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EXISTENCE OF PARETO EQUILIBRIA FOR NON-COMPACT CONSTRAINED MULTI-CRITERIA GAMES

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Abstract. In this paper, an existence result of quasi-equilibrium problem is proved and used to establish the existence of weighted Nash equilibria for constrained multi-criteria games under a generalized quasi-convexity condition and a coercivity type condition on the payoff functions. As consequence, we prove the existence of Pareto equilibria for constrained multi-criteria games with non-compact strategy sets in topological vector spaces.

1. Introduction

A number of classical problems in game theory are formulated as games with multiple non-comparable criteria (or vector payoffs) (see Prasad and Ghose [5]). The aim of this work is to study the existence of Pareto equilibria in such games.

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In Section 2, we firstly prove a result on the existence of quasi-equilibrium problem in topological vector spaces with generalized coercivity and convexity conditions. Our argument is based on a fixed point theorem recently obtained by Ben-El-Mechaiekh, Chebbi and Florenzano in [1]. Secondly, this result is used to prove the existence of weighted Nash equilibria in constrained multi-criteria games when strategy sets players are convex and not necessarily compacts.

In Section 3 and as consequence, we prove the existence of Pareto equilibria for constrained multi-criteria games under a generalized coercivity type condition on the vector payoffs functions.

Let $\mathbb{N} = \{1, 2, \dots, n\}$ and

$$x = (x_1, x_2, \dots, x_n) \in X = \prod_{i \in \mathbb{N}} X^i.$$

For each $i \in \mathbb{N}$, denote

$$X^{-i} = \prod_{j \in \mathbb{N} \setminus \{i\}} X^j.$$

An element of X^{-i} is $x^{-i} = (x^1, x^2, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$. We shall use (x^i, y^{-i}) to denote $z = (z^1, z^2, \dots, z^n)$ such that $z^i = x^i$ and $z^{-i} = y^{-i}$.

First, let us introduce the game model considered in this paper. If $\mathbb{N} = \{1, 2, \dots, n\}$ is a set of players, a collection $(X^i, A^i, F^i)_{i \in \mathbb{N}}$ is called a *constrained multi-criteria game* if each player i has a non-empty strategy set X^i, a constrained correspondence $A^i \colon X^{-i} \to X^i$ and a vector payoff function (or multi-criteria) $F^i \colon X \to \mathbb{R}^{k_i}$ defined for each $x = (x^1, x^2, \dots, x^n)$ by

$$F^i(x) = \left(f_1^i(x), f_2^i(x), \dots, f_{k_i}^i(x)\right),$$

where f_j^i, for $i \in \mathbb{N}$ and $j = 1, 2, \dots, k_i$, k_i is a positive integer, represent the non-commensurable outcomes. If a strategy $x = (x^1, x^2, \dots, x^n)$ is played, each player i is trying to minimize her/his own payoff function F^i. Note that for the games with payoff functions, there does not exist a strategy $x \in X$ minimizing all f_j^i (see [8]).

To give the concept of equilibrium used in this paper, we need the following notations:

We denote by $\mathbb{R}_+^m$ the non-negative orthant of $\mathbb{R}^m$,

$$\mathbb{R}_+^m = \left\{u = (u^1, u^2, \dots, u^m) \in \mathbb{R}^m \colon u^j \geq 0, \ \forall j = 1, 2, \dots, m\right\}$$

and by $\operatorname{int} \mathbb{R}_+^m$, its non-empty interior,

$$\operatorname{int} \mathbb{R}_+^m = \{u = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m \colon \ u_j > 0, \ \forall j = 1, 2, \dots, m\}.$$

We denote by S_+^m the simplex of $\mathbb{R}_+^m$,

$$S_+^m = \left\{ u = (u_1, u_2, \dots, u_m) \in \mathbb{R}_+^m \colon \ \sum_{j=1}^m u_j = 1 \right\},$$

and $\operatorname{int} S_+^m$, its relative interior,

$$\operatorname{int} S_+^m = \left\{ u = (u_1, u_2, \dots, u_m) \in \operatorname{int} \mathbb{R}_+^m \colon \ \sum_{j=1}^m u_j = 1 \right\}.$$

Definition 1. A strategy $\hat{x}^i \in X^i$ of player i is said to be a *Pareto efficient strategy* (resp. *a weak Pareto efficient strategy*) with respect to $\hat{x} = (\hat{x}^1, \hat{x}^2, \dots, \hat{x}^n) \in X$ if $\hat{x}^i \in A^i(\hat{x}^{-i})$ and there is no strategy $x^i \in A^i(\hat{x}^{-i})$ such that

$$F^i(\hat{x}) - F^i(x^i, \hat{x}^{-i}) \in \mathbb{R}_+^{k_i} \setminus \{0\} \quad (\text{resp. } F^i(\hat{x}) - F^i(x^i, \hat{x}^{-i}) \in \operatorname{int} \mathbb{R}_+^{k_i}).$$

Definition 2. Let $(X^i, A^i, F^i)_{i \in \mathbb{N}}$ be a constrained multi-criteria game. A strategy $\hat{x} \in X$ is said to be a *Pareto equilibrium* (resp. *a weak Pareto equilibrium*) of the game if for each player i, $\hat{x}^i$ is a Pareto efficient strategy (resp. a weak Pareto efficient strategy) with respect to $\hat{x}$.

