right)\right)\\ &\le J_k\left(t_k,r_1(t_k;t_0,w_0)\right)\le J_k\left(t_k,r_1^{k-1}(t_k;t_{k-1},w_{k-1}^+)\right)=w_k^+, \end{aligned}$$

where

$$r_1(t;t_0,w_0)=\begin{cases} r_1^0(t;t_0,w_0), & t_0\le t\le t_1,\\ r_1^1(t;t_0,w_1^+), & t_1<t\le t_2,\\ \vdots & \vdots\\ r_1^{k-1}(t;t_0,w_{k-1}^+), & t_{k-1}<t\le t_k,\\ \vdots & \vdots. \end{cases}$$

Suppose that

$$V_1\left(t_1^+, U(t_0,\Theta_0)(t_1^+), U_{t_1^+}(t_0,\Theta_0)\right) + \int_{t_0}^{t_1} P(s,U,\Theta_0)ds \le w_1^+,\ t_1 < t \le t_2.$$

In view of (B$_3$) and Theorem 2.1, we have

$$V_1(t,U,\Theta) + \int_{t_0}^{t} P(s,U,\Theta)ds \le r_1\left(t; t_0, V_1(t_0,\Theta_0,\Theta)\right), \quad t_1 < t \le t_2.$$

Hence, by induction, we get

$$V_1(t,U,\Theta) + \int_{t_0}^{t} P(s,U,\Theta)ds \le r_1\left(t; t_0, V_1(t_0,\Theta_0,\Theta)\right), \quad t \ge t_0.$$

Thus it follows that

$$\begin{aligned} 0 \le V_1(t_0+T,U,\Theta) &\le r_1(t_0+T;t_0,w_0) - \int_{t_0}^{t_0+T} P(s,U,\Theta)ds \\ &\le r_1(t_0+T;t_0,w_0) - \int_{t_0}^{t_0+T} b_2\left(\delta(\varepsilon)\right)ds \\ &< \frac{\delta'}{2} - b_2\left(\delta(\varepsilon)\right)T < 0, \end{aligned}$$

which leads to a contradiction.

This implies that there is a $t^* \in [t_0, t_0+T]$ such that $P(t^*,U,\Theta_0) < b_2\left(\delta(\varepsilon)\right)$ for any solution of (2.1) provided $D_0[\Theta_0,\phi] < \delta(t_0,\sigma)$. Also, $b_2\left(\delta(\varepsilon)\right) > P(t^*,U,\Theta_0) \ge b_2\left(D[U(t_0,\Theta_0)(t^*),\phi]\right)$, then $D[U(t_0,\Theta_0)(t^*),\phi] < \delta(\varepsilon)$ for some $t^* \in [t_0,t_0+T]$. We assert that $D[U(t_0,\Theta_0)(t),\phi] < \varepsilon$, $t \ge t_0+T$. Suppose that our assertion is not true. This turns out to be a contradiction employing the procedure used in Theorem 3.1. Hence the proof is complete. □

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References

[1] Bainov, D. D., Simeonov, P. S., *Systems with Impulse Effect*, Ellis Horwood, Chichester, 1989.

[2] Gnana Bhaskar, T., Lakshmikantham, V., *Set differential equations and flow invariance*, Appl. Anal. **82** (2003), 357–368.

[3] Gnana Bhaskar, T., Lakshmikantham, V., *Lyapunov stability for set differential equations*, Dynam. Systems Appl. **13** (2004), 1–10.

[4] Hale, J., *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[5] Koksal, S., *Stability properties and perturbing Lyapunov functions*, J. Appl. Anal. **43** (1992), 99–107.

[6] Lakshmikantham, V., Bainov, D. D., Simeonov, P. S., *Theory of Impulsive Differential Equations*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[7] Lakshmikantham, V., Gnana Bhaskar, T., Devi, J. V., *Theory of Set Differential Equations*, Cambridge Scientific Publishers, Cambridge, 2006.

[8] Lakshmikantham, V., Leela, S., *On perturbing Lyapunov functions*, Math. Systems Theory **10** (1976), 85–90.

[9] Lakshmikantham, V., Leela, S., Vatsala, A. S., *Set-valued hybrid differential equations and stability in terms of two measures*, Internat. J. Hybrid Systems **2** (2002), 169–187.

[10] Lakshmikantham, V., Leela, S., Vatsala, A. S., *Interconnection between set and fuzzy differential equations*, Nonlinear Anal. **54** (2003), 351–360.

[11] Lakshmikantham, V., Leela, S., Vatsala, A. S., *Stability theory for set differential equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **11** (2004), 181–189.

[12] Lakshmikantham, V., Matrosov, V. M., Sivasundaram, S., *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Kluwer Academic Publishers, Dordrecht, 1991.

[13] Lakshmikantham, V., Tolstonogov, A., *Existence and interrelation between set and fuzzy differential equations*, Nonlinear Anal. **55** (2003), 255-268.

[14] Lopes Pinto, A. J. Brandão, De Blasi, F. S., Iervolino, F., *Uniqueness and existence theorems for differential equations with compact convex valued solutions*, Boll. Un. Mat. Ital. (4) **3** (1970), 47–54.

[15] McRae, F. A., Devi, J. V., *Impulsive set differential equations with delay*, Appl. Anal. **84** (2005), 329–341.

[16] Samoilenko, A. M., Perestyuk, N. A., *Impulsive Differential Equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[17] Soliman, A. A., *Lipschitz stabilty with perturbing Lyapunov functionals*, Appl. Math. Lett. **17** (2004), 939–944.

[18] Stutson, D., Vatsala A. S., *Composite boundedness and stability results by perturbing Lyapunov functions*, Nonlinear Anal. **26** (1996), 761–766.

[19] Tolstonogov, A., *Differential Inclusions in a Banach Space*, Kluwer Academic Publishers, Dordrecht, 2000.

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