

WHEN AN ATOMIC AND COMPLETE ALGEBRA OF SETS IS A FIELD OF SETS WITH NOWHERE DENSE BOUNDARY

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Abstract. We consider pairs $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ where \mathcal{A} is an algebra of sets from some class called the class of algebras of type $\langle \kappa, \lambda \rangle$ and where $\mathcal{H}(\mathcal{A})$ is the ideal of hereditary sets of \mathcal{A} . We characterize which of the above pairs are topological, that is, which are fields of sets with nowhere dense boundary for some topology together with the ideal of nowhere dense sets for this topology. Making use of a theorem of Fichtenholz and Kantorovich which says that in $\mathcal{P}(\kappa)$ there is an independent family of cardinality 2^κ , we construct an example of a pair $\langle \text{algebra, ideal} \rangle$ with complete quotient algebra and the hull property but not topological. This counterexample, given in ZFC, provides the complete solution of a problem posed in [1]. Such an algebra was constructed in [5] under some additional set theoretic assumption.

1. INTRODUCTION

For an ideal \mathcal{I} and an algebra \mathcal{A} of subsets of a set Y such that $\mathcal{I} \subset \mathcal{A}$, we say that:

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- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has the hull property if and only if for every $U \subset Y$ there is a $V \in \mathcal{A}$ such that $U \subset V$ and for every $W \in \mathcal{A}$ if $U \subset W$ then $V \setminus W \in \mathcal{I}$;
- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ is complete if and only if the quotient algebra \mathcal{A}/\mathcal{I} is complete;
- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ is topological if and only if there exists a topology τ on Y such that \mathcal{I} is the ideal of nowhere dense sets and \mathcal{A} is the algebra of all sets with nowhere dense boundary in the topology τ .

The case when one considers only \mathcal{I} and asks if it is the ideal of nowhere dense sets with respect to some topology was investigated in [8], where it is proved that it is always possible to find such a topology.

Recently, another related property of pairs $\langle \mathcal{A}, \mathcal{I} \rangle$ as above has been considered in several papers ([1], [2], [3], [7], [9]). Following the idea of Burstin and Marczewski, for any nonempty family \mathcal{F} of nonempty subsets of Y , one can define the collections of sets:

$$S(\mathcal{F}) = \{A \subset Y : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset A \cap P \text{ or } Q \subset P \setminus A)\}$$

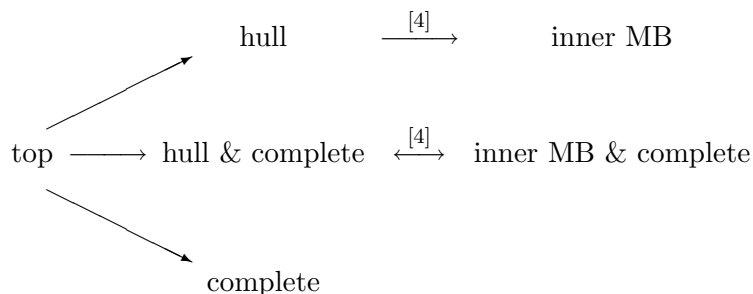
and

$$S_0(\mathcal{F}) = \{A \subset Y : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset P \setminus A)\}$$

Then $S(\mathcal{F})$ is an algebra and $S_0(\mathcal{F})$ is an ideal of subsets of Y (see [12, 2]). Adopting the notation as above, we say that

- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has an inner MB-representation provided there exists an $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{A} = S(\mathcal{F})$ and $\mathcal{I} = S_0(\mathcal{F})$.
- an algebra \mathcal{A} is MB-representable if and only if $\langle \mathcal{A}, \mathcal{I} \rangle$ has an inner MB-representation for some ideal \mathcal{I} of \mathcal{A} .

