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An Attempt of Characterization of Functions With Sharp Weakly Complete Epigraphs

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We attempt to characterize the property, for a convex function f, "Epi$f$ is sharp and weakly complete" by a property of the polar function. Some examples show that a result completely analogous to the J.L. Joly characterization of "Epi$f$ is sharp and weakly locally compact" seems impossible. We use an old (1962) result of G. Choquet.

Keywords : Weakly complete sets, sharp epigraphs, locally convex linear spaces, duality

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1. Introduction

G. Choquet studied in 1962 ([4]–[6]) the weakly complete convex subsets of a Hausdorff locally convex topological vector space (briefly *lctvs*). Those which are *sharp* (i.e. which contain no lines) have some of the properties of sharp weakly locally compact ones, this in spite of the fact that they are more numerous. This comes from the fact that they can be embedded in a product $[0,+\infty[^J$ where the compactness of closed bounded intervals is a useful argument. C. Lescarret [9] used G. Choquet's results some years ago (he used the fact that when the sum of two epigraphs is closed the infimum convolution of the functions is exact and lower semi-continuous — briefly *lsc*). In a forthcoming paper [12] M. Volle also uses G. Choquet's results. We will consider, for a convex function f, the property "Epi$f$ is sharp and weakly complete" and will try to give a characterization of this property using the polar function f^*. The result is not as good as J.L. Joly's characterization of "Epi$f$ is sharp and weakly locally compact". The Joly characterization is ([8, Ch.I §§4–5], [3, Th.I.14 pp. 15–16]) : "the polar f^* is finite and continuous for the Mackey topology at least in one point." We do not obtain an equivalence such as "Epi$f$ is sharp and weakly complete" $\Longleftrightarrow$ "f^* is finite on an affinely generative convex set," but the result is not so far from this equivalence.

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2. Notations

If E is a Hausdorff lctvs, E' denotes its topological dual and E^* its algebraic dual. Hence

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E' is the set of all continuous linear forms on E. There is a canonical injection of E into E'^* and the weak topology $\sigma(E'^*, E')$ induces on E the weak topology $\sigma(E, E')$. Moreover E'^* is weakly complete and can be identified to the completion[1] of $(E, \sigma(E, E'))$ (we will not use this result; more generally see the Grothendieck theorem about the completion of a lctvs [2, Ch. III 3 n°6 p. III.20]). We denote by $\mathrm{Vect}(A)$ the vector subspace of E generated by a subset A of E. Let K be a given set. Then $\mathbb{R}^K$ denotes the classical product space of all families $(x_k)_{k\in K}$ of real numbers and $\mathbb{R}^{(K)}$ the vector subspace of $\mathbb{R}^K$ (which is a strict subspace only when K is an infinite set) of families $(y_k)_{k\in K}$ with finite supports. The spaces $\mathbb{R}^K$ and $\mathbb{R}^{(K)}$ are in canonical duality thanks to the bilinear form

$$((x_k)_{k\in K}, (y_k)_{k\in K}) \mapsto \sum_{k\in K} x_k\, y_k$$

With this bilinear form, $\mathbb{R}^K$ is a representation[2] of the algebraic dual of $\mathbb{R}^{(K)}$. Observe that the weak topology $\sigma(\mathbb{R}^K, \mathbb{R}^{(K)})$ is nothing else than the product topology of $\mathbb{R}^K$. If F is a vector space on the field $\mathbb{R}$ and $(f_k)_{k\in K}$ is an (algebraic) basis of F, then the dual pair (F^*, F) is isomorphic to the pair $(\mathbb{R}^K, \mathbb{R}^{(K)})$. A convex set is *sharp* if it does not contain any line. The *support function* of a subset X of E is

$$\delta^*(x' \mid X) = \sup\{\langle x', x\rangle : x \in X\}\ .$$

The notation $\Gamma_0(E)$ denotes the set of convex lsc functions on E which are *proper* (i.e. with values in $]-\infty, +\infty]$ and not identically the constant $+\infty$). If f is a function from E to $[-\infty, +\infty]$, its *domain* $\mathrm{Dom} f$ is $\{x \in E : f(x) < +\infty\}$ and its *polar* f^* is the function on E' defined by

$$f^*(x') = \sup\{\langle x', x\rangle - f(x) : x \in E\}\ .$$

3. Main result and examples

Before giving our characterization, recall that C. Lescarret [9, just before Prop.3] says "Epi$f$ is sharp and weakly complete is equivalent to : the closed affine subspace spanned by $\mathrm{Dom} f^*$ is E' and the set of affine continuous functions less than f^* is closed in $\mathbb{R}^{E'}$ for the pointwise convergence topology.""

