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Epiconvergence and ε-Subgradients of Convex Functions

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We study, in the context of normed spaces (not necessarily Banach), equivalences between the epiconvergence of convex functions and the set-convergence of their ε-subdifferentials.

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1. Introduction

Recently several new topologies on the set of convex subsets of a normed space were introduced and studied, the two most important ones being the Attouch-Wets topology (see e.g. [4], [5], [6]) and the slice topology (see e.g. [2], [7], [8]). What is notable about both of them is that most of their properties (e.g. continuity of the polarity and conjugacy maps) not only hold in Banach spaces, but also in general normed spaces. However, there are other notions, (e.g. the subdifferential map) which are not useful outside the Banach context but were used to characterize these topologies. For example (in the Banach case), convergence in the Attouch-Wets topology was characterized in [10] in terms of the behaviour of a certain operator involving the subdifferential map, while in [2] convergence in the slice topology was characterized in terms of the Kuratowski-Painlevé convergence of the subdifferential map. Thus it seems appropriate to reformulate these characterizations in the context of normed spaces, not necessarily Banach. This is what we do in this paper: we show that the results of Attouch and Beer, and Beer and Thera, as well as other related results, can be extended to normed spaces if subdifferentials are replaced by ε-subdifferentials. These extensions are possible because in [13] we proved that Borwein's variational principle and Rockafellar's maximal monotonicity theorem are true in normed spaces, provided that we replace subdifferentials with ε-subdifferentials.

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2. Notation and preliminary results

In this first part we shall introduce the notation and recall several known results used in the paper. We shall also extend to ε-subgradients several results concerning subgradients.

Let X be a normed space, B be its unit ball, X^* be the topological dual of X, and B^* be the unit ball in X^* (with respect to the dual norm). On X^* we consider only the strong topology and on products (such as $X \times X^*$ or $X \times R$) we consider the box norm. We write $\Gamma(X)$ (resp. $\Gamma(X^*)$) for the set of all proper, lower semicontinuous, convex functions defined on X (resp. on X^*) and with values in $(-\infty, +\infty]$. As usual, we denote a multivalued map by a double arrow.

Given $f \in \Gamma(X)$, $x \in dom f$, and $\varepsilon \geq 0$, we denote by $\partial_\varepsilon f(x)$ the set of all ε-subgradients to f at x and by $\partial_\varepsilon f$ the set of all pairs (x, x^*) such that $x^* \in \partial_\varepsilon f(x)$. The *conjugate* $f^* \in \Gamma(X^*)$ of f is defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x);\ x \in X\}.$$

Then

$$\langle x^*, x\rangle \leq f^*(x^*) + f(x) \quad \text{for any} \quad x \in X,\ x^* \in X^* \quad \text{(Fenchel's inequality)} \tag{1}$$

and, if $\varepsilon \geq 0$,

$$f^*(x^*) + f(x) \leq \langle x^*, x\rangle + \varepsilon \quad \text{if and only if} \quad (x, x^*) \in \partial_\varepsilon f. \tag{2}$$

Also

$$f(x) = \sup\{\langle x^*, x\rangle - f^*(x^*);\ x^* \in X^*\}. \tag{3}$$

The following Brøndsted-Rockafellar type result was proved in [13].

Proposition 2.1. *Let $f \in \Gamma(X)$, $\varepsilon \geq 0$, $\delta > 0$ ($\delta \geq 0$ if X is a Banach space), $(x, x^*) \in \partial_{\varepsilon+\delta} f$, and $\lambda > 0$. Then there exists $(y, y^*) \in \partial_\delta f$ such that:*

(a) $\|y - x\| \leq \varepsilon/\lambda$;

(b) $\|y^* - x^*\| \leq \lambda$;

(c) $|f(y) - f(x)| \leq \varepsilon(1 + \dfrac{\|x^*\|}{\lambda})$.

Definition 2.2.

(1) For $f \in \Gamma(X)$, let $\partial_\bullet f$ denote the set of all triples $(\varepsilon, x, x^*) \in R \times X \times X^*$ such that $\varepsilon \geq 0$ and $x^* \in \partial_\varepsilon f(x)$.

(2) For $f \in \Gamma(X)$, define $d_f : dom f \times dom f^* \to [0, +\infty)$ by $d_f(u, u^*) = f^*(u^*) + f(u) - \langle u^*, u\rangle$. Thus $u^* \in \partial_{d_f(u,u^*)} f(u)$ but $u^* \notin \partial_\varepsilon f(u)$ if $\varepsilon < d_f(u, u^*)$.

(3) For $f \in \Gamma(X)$, let $\partial_\sharp f$ denote the subset of all $(\varepsilon, u, u^*) \in \partial_\bullet f$ with $\varepsilon = d_f(u, u^*)$.

Later we shall need the following approximation result.

Lemma 2.3. *Let $f \in \Gamma(X)$ and $(\varepsilon, x^*, x^{**}) \in \partial_\sharp f^*$. Then there exist a bounded net $\{x_\iota\}$ in X, weak**-convergent to x^{**}, and a net $\{\varepsilon_\iota\}$ converging to ε such that $(\varepsilon_\iota, x_\iota, x^*) \in \partial_\sharp f$ for each ι. A similar assertion with $\partial_\bullet f^*$ instead of $\partial_\sharp f^*$ is also true.*

Proof. Since $epi f$ is weak**-dense in $epi f^{**}$ (see [12]), by Lemma D in [13], there exists a bounded net $\{x_\iota\}$ in X which is weak**-convergent to x^{**} and such that the net $\{f(x_\iota)\}$ converges to $f^{**}(x^{**})$. Take $\varepsilon_\iota = f^*(x^*) + f(x_\iota) - \langle x^*, x_\iota\rangle$. □

As mentioned in the introduction, we shall need some facts from [13]. Recall that a family of multivalued maps $(T_\varepsilon)_{\varepsilon\geq 0}$ with $T_\varepsilon : X \rightrightarrows X^*$ is called a *monotone family of operators* if for any $x, y \in X$, $\varepsilon, \delta \geq 0$, $x^* \in T_\varepsilon(x)$, and $y^* \in T_\delta(y)$ we have $\langle x-y, x^*-y^*\rangle \geq -\varepsilon-\delta$. A monotone family of operators $(T_\varepsilon)_{\varepsilon\geq 0}$ is called *maximal* if given any monotone family of operators $(S_\varepsilon)_{\varepsilon\geq 0}$ such that $T_\varepsilon(x) \subseteq S_\varepsilon(x)$ for all $x \in X$ and $\varepsilon \geq 0$, then $T_\varepsilon(x) = S_\varepsilon(x)$ for all $x \in X$ and $\varepsilon \geq 0$. One can check easily that $(\partial_\varepsilon f)_{\varepsilon\geq 0}$ is a monotone family of operators for any $f \in \Gamma(X)$. In [13] we proved that such a family of monotone operators is maximal.

