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A New Approach to a Hyperspace Theory

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This article proposes a new method for a unified description of hypertopologies on the closed subsets of an arbitrary topological space X. In the metric case, it represents a natural generalization of the description of hypertopologies as weak topologies induced by gap functionals. The tools are a family of sets of real valued functions on X and a family of subsets of X. This method allows us to recover most of the standard hypertopologies, and naturally leads us to the definition of new hypertopologies.

Keywords : hyperspace, gap functional, weak topologies, Fell topology, Wijsman topology, Vietoris topology, Hausdorff metric topology, Mosco topology.

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1. Introduction

By hyperspace of a topological space (X, t) we mean the set of the closed subsets of X, $CL(X)$, endowed with a topology τ such that the function $i : (X, t) \to (CL(X), \tau)$ defined as $i(x) = \{x\}$ is a homeomorphism onto its image. Since the beginning of the century some hyperspace topologies, also called hypertopologies, have been introduced and investigated; in particular the Hausdorff metric and Vietoris topologies. These two topologies are very fine, at least in view of some applications, and this explains why in the last years several new hypertopologies were defined, aimed at applications in probability, statistics and variational problems, for instance. In minimum problems, also, functions are regarded as sets by identifying them with their epigraphs, and classical convergence notions either are too strong or do not have a good behaviour with respect to stability.

The impressive growth of the number of hypertopologies recently introduced and used in particular problems and the increasing interest to them for their great potentiality in different fields of applications explains the effort in understanding more sharply their

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structure and in finding general patterns to describe them. About this last aspect, we mention the papers [2], [9], [10], devoted to a description and classification of the hypertopologies as initial topologies, namely as the weakest topologies making continuous some families of real-valued functionals defined on $CL(X)$. Not only this is useful in order to have a common description of the hypertopologies, but also allows us to treat some application in a fairly general way ([3], [8]).

All the mentioned papers deal with the hyperspace of a metric space. The aim of this paper is to give a general classification of hypertopologies for a topological space X which is not necessarily metrizable. Most of our results hold for completely regular spaces, or even for spaces with least separation properties. It seems to us that the most striking motivation for doing this is the following: the space we could have to consider can already be a hyperspace! In this case we should work with $CL(CL(X))$. Let us mention an interesting possible application for this: information structures are usually defined as families of closed sets, namely as subsets of $CL(X)$. And observe that, starting from a metrizable space X, most of the known hypertopologies are at least completely regular (as initial topologies), while they are not usually metrizable. Moreover, our approach, being different from the previous ones, offers a new way of describing topologies which are not necessarily completely regular, as the Fell topology, when X is not locally compact. Finally, our method allows us to define in a natural way new topologies.

The paper is organized as follows: Section 2 introduces notions and the background material. In Section 3 we give the definitions leading to the idea of a hypertopology, and we collect the first related elementary properties. Sections 4 and 5 are dedicated to relate our definitions of topologies to hit-and-miss topologies and hit-and-strongly-miss topologies and to show how well-known hypertopologies can be described in our setting. Finally, Section 6 introduces some possible developments of the material of this paper.

2. Preliminaries and background material

In the sequel (X, t) is a topological space, supposed to be at least Hausdorff, and not consisting of a single point. $CL(X)$ and $K(X)$ denote respectively the set of the closed and of the compact subsets of X; if X is metrizable with complete distance d, $CLB(X)$ indicates the set of all closed and d-bounded subsets of X; if X is a normed linear space, $K_w(X)$ denotes the set of all weakly compact subsets of X, $CC(X)$ the closed convex subsets and $CBC(X)$ the closed convex bounded subsets. When the empty set is excluded, the previous sets are denoted $CL_0(X)$, $K_0(X)$, $CLB_0(X)$ and $K_{w,0}(X)$ respectively.

When X is metrizable with compatible distance d, $B(x, r)$ denotes the closed ball centered at x and with radius r, and $S(x, r)$ is the open ball with same center and radius. For a nonempty set $A \in CL(X)$ its distance from a point $x \in X$ is defined, as usual, as $d(x, A) = \inf_{a \in A} d(x, a)$. We define $d(x, \emptyset) = \sup_{y \in X} d(x, y)$, a definition which works better than the usual one $d(x, \emptyset) = +\infty$ when the metric d is bounded. The closed (open) ε-enlargements of a set A are, respectively, $B_\varepsilon[A] = \{x \in X : d(x, A) \leq \varepsilon\}$ and $S_\varepsilon[A] = \{x \in X : d(x, A) < \varepsilon\}$.