It is clear that each Pareto equilibrium is a weak Pareto equilibrium. The following definition of weighted Nash equilibrium as given by Wang in [6] will be also used:

Definition 3. A strategy $\hat{x} \in X$ is called a *weighted Nash equilibrium* with respect to the weight vector $W = (W^1, W^2, \dots, W^n)$ of a constrained multi-criteria game $(X^i, A^i, F^i)_{i \in \mathbb{N}}$ if for each player i, we have:

(a) $\hat{x}^i \in A^i(\hat{x}^{-i})$.
(b) $W^i \in \mathbb{R}_+^{k_i} \setminus \{0\}$.
(c) For all $x^i \in A^i(\hat{x}^{-i})$, $\langle W^i, F^i(\hat{x}) \rangle \leq \langle W^i, F^i(x^i, \hat{x}^{-i}) \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product.

In particular, if $W^i \in S_+^{k_i}$ for all $i \in \mathbb{N}$, then the strategy $\hat{x}$ is said to be a *normalized weighted Nash equilibrium* with respect to W.

Throughout the paper, vector spaces are real and topological spaces are assumed to be Hausdorff. The convex hull of a subset A of a vector space is denoted by $\operatorname{co} A$. A subset B of a topological space E is called *compactly closed* (*open* respectively) if for every compact set K of X, $B \cap K$ is closed (open, respectively) in K. Set-valued maps will be simply called *correspondences* and represented by capital letters F, G, $\dots$. Functions in the usual sense will be represented by small letters.

2. Existence of weighted Nash equilibria

From Definition 3, it is easy to see that a strategy $\hat{x} \in X$ is a weighted Nash equilibrium with respect to W of the game $(X^i, A^i, F^i)_{i\in\mathbb{N}}$ if and only if $\hat{x} \in X$ is a solution of the following constrained optimization problem:

$$\langle W^i, F^i(\hat{x})\rangle = \min_{x^i \in A^i(x^{-i})} \langle W^i, F^i(x^i, \hat{x}^{-i})\rangle$$
$$\hat{x}^i \in A^i(\hat{x}^{-i}).$$

Since this problem is a *quasi-equilibrium problem*, we firstly prove the following result:

Lemma 1. *Let X be a non-empty convex subset of a topological vector space E, $A\colon X \to X$ be a correspondence with non-empty convex values and $f\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function such that:*

(1) *For all $y \in X$, $A^{-1}(y)$ is compactly open in X.*
(2) *The set $D = \{x \in X : x \in A(x)\}$ is compactly closed in X.*
(3) *For each fixed $x \in X$, the function: $y \to f(x, y)$ is upper semi-continuous on each non-empty compact subset of X and the function: $x \to f(x, x)$ is lower semi-continuous on each non-empty compact subset of X.*
(4) *For all finite subset A of X,*

$$\sup_{x\in \operatorname{co} A} \min_{y\in A} \{f(x, x) - f(y, x)\} \leq 0.$$

(5) *There exists a family $\{(C_\alpha, K_\alpha)\}_{\alpha\in I}$ satisfying:*
   (a) *For each $\alpha \in I$, C_α is contained in a compact convex subset of X and K_α is a compact subset of X.*
   (b) *For each $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $C_\alpha \bigcup C_\beta \subseteq C_\gamma$.*
   (c) *For each $\alpha \in I$, there exists $\beta \in I$ such that for all $x \in X \setminus K_\alpha \cup D$, $A(x) \cap C_\beta \neq \emptyset$ and for each $x \in D \setminus K_\alpha$, $\{y \in A(x) \colon f(y, x) < f(x, x)\} \cap C_\beta \neq \emptyset$.*

Then there exists $\hat{x} \in X$ such that

$$\hat{x} \in A(\hat{x}),$$
$$f(\hat{x}, \hat{x}) \leq f(y, \hat{x}),\ \forall y \in A(\hat{x}).$$

Proof. Let us consider the correspondence $F\colon X \to X$ defined by

$$F(x) = \{y \in X \colon f(x, x) - f(y, x) > 0\}.$$

By condition (3), for all $y \in X$, $F^{-1}(y)$ is compactly open in X. Now assume that for each $x \in D$, $A(x) \cap F(x) \neq \emptyset$ and define the correspondence