The following result is a variant of the Rockafellar integration formula.

Lemma 2.4. *Let $f \in \Gamma(X)$ and $(z, z^*) \in dom f \times dom f^*$. Then, for any $x \in X$,*

$$f(x) - f(z) = \sup\left\{\sum_{i=1}^{k} \left(\langle x_i^*, x_{i-1} - x_i\rangle - d_f(x_i, x_i^*)\right)\right\},$$

where $x_0 = x$, $x_k = z$, $x_k^* = z^*$ and the supremum is taken with respect to all $k \geq 2$, $(x_i, x_i^*) \in dom f \times dom f^*$, $i = 1, \ldots, k-1$.

Proof. Let

$$g(x) = \sup\left\{\sum_{i=1}^{k}(\langle x_i^*, x_{i-1} - x_i\rangle - \varepsilon_i)\right\},$$

where $x_0 = x$, $x_k = z$, $x_k^* = z^*$, $\varepsilon_k = d_f(z, z^*)$ and the supremum is taken with respect to all $k \geq 2$, $(\varepsilon_i, x_i, x_i^*) \in \partial_\bullet f$, $i = 1, \ldots, k-1$. It is easy to see that $g(x)$ is also equal to the right side of the formula we want to prove. It was shown in [13] that $g \in \Gamma(X)$, $g(z) \leq 0$, $\partial_\alpha g = \partial_\alpha f$ for any $\alpha \geq 0$, and that g and f differ by a constant. Thus the assertion of the lemma will follow if we prove that $g(z) = 0$. This is trivial if $d_f(z, z^*) = 0$. So assume that $\varepsilon = d_f(z, z^*) > 0$ and $g(z) < 0$. Then $g(z) \leq -\beta < 0$ for some $0 < \beta < \varepsilon$. For any $\delta \geq 0$ and any $y^* \in \partial_\delta f(y)$ we have

$$-\beta \geq g(z) \geq \langle y^*, z - y\rangle - \delta + \langle z^*, y - z\rangle - \varepsilon$$

implying that $\langle y^* - z^*, y - z\rangle \geq -\delta - (\varepsilon - \beta)$. From the maximal monotonicity of the family $(\partial_\delta f)$ (see [13], Theorem 2.2) it follows that $z^* \in \partial_{\varepsilon-\beta} f(z)$, which contradicts the fact that $\varepsilon = d_f(z, z^*)$. Thus $g(z) = 0$ and the lemma is proved. □

Remark 2.5. We shall also need the following dual result. Let f, ε, z, and z^* be as in the previous lemma and let $x^* \in X^*$. Then

$$f^*(x^*) - f^*(z^*) = \sup\left\{\sum_{i=1}^{k}(\langle x_{i-1}^* - x_i^*, x_i\rangle - d_f(x_i, x_i^*))\right\},$$

where $x_0^* = x^*$, $x_k = z$, $x_k^* = z^*$ and the supremum is taken with respect to all $k \geq 2, (x_i, x_i^*) \in dom f \times dom f^*$, $i = 1, \ldots, k-1$. If we apply Lemma 2.4 to f^*, we obtain the above formula, but with the supremum taken with respect to all couples $(x_i^*, x_i^{**}) \in dom f^* \times dom f^{**}$. To obtain our result it is enough to use Lemma 2.3.

For any real number $\lambda > 0$ we denote by K_λ the closed, convex cone $\{(x, t) \in X \times R;\, t \leq -\lambda\|x\|\}$; the interior of K_λ is denoted $\mathring{K}_\lambda$. For $\lambda > 0$, the *Lipschitz regularization with parameter* λ of f is defined as follows (see for example [11]):

$$f_\lambda(x) = \inf\{f(u) + \lambda\|x - u\|;\, u \in X\}.$$

Here are some of the properties of f_λ we need:

$$f_\lambda(x) \leq f_\mu(x) \text{ if } \lambda \leq \mu; \quad f(x) = \lim_{\lambda\to\infty} f_\lambda(x) \text{ for any} x \in X; \tag{4}$$

$$\left.\begin{array}{l}\text{either } f_\lambda(x) = -\infty \text{ for all } x \in X, \text{or } f_\lambda \text{ is Lipschitz on } X \text{with Lipschitz constant } \lambda;\\ \text{the second alternative occurs iff there exist } \varepsilon \geq 0 \text{ and } (x, x^*) \in \partial_\varepsilon f \text{ with} \|x^*\| \leq \lambda.\end{array}\right\} \tag{5}$$

For (x, x^*) as above we have

$$f_\lambda(z) \geq \langle x^*, z - x\rangle + f(x) - \varepsilon, \quad \text{for any} \quad z \in X. \tag{6}$$

If $f_\lambda \neq -\infty$ and $x \in X$, one can look at $f_\lambda(x)$ as the largest real number t such that $epi f$ and the interior of $K_\lambda + (x, t)$ are disjoint, i.e.

$$f_\lambda(x) = \max\{t;\, (\mathring{K}_\lambda + (x, t)) \cap epi f = \emptyset\} = \sup\{t;\, (K_\lambda + (x, t)) \cap epi f = \emptyset\}. \tag{7}$$

Following the terminology introduced in [10], an ε-*estimator* for $f_\lambda(z)$ is a point $x \in X$ such that

$$f(x) + \lambda\|z - x\| < f_\lambda(z) + \varepsilon. \tag{8}$$

Lemma 2.6. *Let $f \in \Gamma(X)$, $f(0) = 0$. Let $r > 1$, $z \in rB$, $0 < \varepsilon < 1$, $0 \leq \delta_0 < 1$, $x_0^* \in \partial_{\delta_0} f(0)$, $\lambda_0 = \|x_0^*\|(1 + 2r) + 2$, and $\lambda \geq \lambda_0$. Then:*

(a) *There exists $(x, x^*) \in \partial_\varepsilon f$ such that:*

(i) *x is an ε-estimator for $f_\lambda(z)$;* (ii) $\|x^*\| \le \lambda$;
(iii) $\|z-x\| \le r+1$ *(hence* $\|x\| \le 2r+1$*)*; (iv) $|f(x)| \le \lambda r + 1$;
(v) $f_\lambda(z) - f(x) \le \langle x^*, z-x\rangle$.