Given two sets A and B, A nonempty, we define the gap between them as

$$d(A, B) = \inf_{a \in A} d(a, B).$$

Observe that $d(A, B) > 0$ implies that two closed sets are disjoint and the vice versa holds too whenever one of the two sets is compact.

Now we introduce some terminology and notations to describe hypertopologies. For $V \subset X$, let:

$$V^- = \{A \in CL(X) : A \cap V \neq \emptyset\}$$

$$V^+ = \{A \in CL(X) : A \subset V\}$$

and if X is metrizable with metric d:

$$V^{++} = \{A \in CL(X) : \ S_\varepsilon[A] \subset V \ \text{for} \ \ \text{some} \ \varepsilon > 0\}.$$

Observe that V^{++} is a set depending on the choice of the metric, and not only on the topology on X. Moreover, if V is nonempty, then $V^{++} = \{A \in CL(X) : \ d(A, V^c) > 0\}$.

All the hypertopologies we shall introduce are the supremum of a lower part and an upper part. The former one can be characterized in the following way: an open set containing A contains also all the closed supersets of A. The opposite holds with upper topologies, that have also the property that if A and B belong to an open set, then their union too belongs to the same open set.

Clearly, sets as V^- are suitable to define lower topologies, while V^+ and V^{++} define upper topologies. More precisely, the family

$$CL(X) \cup \{V^- : V \text{ is open in } X\}$$

is a subbase for the lower Vietoris topology τ_V^-, which is also the lower topology for the bounded Vietoris τ_{bV}^-, proximal τ_p^-, bounded proximal τ_{bp}^-, ball τ_B^-, ball proximal τ_{pB}^-, Wijsman τ_W^- and Fell τ_F^- topologies. The upper Vietoris topology τ_V^+ has as a base the family of open sets

$$\{V^+ : V \text{ is open in X}\};$$

the upper bounded Vietoris τ_{bV}^+ (defined when a metric d on the space X is given)

$$\{(L^c)^+ : L \text{ is closed and bounded}\};$$

the upper ball topology τ_B^+ has as a subbase

$$\{(B^c)^+ : B \text{ is a closed ball}\};$$

the upper Fell topology τ_F^+ has as a base

$$\{(K^c)^+ : K \text{ is compact}\}.$$

All these upper topologies are miss topologies: a basic open set is made by all elements missing a particular closed set.

Not all known topologies can be described as the supremum of hit and miss topologies. For instance, the Hausdorff metric topology τ_H is rather defined in terms of open basic upper neighborhoods of a set A

$$\{B \in CL(X) : B \subset S_\varepsilon[A]\},$$

and open basic lower neighborhoods of a set A

$$\{B \in CL(X) : A \subset S_\varepsilon[B]\},$$

for $\varepsilon > 0$. Observe that the empty set is isolated in the upper topology, whereas $CL(X)$ is the only neighborhood of $\emptyset$ in the lower topology. It turns out that the upper part of the Hausdorff metric topology τ_H^+ can be also described in terms of strongly miss topology generated by the sets

$$\{V^{++} : \text{ V open in } X\}$$

(cf. [4]).

The bounded-Hausdorff topology can be defined in the following way: the open neighborhoods of a set $A \in CL(X)$ in the upper part τ_{bH}^+ are the sets of the form

$$\{B \in CL(X) : B \cap L \subset S_\varepsilon[A]\},$$

and dually, in the lower part τ_{bH}^- are the sets of the form

$$\{B \in CL(X) : A \cap L \subset S_\varepsilon[B]\}$$

when L ranges among the bounded subsets of X and $\varepsilon > 0$. τ_{bH}^+ coincides with the topology generated by $\{\emptyset\}$ and $(L^c)^{++}$, where L is a bounded set ([2]).

An useful way of generating new hypertopologies is to select a lower part of the previous topologies and an upper part of another topology and then taking their supremum. Thus we have the proximal and bounded proximal topologies having the lower Vietoris topology from one side and the Hausdorff and bounded Hausdorff topologies respectively on the other side.

A recent unifying way of presenting hypertopologies is to show that they are initial topologies for given families of geometric functionals defined on $CL(X)$. We shall see that, in the metric case, our approach describes upper parts of initial topologies in a very simple way.