Using the results of [4], an example given in [1], and elementary properties of topological pairs we obtain the following implications between the introduced properties:



The hull property and the completeness are independent, i.e. neither of them implies the other ([4]). The inner MB-representability does not imply the hull property ([1]). The implications between “top” and the remaining properties are elementary. It is proved in [5] that the existence of a pair $\langle \mathcal{A}, \mathcal{I} \rangle$ which is complete and satisfies the hull property but is not

topological, is consistent with ZFC. This is a partial solution of a problem posed in [1]. More precisely, we have shown the:

Theorem 1. *The following two conditions are equivalent:*

- (I) *there exists a set Y and an algebra $\mathcal{A} \subset \mathcal{P}(Y)$ isomorphic to $\mathcal{P}(\omega)$, with $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ complete, inner MB-representable but not topological;*
- (II) *there exists an ideal $\mathcal{J} \subset \mathcal{P}(\omega)$ such that $\mathcal{P}(\omega)/\mathcal{J}$ is isomorphic to $\mathcal{P}(\omega_1)$.*

Here $\mathcal{H}(\mathcal{A})$ denotes here the ideal of hereditary sets of \mathcal{A} , i.e., sets whose power set is included in \mathcal{A} .

To show the consistency of (II), we have used an almost disjoint family of sets in $\mathcal{P}(\omega)$, called strong- Q -sequence. Steprāns in [13] has proved that the existence of a strong- Q -sequence with cardinality ω_1 is consistent. In January 2007 professor Balcar informed the first author that, by a corollary from the Balcar-Franek theorem on independent families of sets in complete Boolean algebras, we can observe that the condition (II) is equivalent to the equality $2^\omega = 2^{\omega_1}$ (and hence it holds f.e by Martin Axiom). In fact, it is enough to use in this case the older and weaker version of Balcar-Franek theorem: the Fichtenholz-Kantorovich theorem. In the present paper, using the general version of Fichtenholz-Kantorovich theorem we give the complete answer to the title question, and to the question originally posed in [1], i.e., we show in ZFC that there exists an algebra of sets \mathcal{A} , such that $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is complete, has the hull property but is not topological.

2. USEFUL FACTS

For the convenience of the reader we recall some basic facts and notions (presented in almost the same form in [5]).

Definition 1. Let $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{P}(Y)$. We say that $\mathcal{F}_1, \mathcal{F}_2$ are *coinitial* if \mathcal{F}_1 is dense in \mathcal{F}_2 and vice versa, i.e. any set of any one of the families $\mathcal{F}_1, \mathcal{F}_2$ has a subset belonging to the other one.

Fact 1 ([2]). *If $\mathcal{F}_1, \mathcal{F}_2$ are coinitial then $\langle S(\mathcal{F}_1), S_0(\mathcal{F}_1) \rangle = \langle S(\mathcal{F}_2), S_0(\mathcal{F}_2) \rangle$. Conversely, if $\langle S(\mathcal{F}_1), S_0(\mathcal{F}_1) \rangle = \langle S(\mathcal{F}_2), S_0(\mathcal{F}_2) \rangle$ and $\mathcal{F}_i \subset S(\mathcal{F}_i)$ for $i = 1, 2$ then $\mathcal{F}_1, \mathcal{F}_2$ are coinitial.*

Fact 2 ([2]). *$\langle \mathcal{A}, \mathcal{I} \rangle$ is topological with respect to a topology τ on Y if and only if $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\tau_0), S_0(\tau_0) \rangle$ where $\tau_0 = \tau \setminus \{\emptyset\}$.*

Fact 3 ([1]). $\langle \mathcal{A}, \mathcal{I} \rangle$ is inner MB-representable if and only if $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{A} \setminus \mathcal{I}), S_0(\mathcal{A} \setminus \mathcal{I}) \rangle$. Moreover, if $\langle \mathcal{A}, \mathcal{I} \rangle$ is inner MB-representable then $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ has the same property.

Now, let us recall some basic algebraic considerations from [6] concerning the structure of an isomorphism between $\mathcal{P}(X)$ (for some set X) and a field \mathcal{A} of subsets of Y . Let $\Phi: \mathcal{P}(X) \rightarrow \mathcal{A} \subset \mathcal{P}(Y)$ be such an isomorphism. Denote by Z the union of all atoms of \mathcal{A} , i.e.

$$Z = \bigcup_{x \in X} \Phi(\{x\}).$$

For any $A \in \mathcal{P}(X)$, denote

$$\Phi_1(A) = \Phi(A) \cap Z \text{ and } \Phi_2(A) = \Phi(A) \setminus Z.$$

Then Φ_1 is an isomorphism between $\mathcal{P}(X)$ and some field of subsets of Z , and Φ_2 is an epimorphism from $\mathcal{P}(X)$ to some field \mathcal{B} of subsets of $Y \setminus Z$. (We say that $\Phi_2(A)$ is the remainder of $\Phi(A)$ and we call \mathcal{B} the remainder algebra). We can describe Φ_1 by the formula $\Phi_1(A) = \bigcup_{x \in A} \Phi(\{x\})$ for $A \in \mathcal{P}(X)$. Denote by \mathcal{J} , the kernel of the homomorphism Φ_2 . Then

$$\mathcal{J} = \{A \in \mathcal{P}(X) : \Phi(A) = \bigcup_{x \in A} \Phi(\{x\})\}.$$