(b) *The function $\lambda \mapsto f_\lambda(z)$ from $[\lambda_0, \infty)$ to R is Lipschitz with Lipschitz constant $r+1$.*

Proof. From (5) it follows that f_λ is a Lipschitz function on X for any $\lambda \ge \|x_0^*\|$. From (7), there exist $x^* \in X^*$ and $t \in R$ such that the graph of the affine functional x^*+t separates *epi f* and $K_\lambda + (z, f_\lambda(z))$. Then $\|x^*\| \le \lambda$ and for any x

$$f_\lambda(z) - \lambda\|z-x\| \le \langle x^*, x\rangle + t \le f(x).$$

Choose now x to be an ε-estimator for $f_\lambda(z)$. It follows from (8) and the above inequalities that $0 \le f(x) - (\langle x^*, x\rangle + t) \le \varepsilon$, hence $x^* \in \partial_\varepsilon f(x)$. Using (8) again we obtain

$$\begin{aligned}\lambda\|z-x\| &\le f_\lambda(z) + \varepsilon - f(x) \le f(0) + \lambda\|z\| + \varepsilon - \langle x_0^*, x\rangle + \delta_0 \\ &\le \lambda\|z\| + \varepsilon + \langle x_0^*, z-x\rangle - \langle x_0^*, z\rangle + \delta_0 \le \lambda r + \varepsilon + \|x_0^*\|\|z-x\| + \|x_0^*\| r + \delta_0\end{aligned}$$

hence

$$\|z-x\| \le \frac{r(\lambda + \|x_0^*\|) + \varepsilon + \delta_0}{\lambda - \|x_0^*\|}.$$

from which (iii) follows. To prove (iv) it remains to notice that

$$-\lambda \le -\|x_0^*\|(2r+1) - 2 \le \langle x_0^*, x\rangle - \delta_0 \le f(x) \le f_\lambda(z) + \varepsilon \le f(0) + \lambda\|z\| + \varepsilon \le \lambda r + 1.$$

To prove (v), it is enough to see (by what precedes) that

$$f_\lambda(z) \le \langle x^*, z\rangle + t = \langle x^*, z-x\rangle + \langle x^*, x\rangle + t \le \langle x^*, z-x\rangle + f(x).$$

To prove (b) let $\mu > \nu \ge \lambda_0$, $\varepsilon > 0$, and x be an ε-estimator for $f_\nu(z)$. We have

$$0 \le f_\mu(z) - f_\nu(z) \le f(x) + \mu\|z-x\| - f(x) - \nu\|z-x\| + \varepsilon = (\mu-\nu)\|z-x\| + \varepsilon.$$

By part (i), $\|z-x\| \le r+1$, hence

$$|f_\mu(z) - f_\nu(z)| \le (\mu-\nu)(r+1) + \varepsilon.$$

Since ε was arbitrary, this proves part (b). □

3. Slice convergence and Kuratowski–Painlevé convergence of ε-subdifferentials

If X is a Banach space, a result of Attouch and Beer [2] (see also [1, Theorem 3.66]) links slice convergence of epigraphs of convex functions with the Kuratowski-Painlevé convergence of their subdifferentials. In this section we prove that a similar result is true in a normed space, provided that we replace subdifferentials with ε-subdifferentials. The main ideas of the proof are the same as in [2], with some changes due to the fact that we work with ε-subdifferentials. We begin by recalling the necessary notions.

Given a topological space T and a sequence $\{C_n\}$ of nonempty subsets of T we denote by $\mathrm{Li}\, C_n$ the set of those $x \in T$ such that there exists a sequence $\{x_n\}$ converging to x, with $x_n \in C_n$ for all n and denote by $\mathrm{Ls}\, C_n$ the set of those $x \in T$ such that there exists a sequence $0 < n_1 < n_2 < \ldots$ and a sequence $\{x_k\}$ converging to x, with $x_k \in C_{n_k}$ for all k. The sequence $\{C_n\}$ is said to be *Kuratowski-Painlevé convergent* if $\mathrm{Li}\, C_n = \mathrm{Ls}\, C_n$; this common value is denoted $\mathrm{Lim}\, C_n$.

Another notion we need is that of *slice topology* on $\Gamma(X)$, denoted τ_S (see [7], [9]). Recall that a subbase for τ_S consists of

> all sets of the form $(V \times (-\infty, t))^-$, with V open in X and $t \in R$, where $(V \times (-\infty, t))^-$ is the set of all $f \in \Gamma(X)$ such that $epi f \cap (V \times (-\infty, t)) \neq \emptyset$;

and

> all sets of the form $(s(\mu, x^*, \eta)^c)^{++}$, with $\mu > 0$, $x^* \in X^*$, and $\eta \in R$, where $s(\mu, x^*, \eta) = \{(x, t) \in X \times R; \|x\| \leq \mu, t = \langle x^*, x \rangle - \eta\}$ and $(s(\mu, x^*, \eta)^c)^{++}$ is the set of all $f \in \Gamma(X)$ such that $((epi f) + \varepsilon(B \times [-1, 1])) \cap s(\mu, x^*, \eta) = \emptyset$ for some $\varepsilon > 0$.

The following characterization of convergence in this topology is an adaptation to the context of normed spaces of Theorem 3.1 in [2].