The following Lemma is an easy result, even in infinite dimension.

Lemma 3.1. *Let F be a vector space and $(e_i)_{i\in I}$ a family in F. The following properties are equivalent:*

(i) *the family $(e_i)_{i\in I}$ is affinely free and generative in F (i.e., for any fixed i_0, the family $(e_i - e_{i_0})_{i\in I\setminus\{i_0\}}$ is a basis of F).*

[1] It would be better to say E'^* is a *realization* of the completion of E.

[2] For the notion of *representation* of a dual and the notion of *isomorphism* of duality pairs see [10, pp.2–3].

(ii) *the family* $((e_i, -1))_{i\in I}$ *is an algebraic basis of* $F \times \mathbb{R}$.

Theorem 3.2. *Let* $f \in \Gamma_0(E)$. *Consider the following property:*

(P) *there exists a family* $(e_i)_{i\in I}$ *which is affinely generative in* E' *and which verifies* $\forall i \in I$, $f^*(e_i)$ *is finite.*

Then (P) *is a necessary condition for* Epi$f$ *to be sharp and weakly complete. Conversely if* E *is weakly complete,* (P) *is a sufficient condition for* Epi$f$ *to be sharp and weakly complete.*

Proof. 1) First suppose E weakly complete and (P). Replacing if necessary I by a smaller set (see for example [1, 7 n°1 Th.2 p.147]) one may suppose that the family $(e_i)_{i\in I}$ is affinely free and generative. By the Lemma, $((e_i, -1))_{i\in I}$ is a basis of $E' \times \mathbb{R}$. One knows that $f^*(e_i) = \delta^*((e_i, -1) \mid \text{Epi} f)$. Let identify $E' \times \mathbb{R}$ with $\mathbb{R}^{(I)}$, then $E \times \mathbb{R}$ may be identified with a subset of $\mathbb{R}^I$. Then Epi$f$ is sharp thanks to the inclusion

$$-\text{Epi} f \subset \prod_{i\in I} [-f^*(e_i), +\infty[\ .$$

Since E is weakly complete, $E \times \mathbb{R}$ and its weakly closed subset Epi$f$ are also weakly complete (see Example 2 below which shows that the completeness of E is an essential hypothesis).

2) Now we suppose Epi$f$ sharp and weakly complete. From G. Choquet's result [4, Prop.1] (see the Comment below for the proof), there exists a basis $((e_i, r_i))_{i\in I}$ of $E' \times \mathbb{R}$ such that $\forall i$, $\delta^*((e_i, r_i) \mid \text{Epi} f)$ is finite. Necessarily the r_i are ≤ 0. If all the r_i are < 0, one has still a basis replacing each (e_i, r_i) by

$$\frac{1}{|r_i|}(e_i, r_i) = (\frac{1}{|r_i|}e_i, -1)$$

and the support function of Epi$f$ remains finite at these points. Hence we may suppose $r_i = -1$. Then $f^*(e_i)$ is finite and, thanks to the Lemma, the e_i are affinely free and generative. Now consider the case when some r_i are null. They cannot be all 0 since then $((e_i, r_i))_{i\in I}$ would span only a strict subspace of $E' \times \mathbb{R}$. Hence there exists i_0 verifying $r_{i_0} < 0$. Then the family :

$$((\varepsilon_i, \rho_i))_{i\in I} := ((e_i, r_i) + (e_{i_0}, r_{i_0}))_{i\in I}$$

is still a basis of $E' \times \mathbb{R}$. Indeed, for any finite subset A of I containing i_0, $((e_i, r_i) + (e_{i_0}, r_{i_0}))_{i\in A}$ is still a basis of Vect$(\{(e_i, r_i) : i \in A\})$ (elementarily, if $(b_1, \ldots, b_n)$ is a basis of a vector space, $(b_1 + b_n, b_2 + b_n, \ldots, b_n + b_n)$ is also a basis of the same space). Finally $\forall i$, $\delta^*((\varepsilon_i, \rho_i) \mid \text{Epi} f) < +\infty$, since the support function $\delta^*(\, . \mid \text{Epi} f)$ is sub-additive.