Note that \mathcal{J} contains all finite subsets of X . Moreover, $\mathcal{J} = \mathcal{P}(X)$ if and only if $Z = Y$, or (what is equivalent) if Φ_2 is the zero-homomorphism. By standard algebraic considerations we have that the field \mathcal{B} consisting of the remainders $\Phi_2(A)$ for $A \in \mathcal{P}(X)$ is isomorphic to the quotient algebra $\mathcal{P}(X)/\mathcal{J}$. Observe, as in [6], that for the algebra $\mathcal{A} = \Phi[\mathcal{P}(X)]$, we have

$$\mathcal{H}(\mathcal{A}) = \{\Phi(A) : A \in \mathcal{J}, \forall x \in A \ |\Phi(\{x\})| = 1\}.$$

The symbols $Z, \Phi, \Phi_1, \Phi_2, \mathcal{J}, \mathcal{B}$ will retain their meaning throughout the paper. Note that given sets $z \subset y$ and a monomorphism $\phi_1: \mathcal{P}(X) \rightarrow \mathcal{P}(z)$ such that the images of singletons cover z and a homomorphism $\phi_2: \mathcal{P}(X) \rightarrow \mathcal{P}(y \setminus z)$ such that the kernel of ϕ_2 contains all finite subsets of X one can define an isomorphism onto its image $\Phi: \mathcal{P}(X) \rightarrow \mathcal{P}(y)$ by putting

$$\Phi(A) = \phi_1(A) \cup \phi_2(A)$$

for $A \subset X$. Note that in such a case we have $\Phi_i = \phi_i$ for $i = 1, 2$, $Y = y$, $Z = z$. In these cases we will use only the notation Φ_i, Y, Z instead of ϕ_i, y and z for $i = 1, 2$. That is, unlike before, Φ will be obtained from Φ_1 and Φ_2 .

Remark. Observe that if an algebra $\mathcal{A} \subset \mathcal{P}(Y)$ is isomorphic to $\mathcal{P}(X)$, then the isomorphism Φ is uniquely determined by an enumeration of the atoms of the algebras $\mathcal{P}(X)$ and \mathcal{A} . Given an enumeration $\{\{x_\xi\} : \xi < \kappa\}$

in $\mathcal{P}(X)$ and an enumeration $\{C_\xi: \xi < \kappa\}$ in \mathcal{A} and putting $C_\xi = \Phi(\{x_\xi\})$ we define Φ for all subsets of X . It is clear that all the properties of \mathcal{A} we consider do not depend on the choice of the isomorphism Φ .

Fact 4 ([6]). *Let $\mathcal{A} = \Phi[\mathcal{P}(X)]$ be a field of subsets of Y where Φ is an isomorphism. Then \mathcal{A} is inner MB-representable (and consequently, $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is inner MB-representable) if and only if the following two conditions are satisfied simultaneously:*

- (*) *the set of all $x \in X$ such that $|\Phi(\{x\})| > 1$, belongs to \mathcal{J} ;*
- (**) *the remainder algebra \mathcal{B} is atomic and the atoms of \mathcal{B} cover $Y \setminus Z$.*

Moreover, if (*) and (**) are satisfied then $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ for $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where

$$\mathcal{F}_1 = \{\Phi(\{x\}): |\Phi(\{x\})| > 1\},$$

$$\mathcal{F}_2 = \{\Phi_1(A) \cup \Phi_2(A): \Phi_2(A) \in At(\mathcal{B})\}$$

and $At(\mathcal{B})$ is the set of atoms of algebra \mathcal{B} .