Proposition 3.1. *Let X be a normed space, f, $f_n \in \Gamma(X)$, $n \geq 1$. The following are equivalent:*

(i) $f = \tau_{\mathrm{S}}\text{-}\lim f_n$;

(ii) *for any $(x, x^*) \in X \times X^*$ there exist a sequence $\{x_n\}$ in X and a sequence $\{x_n^*\}$ in X^* converging to x and to x^* respectively and such that $f(x) = \lim_{n \to \infty} f_n(x_n)$, and $f^*(x^*) = \lim_{n \to \infty} f_n^*(x_n^*)$.*

(iii) *for any $(x, x^*) \in dom f \times dom f^*$ and any $\alpha > 0$ there exist a sequence $\{x_n\}$ in X and a sequence $\{x_n^*\}$ in X^* converging to x and to x^* respectively and such that $f(x) + \alpha \geq \limsup_{n \to \infty} f_n(x_n)$ and $f^*(x^*) + \alpha \geq \limsup_{n \to \infty} f_n^*(x_n^*)$.*

Proof. The fact that (i) implies (ii) is proved in [2], Theorem 3.1. Clearly (ii) implies (iii). It remains to show that (iii) implies (i). So assume that (iii) is true, but (i) is false. Then, by restricting our attention to a subsequence, we can assume that either

(a) there exist an open subset V of X and $t \in R$ such that $epi f \cap (V \times (-\infty, t)) \neq \emptyset$ but $epi f_n \cap (V \times (-\infty, t)) = \emptyset$ for all n

or

(b) there exit $\mu > 0$, $z^* \in X^*$, $\eta \in R$ and $\varepsilon > 0$ such that $((epi f) + \varepsilon(B \times [-1, 1])) \cap s(\mu, z^*, \eta) = \emptyset$ and $((epi f_n) + \varepsilon_n(B \times [-1, 1])) \cap s(\mu, z^*, \eta) \neq \emptyset$ for any $\varepsilon_n > 0$ and any n;

is true. It is not difficult to see that, because of (iii) (the part involving x), (a) cannot be true. So (b) must be true. From Lemma 8.1.1 in [9] it follows that there exists $(x^*, t) \in epi f^*$ such that the graph of the affine functional $x^* - t$ strongly separates $epi f$ and $s(\mu, z^*, \eta)$, i.e.

$$\langle x^*, u\rangle - t \le f(u), \quad u \in X$$

and

$$\langle z^*, u\rangle - \eta \le \langle x^*, u\rangle - t - \delta, \quad \|u\| \le \mu.$$

for some $\delta > 0$. It is easy to see that there exists $\mu' > \mu$ such that the last inequality becomes

$$\langle z^*, u\rangle - \eta \le \langle x^*, u\rangle - t - \delta/2, \quad \|u\| \le \mu'.$$

By hypothesis there exists a sequence $\{x_n^*\}$ in X^* converging to x^* and such that $f^*(x^*) + \delta/16 \ge \limsup_{n\to\infty} f_n^*(x_n^*)$. Choose n large enough such that $\|x^* - x_n^*\| < \delta/(8\mu')$ and $f^*(x^*) + \delta/8 > f_n^*(x_n^*)$. Let $u \in \mu' B$. By Fenchel's inequality, our choices, and the previous inequalities we have

$$\begin{aligned} f_n(u) &\ge \langle x_n^*, u\rangle - f_n^*(x_n^*) = \langle x_n^* - x^*, u\rangle + \langle x^*, u\rangle - f_n^*(x_n^*) \\ &\ge -\delta/8 + \langle x^*, u\rangle - f^*(x^*) - \delta/8 \ge \langle x^*, u\rangle - t - \delta/4 \\ &\ge \langle z^*, u\rangle - \eta + \delta/4. \end{aligned}$$

The resulting inequality shows that, if $\varepsilon_n > 0$, $\varepsilon_n < \mu' - \mu$ and $4\varepsilon_n(1 + \|z^*\|) < \delta$, then $(epi f_n + \varepsilon_n(B \times [-1,1])) \cap s(\mu, z^*, \eta) = \emptyset$, which contradicts (b). It follows that (iii) implies (i) and the proposition is proved. □

Let now f, $f_n \in \Gamma(X)$, $n \ge 1$. Recall that a sequence $\{(x_n, x_n^*)\}$ with $(x_n, x_n^*) \in dom f_n \times dom f_n^*$ is called *normalizing* if $\lim_{n\to\infty}(x_n, x_n^*) = (x, x^*) \in dom f \times dom f^*$, $\lim_{n\to\infty} f_n(x_n) = f(x)$, and $\lim_{n\to\infty} f_n^*(x_n^*) = f^*(x^*)$.

Lemma 3.2. *Let f, $f_n \in \Gamma(X)$, $n \ge 1$. Let also $\{(x_n, x_n^*)\}$ be a normalizing sequence and let $\lim_{n\to\infty}(x_n, x_n^*) = (x, x^*)$. Then*

(1) $\lim_{n\to\infty} d_{f_n}(x_n, x_n^*) = d_f(x, x^*)$.

(2) *Assume that $\partial_\varepsilon f \subseteq \mathrm{Li}\,\partial_\varepsilon f_n$ for every $\varepsilon > 0$. Let $(u, u^*) \in dom f \times dom f^*$ and $(u_n, u_n^*) \in X \times X^*$, $n \ge 1$, be such that $\lim_{n\to\infty}(u_n, u_n^*) = (u, u^*)$. Let $\alpha = d_f(u, u^*)$ and $\alpha_n = d_{f_n}(u_n, u_n^*)$. Then*

$$\limsup_{n\to\infty} f_n(u_n) + \alpha - \limsup_{n\to\infty} \alpha_n \le f(u) \le \liminf_{n\to\infty} f_n(u_n) + \limsup_{n\to\infty} \alpha_n - \liminf_{n\to\infty} \alpha_n$$

and

$$\limsup_{n\to\infty} f_n^*(u_n^*) + \alpha - \limsup_{n\to\infty} \alpha_n \le f^*(u^*) \le \liminf_{n\to\infty} f_n^*(u_n^*) + \limsup_{n\to\infty} \alpha_n - \liminf_{n\to\infty} \alpha_n$$

(3) *If in addition to the conditions in (2) we have $\limsup_{n\to\infty} d_{f_n}(u_n, u_n^*) \le d_f(u, u^*)$, then $\{(u_n, u_n^*)\}$ is a normalizing sequence, i.e. $\lim_{n\to\infty} f_n(u_n) = f(u)$ and $\lim_{n\to\infty} f_n^*(u_n^*) = f^*(u^*)$.*