A fundamental topology cannot be included in the previous framework, since its upper part can be described as a miss topology only in particular cases: this is the Wijsman topology. Recalling its original definition, a net $\{A_t\}_{t\in T}$, T a directed set, converges to $A \in CL(X)$ if $\lim d(x, A_t) = d(x, A)$ for every $x \in X$. Again, the lower part can be defined by considering the relation $\limsup d(x, A_t) \leq d(x, A)$ for every $x \in X$, and, dually, the upper part is provided by the inequalities $\liminf d(x, A_t) \geq d(x, A)$ for every $x \in X$. It turns out that the lower part coincides with the lower Vietoris topology, while the upper part in particular spaces (for instance spaces in which the balls are totally bounded) is the upper part of the ball proximal topology, generated by the sets $(B^c)^{++}$ when B ranges among the closed balls. Of course, the Wijsman topology is an initial topology.

There are other topologies, defined on the closed convex subsets of a linear space X and that are worth to be mentioned : the Mosco, slice and linear topologies. They have the lower Vietoris topology as their lower part, while the upper part of the Mosco topology is generated by the family

$$\{(wK^c)^+ : wK \text{ is a weakly compact set}\},$$

the upper part of the linear topology by the family

$$\{(S^c)^{++} : S \text{ is a convex set}\},$$

and the upper part of the slice topology by the family

$$\{(S^c)^{++} : S \text{ is a bounded convex set}\}.$$

For easy reference, we collect the hypertopologies introduced above together with their subdivision into lower and upper parts in the following table:

Topology	**Notation**	**Lower Part**	**Upper Part**	**Space**
Fell	τ_F	τ_V^-	τ_F^+	*topological*
Wijsman	τ_W	τ_V^-	τ_W^+	*metric*
Ball Proximal	τ_{pB}	τ_V^-	τ_{pB}^+	*metric*
Ball	τ_B	τ_V^-	τ_B^+	*metric*
b-Proximal	τ_{bp}	τ_V^-	τ_{bH}^+	*metric*
Proximal	τ_p	τ_V^-	τ_H^+	*metric*
b-Vietoris	τ_{bV}	τ_V^-	τ_{bV}^+	*metric*
Vietoris	τ_V	τ_V^-	τ_V^+	*topological*
b-Hausdorff	τ_{bH}	τ_{bH}^-	τ_{bH}^+	*metric*
Hausdorff	τ_H	τ_H^-	τ_H^+	*metric*
Slice	τ_S	τ_V^-	τ_S^+	*normed*
Mosco	τ_M	τ_V^-	τ_M^+	*normed*
Linear	τ_l	τ_V^-	τ_l^+	*normed*

The last three topologies are considered on the subspace of the closed convex subsets of X.

To conclude, we refer the reader to the book [1] for more about the above topologies, and for a complete reference list.

3. Basic definitions and properties

This section is devoted to the introduction of topological structures on $CL(X)$ and to the investigation of their basic properties.

Let $\mathcal{F}$ be a nonempty family of real-valued, lower bounded functions $f : X \to \mathbb{R}$, let r be a positive real number and P a subset of X. For a nonempty closed subset A of X, define:

$$\mathcal{N}^+(A, \mathcal{F}, r, P) = \{F \in CL(X) : \inf_{F\cap P} f > \inf_A f - r \ \forall f \in \mathcal{F}\},$$

$$\mathcal{N}^-(A, \mathcal{F}, r, P) = \{F \in CL(X) : \inf_{A\cap P} f > \inf_F f - r \ \forall f \in \mathcal{F}\},$$

with the agreement $\inf_\emptyset f = \sup_X f$, and:

$$\mathcal{N}^+(\emptyset, \mathcal{F}, r, P) = (P^c)^+ \cup \{F \in P^- : \inf_{F\cap P} f > \sup_X f - r \ \forall f \in \mathcal{F}\},$$

$$\mathcal{N}^-(\emptyset, \mathcal{F}, r, P) = CL(X).$$

Finally set

$$\mathcal{N}(A, \mathcal{F}, r, P) = \mathcal{N}^+(A, \mathcal{F}, r, P) \cap \mathcal{N}^-(A, \mathcal{F}, r, P).$$