$G\colon X \to X$ by

$$G(x) = \begin{cases} A(x) \cap F(x) & \text{if } x \in D \\ A(x) & \text{if } x \notin D. \end{cases}$$

G has non-empty values and we can see that

$$G^{-1}(y) = \left[A^{-1}(y) \cap F^{-1}(y)\right] \cup \left[(X \setminus D) \cap A^{-1}(y)\right]$$

for any given $y \in X$. Hence, by conditions (1) and (2), $G^{-1}(y)$ is compactly open in X. Condition (5) implies that G satisfies all hypothesis of Theorem 3.2 in [1], then there exists $\hat{x} \in \operatorname{co} G(\hat{x})$. By definition of G and A, $\hat{x}$ must be in D. It follows that $\hat{x} \in \operatorname{co} F(\hat{x})$, wich contradicts condition (4). Therefore, there exists $\hat{x} \in D$ such that $A(\hat{x}) \cap F(\hat{x}) = \emptyset$, that is

$$\hat{x} \in A(\hat{x}),$$
$$f(\hat{x}, \hat{x}) \leq f(y, \hat{x}),\ \forall y \in A(\hat{x}).$$

□

Remark 1.

(a) If a function f satisfies condition (4) of Lemma 1, then following Zhou and Chen in [10] the function g defined by $g(x, y) = f(x, x) - f(y, x)$ is said to be 0-*diagonally quasi-convex* in the second argument. Note that if g is quasi-convex in the second argument, then g is 0-diagonally quasi-convex in the second argument.

(b) Condition (5) is a generalized coercivity type condition firstly introduced in [1] where examples of functions satisfying such condition are given. If X is compact, then condition (5) is automatically satisfied.

(c) It follows by (a) and (b) that Lemma 1 extends Theorem 4.2 in [4].

Lemma 1 is now used to prove the following equilibrium result:

Theorem 1. *Let $(X^i, A^i, F^i)_{i \in \mathbb{N}}$ be a constrained multi-criteria game with for each $i \in \mathbb{N}$, X^i is a non-empty convex subset of a topological vector space E^i and $A^i\colon X^{-i} \to X^i$ has non-empty convex values. Suppose that there exists a vector*

$$W = (W^1, W^2, \ldots, W^n) \in \prod_{i \in \mathbb{N}} \mathbb{R}^{k_i} \setminus \{0\}$$

satisfying the following conditions for each $i \in I$:

(1) *For each $y^i \in X^i$, $(A^i)^{-1}(y^i)$ is compactly open in X^{-i} and the set $D = \{x \in X\colon x \in A(x)\}$, where $A(x) = \prod_{i \in \mathbb{N}} A^i(x^{-i})$, is compactly closed in X.*

(2) *The mapping*

$$x \mapsto \sum_{i=1}^{n} \left\langle W^i, F^i(x) \right\rangle$$

is lower semi-continuous on each non-empty compact subset of X.

(3) *For each fixed* $x \in X$, *the mapping*

$$y \mapsto \sum_{i=1}^{n} \left\langle W^i, F^i(x^i, y^{-i}) \right\rangle$$

is upper semi-continuous on each non-empty compact subset of X.

(4) *For all finite subset* A *of* X, *for each* $x \in \operatorname{co} A$:

$$\min_{y \in A} \left\{ \sum_{i=1}^{n} \left\langle W^i, F^i(x) - F^i(y^i, x^{-i}) \right\rangle \right\} \leq 0.$$

(5) *There exists a family* $\{(C_\alpha, K_\alpha)\}_{\alpha \in I}$ *satisfying* (*a*) *and* (*b*) *of condition* (5) *of Lemma* 1 *and the following one: For each* $\beta \in I$, *there exists* $\alpha \in I$ *such that for all* $x \in X \setminus K_\beta \cup D$, $A(x) \cap C_\alpha \neq \emptyset$ *and for each* $x \in D \setminus K_\beta$,

$$\left\{ y \in A(x) \colon \sum_{i=1}^{n} \left\langle W^i, F^i(x) - F^i(y^i, x^{-i})) \right\rangle > 0 \right\} \cap C_\alpha \neq \emptyset.$$

Then the game $(X^i, A^i, F^i)_{i \in \mathbb{N}}$ *has a weighted Nash equilibrium with respect to* W.