Proof. The proof of (1) is trivial, so we shall prove only (2) and (3). Let $\gamma = \liminf_{n\to\infty} \alpha_n$, and $\beta = \limsup_{n\to\infty} \alpha_n$. Let $k \geq 2$ and $(\varepsilon_i, y_i, y_i^*) \in \partial_\bullet f$, $i = 1, \ldots, k-1$. Let $\mu > 0$ and $\mu_i = \varepsilon_i + \mu/(k-1)$ (this is needed in case $\varepsilon_i = 0$). Then $(\mu_i, y_i, y_i^*) \in \partial_\bullet f$, $i = 1, \ldots, k-1$ and since $\partial_{\mu_i} f \subseteq \mathrm{Li}\, \partial_{\mu_i} f_n$, there exist $(y_{i,n}, y_{i,n}^*) \in \partial_{\mu_i} f_n$ such that $\lim_{n\to\infty} (y_{i,n}, y_{i,n}^*) = (y_i, y_i^*)$, $i = 1, \ldots, k-1$. Let also $y_0 = x$, $y_{0,n} = x_n$, $y_k = u$, $y_k^* = u^*$, $\varepsilon_k = \alpha$, $y_{k,n} = u_n$, and $y_{k,n}^* = u_n^*$, (thus $\limsup_{n\to\infty} \alpha_n = \beta = \varepsilon_k - \alpha + \beta$). From the definition of ε-subgradients it follows that

$$f_n(x_n) \geq f_n(y_{k,n}) + \sum_{i=1}^{k} \langle y_{i,n}^*, y_{i-1,n} - y_{i,n} \rangle - \sum_{i=1}^{k-1} \mu_i - \alpha_n, \quad n \geq 1.$$

By replacing μ_i with $\varepsilon_i + \mu/(k-1)$ and taking lim sup (with respect to n) we obtain

$$f(x) \geq \limsup_{n\to\infty} f_n(u_n) + \sum_{i=1}^{k} \langle y_i^*, y_{i-1} - y_i \rangle - \sum_{i=1}^{k} \varepsilon_i + \alpha - \beta - \mu.$$

Lemma 2.4 (more exactly its proof) implies that

$$f(x) \geq \limsup_{n\to\infty} f_n(u_n) + f(x) - f(u) + \alpha - \beta - \mu$$

and, since μ is arbitrary,

$$f(u) \geq \limsup_{n\to\infty} f_n(u_n) + \alpha - \beta.$$

Similar arguments (the remark following Lemma 2.4 is essential) show that

$$f^*(u^*) \geq \limsup_{n\to\infty} f_n^*(u_n^*) + \alpha - \beta.$$

Since

$$f_n^*(u_n^*) + f_n(u_n) = \langle u_n^*, u_n \rangle + \alpha_n,$$

it follows that

$$\begin{aligned}\liminf_{n\to\infty} f_n(u_n) &\geq \langle u^*, u \rangle + \gamma - \limsup_{n\to\infty} f_n^*(u_n^*) \\ &\geq \langle u^*, u \rangle + \alpha - f^*(u^*) + \gamma - \beta = f(u) + \gamma - \beta.\end{aligned}$$

and therefore

$$f(u) \leq \liminf_{n\to\infty} f_n(u_n) + \beta - \gamma.$$

The corresponding inequality involving the conjugate functions can be proved in the same way. This proves (2).

To prove (3), assume first that $\beta = \gamma \leq \alpha$. From (2) it follows that $\beta = \alpha$, $\lim_{n\to\infty} f_n(u_n) = f(u)$, and $\lim_{n\to\infty} f_n^*(u_n^*) = f^*(u^*)$. To prove the general case ($\beta \leq \alpha$), notice that the above arguments imply that every convergent subsequence of $\{\alpha_n\}$ must converge to α. Therefore the sequence $\{\alpha_n\}$ is convergent (to α) and the lemma is proved. □

Lemma 3.3. *Let $f,\ f_n \in \Gamma(X),\ n \geq 1$. Then*

(1) $\partial_\bullet f \subseteq \mathrm{Li}\,\partial_\bullet f_n$ *if and only if* $\partial_\bullet f = \mathrm{Lim}\,\partial_\bullet f_n$

(2) *Given any $\varepsilon \geq 0$,* $\partial_\varepsilon f \subseteq \mathrm{Li}\,\partial_\varepsilon f_n$ *if and only if* $\partial_\varepsilon f = \mathrm{Lim}\,\partial_\varepsilon f_n$.

Proof. We shall prove only the first assertion, the proof of the other one being similar. Since trivially $\mathrm{Li}\,\partial_\bullet f_n \subseteq \mathrm{Ls}\,\partial_\bullet f_n$, it is enough to prove that $\mathrm{Ls}\,\partial_\bullet f_n \subseteq \partial_\bullet f$ if $\partial_\bullet f \subseteq \mathrm{Li}\,\partial_\bullet f_n$. To this end let $(\alpha, x, x^*) \in \mathrm{Ls}\,\partial_\bullet f_n$. By definition there exist an increasing sequence of positive integers $n_1 < n_2 < \ldots < n_k < \ldots$ and a sequence $\{(\alpha_k, x_k, x_k^*)\}$ with $(\alpha_k, x_k, x_k^*) \in \partial_\bullet f_{n_k}$ such that

$$\lim_{k\to\infty} (\alpha_k, x_k, x_k^*) = (\alpha, x, x^*).$$

Let $\varepsilon \geq 0$ and let $(y, y^*) \in \partial_\varepsilon f$. Then $(\varepsilon, y, y^*) \in \partial_\bullet f$ and, since $\partial_\bullet f \subseteq \mathrm{Li}\,\partial_\bullet f_n$, there exist $(\varepsilon_n, y_n, y_n^*) \in \partial_\bullet f_n$, $n \geq 1$, such that $\lim\limits_{n\to\infty} (\varepsilon_n, y_n, y_n^*) = (\varepsilon, y, y^*)$. It is easy to see that

$$\langle y_{n_k}^* - x_k^*, y_{n_k} - x_k \rangle \geq -\varepsilon_{n_k} - \alpha_k$$

and, by taking the limit, we obtain

$$\langle y^* - x^*, y - x \rangle \geq -\varepsilon - \alpha.$$

The fact that $(\partial_\varepsilon f)_\varepsilon$ is a maximal family of monotone operators (see [13], Theorem 2) implies that $x^* \in \partial_\alpha f(x)$, showing that $\mathrm{Ls}\,\partial_\bullet f_n \subseteq \partial_\bullet f$. □

We can now state and prove our main result in this section.