The following properties are immediate to verify:

i) $A \in \mathcal{N}(A, \mathcal{F}, r, P)$;

ii) $\mathcal{F} \subset \mathcal{G} \Rightarrow \mathcal{N}^+(A, \mathcal{G}, r, P) \subset \mathcal{N}^+(A, \mathcal{F}, r, P)$
$\mathcal{F} \subset \mathcal{G} \Rightarrow \mathcal{N}^-(A, \mathcal{G}, r, P) \subset \mathcal{N}^-(A, \mathcal{F}, r, P)$;
(antitonicity with respect to the family of functions)

iii) $r \leq s \Rightarrow \mathcal{N}^+(A, \mathcal{F}, r, P) \subset \mathcal{N}^+(A, \mathcal{F}, s, P)$,
$r \leq s \Rightarrow \mathcal{N}^-(A, \mathcal{F}, r, P) \subset \mathcal{N}^-(A, \mathcal{F}, s, P)$;
(isotonicity with respect to the real variable)

iv) $P \subset Q \Rightarrow \mathcal{N}^+(A, \mathcal{F}, r, Q) \subset \mathcal{N}^+(A, \mathcal{F}, r, P)$,
$P \subset Q \Rightarrow \mathcal{N}^-(A, \mathcal{F}, r, Q) \subset \mathcal{N}^-(A, \mathcal{F}, r, P)$.
(antitonicity with respect to set P)

Now, let $\boldsymbol{\mathcal{F}}$ be a (nonempty) family of (nonempty) sets $\mathcal{F}$ of real-valued lower bounded functions on X with the following basic property:

(FU) $\quad \mathcal{F}_1, \mathcal{F}_2 \in \boldsymbol{\mathcal{F}} \Rightarrow \exists\, \mathcal{F} \in \boldsymbol{\mathcal{F}} : \ \mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}$,

and let $\boldsymbol{\mathcal{P}}$ be a (nonempty) family of subsets of X with the basic property:

(PU) $\quad P_1, P_2 \in \boldsymbol{\mathcal{P}} \Rightarrow \exists\, P \in \boldsymbol{\mathcal{P}} : P_1 \cup P_2 \subset P$.

In the sequel we shall always assume that (FU) and (PU) are verified.

Given $A \in CL(X)$, we consider the following families of (closed) sets:

$$\{\mathcal{N}^+(A, \mathcal{F}, r, P)\}_{(\mathcal{F} \in \boldsymbol{\mathcal{F}}, P \in \boldsymbol{\mathcal{P}}, r>0)},$$

$$\{\mathcal{N}^-(A, \mathcal{F}, r, P)\}_{(\mathcal{F} \in \boldsymbol{\mathcal{F}}, P \in \boldsymbol{\mathcal{P}}, r>0)},$$

$$\{\mathcal{N}(A, \mathcal{F}, r, P)\}_{(\mathcal{F} \in \boldsymbol{\mathcal{F}}, P \in \boldsymbol{\mathcal{P}}, r>0)} = \\ = \{\mathcal{N}^+(A, \mathcal{F}, r, P) \cap \mathcal{N}^-(A, \mathcal{F}, r, P)\}_{(\mathcal{F} \in \boldsymbol{\mathcal{F}}, P \in \boldsymbol{\mathcal{P}}, r>0)}.$$

These three families of closed sets are easily seen to form a local filter base at $A \in CL(X)$[2]. The first one gives rise to an upper convergence, denoted $c^+(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}})$, the second one to a lower convergence, denoted $c^-(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}})$, the third one to a convergence, denoted $c(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}})$, which is the supremum of the first two. When c is a topology, we shall rather use the symbol τ. We shall also indicate by $\mathcal{N}^+(A, \boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}})$ (resp. $\mathcal{N}^-(A, \boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}})$ and $\mathcal{N}(A, \boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}})$) the

[2] For this and other concepts in general topology, we refer to [6] and [5], especially Ch.2.

local filter at $A \in CL(X)$. Finally, for easy notation, we shall write $c(\boldsymbol{\mathcal{F}})$ (resp. $\tau(\boldsymbol{\mathcal{F}})$) when we use the family $\boldsymbol{\mathcal{P}}=\{X\}$.[3]

The next elementary result, which follows from the monotonicity properties mentioned above, is useful to compare topologies and/or convergences.