By Facts 1, 2, 4 we have:

Fact 5. *If $\mathcal{A} = \Phi[\mathcal{P}(X)]$ is isomorphic to $\mathcal{P}(X)$ then $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is topological if and only if \mathcal{A} satisfies (*) and (**) and there exists a topology τ on Y such that $\tau \setminus \{\emptyset\}$ is coinital with \mathcal{F} .*

Fact 6 ([6]). *If the algebra $\mathcal{A} = \Phi[\mathcal{P}(X)]$ is isomorphic to $\mathcal{P}(X)$ and $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ has the hull property then the quotient algebras $\mathcal{A}/\mathcal{H}(\mathcal{A})$ and $\mathcal{P}(X)/\mathcal{J}$ are complete. If \mathcal{A} satisfies (*), then the completeness of $\mathcal{A}/\mathcal{H}(\mathcal{A})$ and $\mathcal{P}(X)/\mathcal{J}$ are equivalent.*

By Facts 4, 5, 6 in the class of atomic, complete Boolean algebras of sets (i.e. in the class of algebras isomorphic to some power set $\mathcal{P}(X)$) we can find topological pairs $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ only if \mathcal{A} satisfies (*), (**) and the remainder algebra \mathcal{B} is isomorphic to $\mathcal{P}(\lambda)$ for some cardinal number λ . So it seems reasonable to introduce the following:

Definition 2. We say that an algebra of sets \mathcal{A} is of the type $\langle \kappa, \lambda \rangle$ for some cardinal numbers κ and λ ($\mathcal{A} \in \langle \kappa, \lambda \rangle$) if

- (a) there exists an isomorphism Φ between $\mathcal{P}(X)$ and \mathcal{A} for some set X with cardinality κ .
- (b) the conditions (*) and (**) are satisfied.
- (c) there exists an isomorphism Ψ between the remainder algebra \mathcal{B} and $\mathcal{P}(\lambda)$.

Observe that we can always take $X = \kappa$ and that κ can be strictly less than λ as we can see in the following proposition.

3. RESULTS

Proposition 1. *There exists an algebra $\mathcal{A} \in \langle \kappa, \lambda \rangle$ if and only if $2^\kappa \geq 2^\lambda$.*

Proof. By the condition (c) of the last definition the inequality $2^\kappa \geq 2^\lambda$ is necessary.

So, let us assume this inequality. Then, by the Fichtenholz-Kantorovich theorem $\mathcal{P}(\kappa)$ has an independent family (consisting of infinite sets) of size $2^\kappa \geq 2^\lambda$, so by Sikorski's theorem (see [11]) there exists an epimorphism Φ_2 from $\mathcal{P}(\kappa)$ onto $\mathcal{P}(\lambda)$ such that all finite subsets of κ are in its kernel. Put $Z = \kappa \times \{0\}$ and $Y = Z \cup \lambda$. Let $\Phi: \mathcal{P}(\kappa) \rightarrow \mathcal{P}(Y)$ be given by the formula

$$\Phi(A) = (A \times \{0\}) \cup \Phi_2(A)$$

Then \mathcal{J} is the kernel of the homomorphism Φ_2 . Since $\Phi(\{x\}) = \{x\} \times \{0\}$ for each $x \in \kappa$ we obtain (*). Moreover $Y \setminus Z = \lambda$ and so $\mathcal{A} = \Phi[\mathcal{P}(\kappa)]$ satisfies all desired conditions. \square

Definition 3. Let $\mathcal{A} \in \langle \kappa, \lambda \rangle$ and $\{B_\xi: \xi < \lambda\}$ be one-to-one enumeration of all atoms of \mathcal{B} . We say that a family $\mathcal{X} = \{X_\xi: \xi < \lambda\}$ of subsets of κ , generates atoms of \mathcal{B} if for every $\xi < \lambda$ we have $\Phi_2(X_\xi) = B_\xi$. We say that \mathcal{X} has a disjoint refinement generating atoms of \mathcal{B} if there exists a family $\mathcal{T} = \{T_\xi: \xi < \lambda\}$ such that for every $\xi < \lambda$ we have $T_\xi \subset X_\xi$, $\Phi_2(T_\xi) = B_\xi$ and the sets in \mathcal{T} are pairwise disjoint.