Theorem 3.4. *Let X be a normed space and $f,\ f_n \in \Gamma(X)$, $n \geq 1$. The following assertions are equivalent:*

(1) $f = \tau_{\mathrm{S}}\text{-}\lim f_n$;

(2) $\partial_\sharp f \subseteq \mathrm{Li}\,\partial_\sharp f_n$ *and there exists a normalizing sequence;*

(3) $\partial_\bullet f = \mathrm{Lim}\,\partial_\bullet f_n$ *and there exists a normalizing sequence;*

(4) $\partial_\varepsilon f = \mathrm{Lim}\,\partial_\varepsilon f_n$ *for every $\varepsilon > 0$ and there exists a normalizing sequence.*

Proof. First we shall prove that (1) implies (2). To this end assume that (1) is true and let $(\varepsilon, x, x^*) \in \partial_\sharp f$. It follows that there exist sequences $\{x_n\}$ and $\{x_n^*\}$ such that $x = \lim\limits_{n\to\infty} x_n$, $x^* = \lim\limits_{n\to\infty} x_n^*$ and $f(x) = \lim\limits_{n\to\infty} f_n(x_n)$, $f^*(x^*) = \lim\limits_{n\to\infty} f_n^*(x_n^*)$. Let $\varepsilon_n = f_n^*(x_n^*) + f_n(x_n) - \langle x_n^*, x_n \rangle$, $n \geq 1$. Then $(\varepsilon_n, x_n, x_n^*) \in \partial_\sharp f_n$ and clearly $\lim\limits_{n\to\infty} \varepsilon_n = \varepsilon$. Thus $(\varepsilon, x, x^*) \in \mathrm{Li}\,\partial_\sharp f_n$ and therefore $\partial_\sharp f \subseteq \mathrm{Li}\,\partial_\sharp f_n$.

Assume now that (2) is true and let $(\varepsilon, u, u^*) \in \partial_\bullet f$. Then $(\alpha, u, u^*) \in \partial_\sharp f$, where $\alpha = d_f(u, u^*)$. Since (2) is assumed true, there exist $(\alpha_n, u_n, u_n^*) \in \partial_\sharp f_n$, $n \geq 1$, such that $\lim_{n\to\infty}(\alpha_n, u_n, u_n^*) = (\alpha, u, u^*)$. Then $(\varepsilon + \alpha_n - \alpha, u_n, u_n^*) \in \partial_\bullet f_n$ and $\lim_{n\to\infty}(\varepsilon + \alpha_n - \alpha, u_n, u_n^*) = (\varepsilon, u, u^*)$, showing that $\partial_\bullet f \subseteq \mathrm{Li}\,\partial_\bullet f_n$. From Lemma 3.3 it follows that (3) is true.

Next we shall show that (3) implies (4). So assume that (3) is true and let $\varepsilon > 0$ and $u^* \in \partial_\varepsilon f(u)$. Our assumption implies that there exist $(\varepsilon_n, u_n, u_n^*) \in \partial_\bullet f_n$, $n \geq 1$, such that $\lim_{n\to\infty}(\varepsilon_n, u_n, u_n^*) = (\varepsilon, u, u^*)$. Let $\delta_n = \min\{\varepsilon, \varepsilon_n\}$. If $\delta_n < \varepsilon_n$, we can use Proposition 2.1 to find $(v_n, v_n^*) \in \partial_{\delta_n} f_n$ such that $\|v_n - u_n\| \leq \sqrt{\varepsilon_n - \delta_n}$ and $\|v_n^* - u_n^*\| \leq \sqrt{\varepsilon_n - \delta_n}$. If $\delta_n = \varepsilon_n$, take $v_n = u_n$ and $v_n^* = u_n^*$. It follows that $\lim_{n\to\infty} v_n = u$ and $\lim_{n\to\infty} v_n^* = u^*$ and, since $\delta_n \leq \varepsilon$, $\partial_{\delta_n} f_n \subseteq \partial_\varepsilon f_n$. Thus $\partial_\varepsilon f \subseteq \mathrm{Li}\ \partial_\varepsilon f_n$. From Lemma 3.3 it follows that (4) is true.

Finally assume that (4) is true. In order to prove that (1) is true, it will be enough to show that for any $(u, u^*) \in dom f \times dom f^*$ and any $\mu > 0$ there exist a sequence $\{u_n\}$ in X converging to u and a sequence $\{u_n^*\}$ in X^* converging to u^* such that $f(u) + \mu \geq \limsup_{n\to\infty} f_n(u_n)$ and $f^*(u^*) + \mu \geq \limsup_{n\to\infty} f_n^*(u_n^*)$ (see Proposition 3.1). To this end, let $u \in dom f$, $u^* \in dom f^*$, $\mu > 0$, and $\varepsilon = d_f(u, u^*)$. Then $(u, u^*) \in \partial_{\varepsilon+\mu} f$ and, by hypothesis, there exists a sequence $\{(u_n, u_n^*)\}$ with $(u_n, u_n^*) \in \partial_{\varepsilon+\mu} f_n$ and such that $\lim_{n\to\infty}(u_n, u_n^*) = (u, u^*)$. By Lemma 3.2, $\{u_n\}$ and $\{u_n^*\}$ have the required properties. $\square$

The following example shows that we cannot expect equality in (2) above.

Example 3.5. Define $f_n \in \Gamma(R)$ by $f_n(x) = -1$ if $x > 1/n$ and $f_n(x) = -nx$ if $0 \leq x \leq 1/n$. Then τ_S-lim$f_n = f$, where $f \in \Gamma(R)$ is given by $f(x) = -1$ if $x \geq 0$. Clearly $(1, 0, 0) \in \partial_\sharp f_n$ for all $n \geq 1$, so $(1, 0, 0) \in \mathrm{Li}\ \partial_\sharp f_n$. However, $d_f(0, 0) = 0$, so $(1, 0, 0)$ is not an element of $\partial_\sharp f$.

Remark 3.6. If X is Banach, it was proved in [2] that assertion (1) of Theorem 3.4 is equivalent to

$\partial f = \mathrm{Lim}\,\partial f_n$ and there exists a normalizing sequence.