Proof. For each $(x, y) \in X \times X$, consider the map

$$f(x, y) = \sum_{i=1}^{n} \left\langle W^i, F^i(x^i, y^{-i}) \right\rangle.$$

It is easy to see that f satisfies all hypothesis of Lemma 1, hence there exists $\hat{x} \in X$ such that $\hat{x} \in A(\hat{x})$ and

$$\sum_{i=1}^{n} \left\langle W^i, F^i(\hat{x}) - F^i(y^i, \hat{x}^{-i}) \right\rangle \leq 0.$$

If we take $y = (y^i, \hat{x}^{-i})$, for $i = 1, 2, \ldots, n$, then we can verify that $\hat{x}$ is a weighted Nash equilibrium with respect to W of the game $(X^i, A^i, F^i)_{i \in \mathbb{N}}$. □

Remark 2. This result extends Theorem 1 of Yuan and Tarafdar in [9] since our conditions (4) and (5) are more general than their conditions (3) and (4) respectively. It also extends Theorem 1 in [2] and Theorem 3.1 in [6] obtained when X is a compact subset of a normed space.

3. Existence of Pareto equilibria

Since a Pareto equilibrium of a multi-criteria game is not necessarily a weighted Nash equilibrium of the game, we will use exactly the same argument as used in the proof of Lemma 2.1 in [6] to obtain:

Lemma 2. *A normalized weighted Nash equilibrium of a game $(X^i, A^i, F^i)_{i\in\mathbb{N}}$ with respect to the weight*

$$W \in \prod_{i\in\mathbb{N}} S_+^{k_i} \quad (\textit{resp. } W \in \prod_{i\in\mathbb{N}} \operatorname{int} S_+^{k_i})$$

is a weak Pareto equilibrium (resp. a Pareto equilibrium) of the game. This result remain valid if the weight

$$W \in \prod_{i\in\mathbb{N}} \mathbb{R}_+^{k_i} \quad (\textit{resp. } W \in \prod_{i\in\mathbb{N}} \operatorname{int} \mathbb{R}_+^{k_i}).$$

By combining Theorem 1 and Lemma 2, we obtain:

Theorem 2. *Let $(X^i, A^i, F^i)_{i\in\mathbb{N}}$ be a constrained multi-criteria game with for each $i \in \mathbb{N}$, X^i is a non-empty convex subset of a topological vector space E^i, $A^i\colon X^{-i} \to X^i$ has non-empty convex values and $F^i = \big(f_1^i, f_2^i, \dots, f_{k_i}^i\big)$. Suppose that for all $i \in \mathbb{N}$, for all $j = 1, 2, \dots, k_i$, the following conditions are satisfied:*

(1) *For each $y^i \in X^i$, $(A^i)^{-1}(y^i)$ is compactly open in X^{-i} and the set $D = \{x \in X\colon x \in A(x)\}$, where $A(x) = \prod_{i\in\mathbb{N}} A^i(x^{-i})$, is compactly closed in X.*

(2) *The function f_j^i is lower semi-continuous on each non-empty compact subset of X.*

(3) *For each fixed $x^i \in X^i$, the function $y^{-i} \to f_j^i(x^i, y^{-i})$ is upper semi-continuous on each non-empty compact subset of X^{-i}.*

(3) *For each fixed $x^{-i} \in X^{-i}$, the function $y^i \to f_j^i(y^i, x^{-i})$ is quasi-convex on X^i.*

(4) *There exists a family $\{(C_\alpha, K_\alpha)\}_{\alpha\in I}$ satisfying (a) and (b) of condition (5) of Lemma 1 and the following one: For each $\beta \in I$, there exists $\alpha \in I$ such that for all $x \in X \setminus K_\beta \cup D$, $A(x) \cap C_\alpha \neq \emptyset$ and for each $x \in D \setminus K_\beta$, $\{y \in A(x)\colon f_j^i(x) - f_j^i(x^{-i}, y^i) > 0\} \cap C_\alpha \neq \emptyset$.*

Then the multi-criteria game $(X^i, A^i, F^i)_{i\in\mathbb{N}}$ has at least one Pareto equilibrium.

Proof. Let $W \in \prod_{i\in\mathbb{N}} \operatorname{int} S^{k_i}$ be a fixed weight vector. From conditions (1)–(4), it follows by Theorem 1 that the game $(X^i, A^i, F^i)_{i\in\mathbb{N}}$ has at least one weighted Nash equilibrium $\hat{x}$ with respect to the weight vector W. Since for each $i \in \mathbb{N}$, $W^i \in \operatorname{int} S_+^{k_i}$, by Lemma 2, $\hat{x}$ is also a Pareto equilibrium of the game. □

Remark 3. Theorem 2 generalizes Theorem 5 of [3], Theorem 3 of [9] and Theorem 6 of [7] obtained in the non-compact case. If for each $i \in I$, X^i is a compact subset and $A(x^{-i}) = X^i$, then Theorem 2 is reduced to Theorem 3.2 of [6].

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