Proposition 2. *For $\mathcal{A} \in \langle \kappa, \lambda \rangle$ the following conditions are equivalent:*

- (i) *the pair $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is topological.*
- (ii) *for every family \mathcal{X} generating atoms of \mathcal{B} there exists a disjoint refinement with the same property.*
- (iii) *for some family \mathcal{X} generating atoms of \mathcal{B} there exists a disjoint refinement with the same property.*

Proof. (i) \Rightarrow (ii). Let τ be the topology on Y such that $\mathcal{H}(\mathcal{A})$ is the family of τ -nowhere dense sets and \mathcal{A} is a field of sets with τ -nowhere dense boundary. Let $\mathcal{X} = \{X_\xi: \xi < \lambda\}$, $\mathcal{B} = \{B_\xi: \xi < \lambda\}$ and Φ_2 be as in Definition 3, i.e., \mathcal{X} generates atoms of \mathcal{B} . Observe that \mathcal{X} is an almost disjoint family with respect to the ideal $\mathcal{J} = \text{Ker } \Phi_2$ (i.e for $\alpha \neq \beta$ we have $X_\alpha \cap X_\beta \in \mathcal{J}$). Denote by A_0 the set of all points $\eta \in \kappa$ for which $|\Phi(\{\eta\})| > 1$. By (*) $A_0 \in \mathcal{J}$, so the sets $\{X_\xi \setminus A_0: \xi < \lambda\}$ form a family of sets generating atoms of \mathcal{B} . Thus without loss of generality, we can assume that all the sets $\{X_\xi: \xi < \lambda\}$ are disjoint with A_0 . Hence for all $\alpha < \beta < \lambda$ we have

$$\Phi_1(X_\alpha) \cap \Phi_1(X_\beta) \in \mathcal{H}(\mathcal{A})$$

By Facts 4, 5, any set of the form $\Phi_1(X_\xi) \cup B_\xi$ (hence, necessarily belonging to \mathcal{F}_2) contains some subset G_ξ open in τ , such that $\{\Phi^{-1}(G_\xi) : \xi < \lambda\}$ is a family generating atoms of \mathcal{B} . Thus $X_\xi \setminus \Phi^{-1}(G_\xi) \in \mathcal{J}$ and consequently $\Phi_1(X_\xi) \setminus G_\xi \in \mathcal{H}(\mathcal{A})$. So, we have $G_\alpha \cap G_\beta \in \mathcal{H}(\mathcal{A})$ for $\alpha \neq \beta$ but $\mathcal{H}(\mathcal{A})$ consists of τ -nowhere dense sets and thus $G_\alpha \cap G_\beta = \emptyset$. The family $\{\Phi^{-1}(G_\xi) : \xi < \lambda\}$ is the disjoint refinement of \mathcal{X} generating atoms.

(ii) \Rightarrow (iii) Obvious. (iii) \Rightarrow (i) Let \mathcal{T} be a disjoint refinement of \mathcal{X} generating atoms of \mathcal{B} . Put

$$\begin{aligned} \tau_1 &= \mathcal{F}_1 = \{\Phi(\{\eta\}) : |\Phi(\{\eta\})| > 1\} \\ \text{and } \tau_2 &= \{\Phi_1(T_\xi \setminus A) \cup B_\xi : \xi < \lambda, A \in \mathcal{J}\} \end{aligned}$$

Let τ be the topology generated by $\tau_1 \cup \tau_2$. Then τ_0 is coinital with \mathcal{F} (definition 1). Hence by Fact 2 the pair $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is topological. \square

Proposition 3. *If $\mathcal{A} \in \langle \kappa, \lambda \rangle$ and $\lambda > \kappa$ then $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is not topological.*

Proof. There is no family of λ disjoint subsets of κ . \square

Proposition 4. *For any $\lambda \leq \kappa$ there exists an algebra of sets $\mathcal{A} \in \langle \kappa, \lambda \rangle$ such that the pair $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is topological.*

Proof. Let us divide κ into λ disjoint subsets $\{X_\xi : \xi < \lambda\}$ with cardinality $|X_\xi| = \kappa$. Let $\{\mathcal{J}_\xi : \xi < \lambda\}$ be a family of maximal ideals on $\mathcal{P}(\kappa)$ with the following properties:

- $[\kappa]^{<\omega} \in \mathcal{J}_\xi$ for any $\xi < \lambda$.
- $X_\xi \notin \mathcal{J}_\xi$ for any $\xi < \lambda$.

Then we have:

- $X_\alpha \in \mathcal{J}_\beta$ for $\alpha \neq \beta$.