The following characterization of the slice topology was proved in [7], Theorem 4.11: *If X is a Banach space, the slice topology on $\Gamma(X)$ is the weakest topology on $\Gamma(X)$ for which the multifunction $\Delta : \Gamma(X) \rightrightarrows X \times R \times X^*$, $\Delta(f) = \{(x, f(x), x^*); (x, x^*) \in \partial f\}$ is lower semicontinuous.* In the case of normed spaces we have:

Theorem 3.7. *Let X be a normed space.*

(1) *The slice topology is the weakest topology on $\Gamma(X)$ for which all multifunctions $\Delta_\varepsilon : \Gamma(X) \rightrightarrows X \times R \times X^*$, $\Delta_\varepsilon(f) = \{(x, f(x), x^*); (x, x^*) \in \partial_\varepsilon f\}$, $\varepsilon > 0$, are lower semicontinuous.*

(2) *The slice topology is the weakest topology on $\Gamma(X)$ for which the multifunction $\Delta_\sharp : \Gamma(X) \rightrightarrows X \times R \times X^* \times R$, $\Delta_\sharp(f) = \{(x, f(x), x^*, \varepsilon); x \in dom f, x^* \in dom f^*, \varepsilon = d_f(x, x^*)\}$, is lower semicontinuous.*

Proof. Recall first that a multifunction $T : A \rightrightarrows B$, with A and B topological spaces, is *lower semicontinuous* if for any net $\{a_\iota\}$ converging to a in A we have $T(a) \subseteq \mathrm{Li}\, T(a_\iota)$. Notice next that in all our results we can replace sequences with nets. Now, arguments similar to those used in the proof of Theorem 3.4 show that the multifunctions $\Delta_\sharp$ and Δ_ε are lower semicontinuous when $\Gamma(X)$ is endowed with the slice topology.

Let τ be a topology on $\Gamma(X)$ for which all multifunctions Δ_ε are lower semicontinuous. In order to prove that the slice topology is weaker than τ it is enough to show that if a net $\{f_\iota\}$ is τ-convergent to f then it is also τ_s-convergent to f. To this end, let $\{f_\iota\}$ be such a net, $(x, x^*) \in dom f \times dom f^*$, $\varepsilon = d_f(x, x^*)$, and $\mu > 0$. The τ-lower semicontinuity of $\Delta_{\varepsilon+\mu}$ implies that $\Delta_{\varepsilon+\mu}(f) \subseteq \mathrm{Li}\,\Delta_{\varepsilon+\mu}(f_\iota)$. Since $(x, f(x), x^*) \in \Delta_{\varepsilon+\mu}(f)$, there exists a net $(x_\iota, f_\iota(x_\iota), x_\iota^*) \in \Delta_{\varepsilon+\mu}(f_\iota)$ converging to $(x, f(x), x^*)$. From this we get that $\lim f_\iota(x_\iota) = f(x)$ and $\limsup f_\iota^*(x_\iota^*) \leq f^*(x^*) + \mu$. Proposition 3.1 implies that $\{f_\iota\}$ τ_s-converges to f and this completes the proof of (1). The proof of (2) is similar. $\square$

4. Attouch–Wets convergence and ε-subdifferentials

Another useful topology on the set of closed, convex sets is that introduced by Attouch and Wets [4]; if convex functions are identified with their epigraphs, we obtain the Attouch-Wets topology on $\Gamma(X)$, denoted τ_{AW}. In [10], Theorem 3.6, this topology is characterized in terms of the "uniform lower semicontinuity on bounded sets" of the multifunction $\Delta : \Gamma(X) \rightrightarrows X \times R \times X^*$ (X is assumed to be Banach). In this section we show that a similar characterization is true in normed spaces, provided that we replace subdifferentials with ε-subdifferentials. The ideas of the proof are those in [10], with changes due to the fact that we work with ε-subdifferentials. We begin by recalling the necessary notions (for details see [4], [7], [5]).

Let X be a normed space. If A, C are subsets of X, the *excess of C over A* is defined by
$$e(C, A) = \sup_{c\in C} \inf_{a\in A} \|c - a\| = \sup_{c\in C} d(c, A).$$

For $\rho > 0$ we write $e_\rho(C, A) = e(C \cap \rho B, A)$. The *$\rho$-Hausdorff distance between A and C* is
$$\mathrm{haus}_\rho(A, C) = \max\{e_\rho(A, C), e_\rho(C, A)\}.$$

Finally we say that a sequence $\{f_n\}$ of functions from $\Gamma(X)$ converges to $f \in \Gamma(X)$ in the Attouch-Wets sense, denoted τ_{AW}-$\lim f_n = f$, if $\lim_{n\to\infty} \mathrm{haus}_\rho(epi f_n, epi f) = 0$ for each $\rho > 0$ (in fact, it is enough that this happens for any ρ larger than some positive number).

Lemma 4.1. *Let $0 < \varepsilon < 1/2$, $\delta > 0$ ($\delta \geq 0$ if X is Banach), and $r > \max\{2, \delta\}$. If f, $g \in \Gamma(X)$ verify $e_r(epi f, epi g) < \varepsilon^2$ and $e_{2r^2}(epi f^*, epi g^*) < \varepsilon^2$ then $e_r(\Delta_\delta(f), \Delta_\delta(g)) \leq \delta + 6r^2\varepsilon$.*

Proof. Let $(x, f(x), x^*) \in \Delta_\delta(f) \cap (rB \times [-r, r] \times rB^*)$. Then $|f^*(x^*)| \leq 2r^2$ and there exist $(y, t) \in epi g$ and $(y^*, t^*) \in epi g^*$ such that
$$\|y - x\| \leq \varepsilon^2, \quad |t - f(x)| \leq \varepsilon^2, \quad \|y^* - x^*\| \leq \varepsilon^2, \quad |t^* - f^*(x^*)| \leq \varepsilon^2.$$

It follows that

$$g(y) - f(x) \le \varepsilon^2, g^*(y^*) - f^*(x^*) \le \varepsilon^2, \text{ ezrm and } |\langle y^*, y\rangle - \langle x^*, x\rangle| \le 3r\varepsilon^2.$$

The above inequalities give

$$\begin{aligned} g^*(y^*) + g(y) &\le f^*(x^*) + f(x) + 2\varepsilon^2 \le \langle x^*, x\rangle + \delta + 2\varepsilon^2 \\ &\le \langle y^*, y\rangle + \delta + 3r\varepsilon^2 + 2\varepsilon^2 \le \langle y^*, y\rangle + \delta + 4r^2\varepsilon^2 \end{aligned}$$

and therefore $y^* \in \partial_{\delta+4r^2\varepsilon^2} g(y)$. From Proposition 2.1 it follows that there exist $(z, z^*) \in \partial_\delta g$ such that

$$\|z - y\| \le 2r\varepsilon, \ \|z^* - y^*\| \le 2r\varepsilon, \text{ ezrm and } |g(z) - g(y)| \le 4r^2\varepsilon.$$