Comment. We give (following Bourbaki [2, exercise 16 in Ch.II §6, p.II.91]; and for sharp cones [7, vol.2 Prop.30.10 p.197]) a proof of G. Choquet's result in the case where the set X is sharp. So consider a sharp weakly complete convex set X in a lctvs G and let us prove that there exists a basis of G' at each element of which the support function of X is finite. Let X° denote the polar set in G'

$$X^\circ = \{x' \in G' : \forall x \in X,\ \langle x', x\rangle \le 1\} .$$

First assume $0 \in X$. Then, since X is $\sigma(G'^*, G')$ closed, its bipolar, $X^{\circ\circ}$, calculated in G'^*, equals X. We prove that $\mathrm{Vect}(X^\circ) = \mathrm{G}'$: Indeed, otherwise there would exist $x \in G'^* \backslash \{0\}$ such that x is null on $\mathrm{Vect}(X^\circ)$. But the line $\mathbb{R}\, x$ would be contained in $X^{\circ\circ}$. Finally there exists a basis $(g_i)_{i\in I}$ of $G' = \mathrm{Vect}(X^\circ)$ contained in X° (again this comes from [1, §7 n°1 Th.2 p.147]) and the support function of X verifies $\forall i,\ \delta^*(g_i \mid X) \le 1$. For a general X, consider $X_0 = X - x_0$, where x_0 is any point of X.

Example 3.3. (This example shows that the property "Epi$f$ is sharp and weakly complete" is strictly weaker than "Epi$f$ is sharp and weakly locally compact" and is more algebraic). Let K be an infinite set, $E = \mathbb{R}^K$ equipped with the product topology (hence $\mathbb{R}^{(K)}$ is a representation of E'), f the indicator function of $[0, +\infty[^K$. Then f^* is the indicator function of the cone $-([0, +\infty[^K \cap\ \mathbb{R}^{(K)})$. But this cone has no interior point for any vector space topology, hence has no interior point for the Mackey topology $\tau(\mathbb{R}^{(K)}, \mathbb{R}^K)$.

Example 3.4. (Showing that for the sufficiency, one cannot remove the hypothesis that E is weakly complete). Let K be an infinite set and $E = \mathbb{R}^{(K)}$ equipped with the topology $\sigma(\mathbb{R}^{(K)}, \mathbb{R}^{(K)})$. Thus $\mathbb{R}^{(K)}$ is a representation of E' and $\mathbb{R}^K$ is a representation of E'^*. Let f be the indicator of

$$C = -([0, +\infty[^K \cap\ \mathbb{R}^{(K)}) .$$

Then f^* is the indicator of the cone $[0, +\infty[^K \cap\ \mathbb{R}^{(K)}$. Condition (P) is verified taking the canonical basis of $\mathbb{R}^{(K)}$ and 0. But Epi$f = C \times [0, +\infty[$ is not weakly complete. This comes from the fact that C (or $-C$) is not closed in $\mathbb{R}^K$ For example, $(x_k)_{k\in K}$, defined by $x_k \equiv 1$, is the $\sigma(\mathbb{R}^K, \mathbb{R}^{(K)})$ limit of the generalized sequence $(x^A)_A$ whose indexes are the finite subsets of K, with $x^A_k = 1$ if $k \in A$, 0 otherwise.

Example 3.5. (Sharpness without weak completeness does not imply (P)). Let $E = \ell^2(\mathbb{N}^*)$ and

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2n}\, x_n^2 .$$

Clearly Epi$f$ is sharp. One can prove [3, Th.I.34 p.33] that Domf^* is the following strict vector subspace of $\ell^2(\mathbb{N}^*)$:

$$\mathrm{Dom} f^* = \{y \in \mathbb{R}^{\mathbb{N}^*} : \sum_{n=1}^{\infty} \frac{n}{2}\, y_n^2 < +\infty\}$$

and that, on Domf^*,

$$f^*(y) = \sum_{n=1}^{\infty} \frac{n}{2}\, y_n^2 \ .$$

Since Domf^* is strictly included in $E' = \ell^2(\mathbb{N}^*)$, (P) is not verified. The existence of a non-convergent Cauchy generalized sequence in Epi$f$ can also be proved as follows: There exists a linear form $\xi \in [\ell^2(\mathbb{N}^*)]^* \backslash \{0\}$ null on the vector subspace Domf^*. So for the bipolar f^{**} calculated in $[\ell^2(\mathbb{N}^*)]^*$, one has $f^{**}(\xi) = 0$. Indeed

$$f^{**}(\xi) := \sup \{\langle \xi, y\rangle - f^*(y) : y \in \mathrm{Dom} f^*\} \ ,$$

is 0 since f^* is ≥ 0 and $f^*(0) = 0$. Thus $(\xi, 0)$ is weakly adherent to Epi$f$. Hence there exists a generalized sequence in Epi$f$ which converges weakly to $(\xi, 0)$ in $[\ell^2(\mathbb{N}^*)]^* \times \mathbb{R}$. This generalized sequence is Cauchy but does not converge in Epi$f$.

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