Let $\mathcal{J} = \bigcap_{\xi < \lambda} \mathcal{J}_\xi$. Observe that $A \in \mathcal{J}$ if and only if for any ξ we have $A \cap X_\xi \in \mathcal{J}$ or (what is equivalent) if for any ξ we have $A \cap X_\xi \in \mathcal{J}_\xi$. Hence the following hold:

- (1) the class of equivalence $[X_\xi]$ modulo \mathcal{J} is an atom in $\mathcal{P}(\kappa)/\mathcal{J}$.
- (2) the quotient algebra $\mathcal{P}(\kappa)/\mathcal{J}$ is atomic.
- (3) the quotient algebra $\mathcal{P}(\kappa)/\mathcal{J}$ is complete.

Condition (3) follows from the disjointness of the family $\{X_\xi : \xi < \lambda\}$.

Thus the quotient algebra $\mathcal{P}(\kappa)/\mathcal{J}$ is isomorphic to $\mathcal{P}(\lambda)$. Put $Z = \kappa \times \{0\}$ and $Y = Z \cup \lambda$. Let $\Phi : \mathcal{P}(\kappa) \rightarrow \mathcal{P}(Y)$ be given by the formula

$$\Phi(A) = (A \times \{0\}) \cup \Psi([A])$$

where Ψ is an isomorphism between $\mathcal{P}(\kappa)/\mathcal{J}$ and $\mathcal{P}(\lambda)$ such that $\Psi([X_\xi]) = \{\xi\}$. Then $\Phi_2(X_\xi) = \{\xi\}$ for $\xi < \lambda$ and $\{X_\xi: \xi < \lambda\}$, as a disjoint family, obviously has a disjoint refinement with the same property. By Proposition 2 this completes the proof. \square

Proposition 5. *If $\kappa \geq \lambda > \omega$ then there exists an algebra $\mathcal{A} \in \langle \kappa, \lambda \rangle$ such that the pair $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is not topological.*

Proof. Let \mathcal{M} be a complete Boolean algebra satisfying the ccc condition with cardinality $|\mathcal{M}| = 2^\kappa$. For example \mathcal{M} may be a measure algebra. For the existence of such an algebra see [10]. Then by the same argument as in the proof of Proposition 1 there exists an epimorphism $f: \mathcal{P}(\kappa) \rightarrow \mathcal{M}$ whose kernel contains all finite subsets of κ . Clearly \mathcal{M} contains an independent family of size 2^κ and so, using the same argument we conclude that there exists an epimorphism $g: \mathcal{M} \rightarrow \mathcal{P}(\lambda)$. Let $\Phi_2 = g \circ f$. Then Φ_2 is an epimorphism from $\mathcal{P}(\kappa)$ onto $\mathcal{P}(\lambda)$ whose kernel contains all finite subsets of κ . Put $Z = \kappa \times \{0\}$ and $Y = Z \cup \lambda$. Let us define $\Phi(A) = (A \times \{0\}) \cup \Phi_2(A)$. As in the proof of Proposition 1, one notes that \mathcal{J} is equal to the kernel of Φ_2 and both (*) and (**) are satisfied concluding that \mathcal{A} is of type $\langle \kappa, \lambda \rangle$.

For any $\xi < \lambda$ let us define $X_\xi = \Phi_2^{-1}(\{\xi\})$. One can see that the family $\chi = \{X_\xi: \xi < \lambda\}$ of subsets of κ generates atoms of \mathcal{B} . However χ does not have a disjoint refinement $\{T_\xi: \xi < \lambda\}$ because $f[T_\xi]$'s would be disjoint and non-zero, which is impossible since \mathcal{M} is ccc and $\lambda > \omega$. \square

Theorem 2. *There exists an algebra of sets \mathcal{A} such that the pair $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is complete, has the hull property but is not topological.*

Proof. Let \mathcal{A} be the algebra of the previous proposition. Since \mathcal{A} is of type $\langle \kappa, \lambda \rangle$, it is inner MB-representable by Fact 4. It is clear from the construction that $\mathcal{P}(\kappa)/\mathcal{J}$ is isomorphic to $\mathcal{P}(\lambda)$ and hence complete. By Fact 6 the quotient algebra $\mathcal{A}/\mathcal{H}(\mathcal{A})$ is also complete. Now, to conclude the theorem, use the result of [4] saying that an algebra \mathcal{A} is MB-representable and complete if and only if it is complete and has the hull property. \square

The above theorem is the complete solution of the problem posed in [1] — is it possible to reverse the implication “top” \rightarrow “hull & complete”? The answer is “NO” and the countrexample is given in ZFC.

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