We also have

$$f(x) - g(y) \le \langle x^*, x\rangle - f^*(x^*) + \delta + g^*(y^*) - \langle y^*, y\rangle \le 3r\varepsilon^2 + \varepsilon^2 + \delta \le 2r^2\varepsilon + \delta.$$

Thus

$$\|z - x\| \le 3r\varepsilon, \|z^* - x^*\| \le 3r\varepsilon, \text{ and } |g(z) - f(x)| \le \delta + 6r^2\varepsilon,$$

meaning that $e_r(\Delta_\delta(f), \Delta_\delta(g)) \le \delta + 6r^2\varepsilon$. □

Lemma 4.2. *Let $f \in \Gamma(X)$, $f(0) = 0$ and choose λ_0 as in Lemma 2.6. Let also $0 < \varepsilon < 1$, $0 < \delta < 1$, $\lambda \ge \lambda_0$, and $\rho = \max\{2r + 1, \lambda, \lambda r + 1\}$. Assume that $g \in \Gamma(X)$ satisfies $e_\rho(\Delta_\delta(f), \Delta_\delta(g)) < \varepsilon + \delta$. Then $\sup\{|f_\lambda(z) - g_\lambda(z)|;\ z \in rB\} \le 5\rho(\varepsilon + \delta)$.*

Proof. Let $z \in rB$ and let x be a δ-estimator for $f_\lambda(z)$. Choose x^* as in Lemma 2.6. Then $(x, f(x), x^*) \in \Delta_\delta(f) \cap (\rho B \times [-\rho, \rho] \times \rho B^*)$ and therefore there exists $(y, g(y), y^*) \in \Delta_\delta(g)$ such that

$$\|x - y\| \le \varepsilon + \delta, \quad |f(x) - g(y)| \le \varepsilon + \delta, \quad \|x^* - y^*\| \le \varepsilon + \delta.$$

Using the definition of $g_\lambda(z)$, the fact that x is a δ-estimator for $f_\lambda(z)$, and the above inequalities, we have

$$\begin{aligned} g_\lambda(z) - f_\lambda(z) &\le g(y) + \lambda\|z - y\| - f(x) - \lambda\|z - x\| + \delta \\ &\le \varepsilon + \delta + \lambda\|x - y\| + \delta \le \varepsilon + 2\delta + \lambda(\varepsilon + \delta) \le 2\rho(\varepsilon + \delta). \end{aligned}$$

From Lemma 2.6 (a) (v) it follows that $f_\lambda(z) \le \langle x^*, z\rangle + f(x) - \langle x^*, x\rangle$. Notice also that $\|y^*\| \le \lambda + \varepsilon + \delta$. Using the previous estimation for $f_\lambda(z)$ and (6), we have

$$\begin{aligned} f_\lambda(z) - g_{\lambda+\varepsilon+\delta}(z) &\le \langle x^*, z\rangle + f(x) - \langle x^*, x\rangle - \langle y^*, z - y\rangle - g(y) + \delta \\ &\le f(x) - g(y) + \langle x^* - y^*, z\rangle + \langle y^* - x^*, y\rangle + \langle x^*, y - x\rangle + \delta \\ &\le \varepsilon + \delta + r(\varepsilon + \delta) + (\varepsilon + \delta)(\rho + \varepsilon + \delta) + \rho(\varepsilon + \delta) + \delta \le 4\rho(\varepsilon + \delta). \end{aligned}$$

Finally, from this and Lemma 2.6 (b) we obtain that $f_\lambda(z) - g_\lambda(z) \le 5\rho(\varepsilon + \delta)$ which completes the proof of the lemma. □

We can now prove the main result of this section.

Theorem 4.3. *Let X be normed space and f, $f_n \in \Gamma(X)$, $n \geq 1$. The following statements are equivalent:*

(1) $f = \tau_{\mathrm{AW}}\text{-lim} f_n$;

(2) *for any $\varepsilon > 0$ there exists $\rho_0 > 0$ such that* $\limsup_{n\to\infty} e_\rho(\Delta_\varepsilon(f), \Delta_\varepsilon(f_n)) \leq \varepsilon$ *if* $\rho \geq \rho_0$.

Proof. The fact that (1) implies (2) follows immediately from Lemma 4.1 and a result established in [6] which asserts that the Fenchel transform (i.e. $f \mapsto f^*$ from $\Gamma(X)$ to $\Gamma(X^*)$) is continuous with respect to the Attouch-Wets topology.

The other implication follows from Lemma 4.2 and another result of Beer [8, Theorem 4.3] which asserts that $f = \tau_{\mathrm{AW}}\text{-lim} f_n$ if and only if there exists λ_0 such that, for any $\lambda \geq \lambda_0$, the sequence $\{(f_n)_\lambda\}$ converges uniformly on bounded sets to f_λ. □

Question. Is it possible to replace (2) in the above theorem with

(2′) *for any $\varepsilon > 0$ there exists $\rho_0 > 0$ such that* $\limsup_{n\to\infty} e_\rho(\Delta_\varepsilon(f), \Delta_\varepsilon(f_n)) = 0$ *if* $\rho \geq \rho_0$.

We conclude with an extension to normed spaces of a result from [3].

Theorem 4.4. *Let X be normed space and f, $f_n \in \Gamma(X)$, $n \geq 1$. Assume that $f = \tau_{\mathrm{AW}}\text{-lim} f_n$. Then, for any $\varepsilon > 0$,* $\partial_\varepsilon f = \text{gph-dist} \lim_{n\to\infty} \partial_\varepsilon f_n$ *(i.e.* $\lim_{n\to\infty} \mathrm{haus}_\rho(\partial_\varepsilon f, \partial_\varepsilon f_n) = 0$ *for ρ sufficiently large).*

Proof. From the proof of Lemma 4.1 it follows that $\lim_{n\to\infty} e_\rho(\partial_\varepsilon f, \partial_\varepsilon f_n) = 0$ for ρ large enough. The arguments given in the proof of Theorem 2.3 of [3] can be adapted to our context as in the cases discussed above and one can show that $\lim_{n\to\infty} e_\rho(\partial_\varepsilon f_n, \partial_\varepsilon f) = 0$ too.

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