Write

$\boldsymbol{\mathcal{F}} \preceq \boldsymbol{\mathcal{G}}$ if for every $\mathcal{F} \in \boldsymbol{\mathcal{F}}$ there is $\mathcal{G} \in \boldsymbol{\mathcal{G}}$ with $\mathcal{F} \subset \mathcal{G}$, and

$\boldsymbol{\mathcal{P}} \preceq \boldsymbol{\mathcal{R}}$ if for every $P \in \boldsymbol{\mathcal{P}}$ there is $R \in \boldsymbol{\mathcal{R}}$ such that $P \subset R$,

$\boldsymbol{\mathcal{P}} \sim \boldsymbol{\mathcal{R}}$ if $\boldsymbol{\mathcal{P}} \succeq \boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{R}} \succeq \boldsymbol{\mathcal{P}}$.

Then we have:

Proposition 3.1.

$c^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}})$ is finer than $c^+(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{P}})$, $\forall\ \boldsymbol{\mathcal{P}}\ \forall\ \boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{G}}$ such that $\boldsymbol{\mathcal{F}} \succeq \boldsymbol{\mathcal{G}}$;

$c^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}})$ is finer than $c^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{R}})$, $\forall\ \boldsymbol{\mathcal{F}}\ \forall\ \boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{R}}$ such that $\boldsymbol{\mathcal{P}} \succeq \boldsymbol{\mathcal{R}}$.

The same holds with lower c^- and c convergences.

As a result, for a fixed $\boldsymbol{\mathcal{F}}$, the choice of $\boldsymbol{\mathcal{P}} = \{X\}$ gives the finest convergence (topology). Moreover, the families $\boldsymbol{\mathcal{P}} \sim \boldsymbol{\mathcal{P}}'$ generate the same topology. As an example, one cannot change topology by playing with $\boldsymbol{\mathcal{P}}$={bounded sets}, $\boldsymbol{\mathcal{P}}'$= {open bounded sets}, $\boldsymbol{\mathcal{P}}''$={closed bounded sets}, $\boldsymbol{\mathcal{P}}'''$={balls}.

A basic family of sets of functions is $\boldsymbol{\mathcal{C}}$:= { all finite (nonempty) sets of continuous lower bounded functions $f : X \to \mathbb{R}$}. Most (but not all) of our results deal with some $\boldsymbol{\mathcal{F}} \subset \boldsymbol{\mathcal{C}}$.

We shall use the following notation:

$$\cup\boldsymbol{\mathcal{F}} =: \{f : X \to \mathbb{R}\ \ : \exists\, \mathcal{F} \in \boldsymbol{\mathcal{F}} \text{ with } f \in \mathcal{F}\}.$$

The first interesting question is when the convergences $c^{\pm}$ are topological. The next two theorems provide an answer.

Theorem 3.2. *Suppose $\boldsymbol{\mathcal{F}} \subset \boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{P}} = \{X\}$. Then the convergences are topological.*

Proof. Let us prove it for the upper part.

Let $F \in \mathcal{N}^+(A,\mathcal{F},r,X) \cap \mathcal{N}^+(B,\mathcal{G},s,X)$. Choose ε with $0 < \varepsilon < \min\{r,s\}$ so that $\inf_F f > \inf_A f - r + \varepsilon\ \forall f \in \mathcal{F}$, and $\inf_F g > \inf_B f - s + \varepsilon\ \forall g \in \mathcal{G}$.

Let $\mathcal{F} \cup \mathcal{G} \subset \mathcal{H} \in \boldsymbol{\mathcal{F}}$ and let $t = \min\{\min\{r,s\} - \varepsilon, \varepsilon\}$. Then it is easy to see that $\mathcal{N}^+(F,\mathcal{H},t,X) \subset \mathcal{N}^+(A,\mathcal{F},r,X) \cap \mathcal{N}^+(B,\mathcal{G},s,X)$. □

[3] We use the following convention: bold (calligraphic) characters are used to indicate a family of subsets of X which is the specialization of $\boldsymbol{\mathcal{P}}$, even if, when merely considered as a set, it is indicated with the notation of Section 2. For instance, $CL(X)$ denotes the family of all closed nonempty subsets of X, but when we select this family as our family $\boldsymbol{\mathcal{P}}$, then we write $\boldsymbol{\mathcal{CL}}$ (having omitted the dependence on X for simplicity).

Theorem 3.3. *Let $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}}$ have the following properties:*

i) $\boldsymbol{\mathcal{F}} \subset \boldsymbol{\mathcal{C}}$;

ii) $\boldsymbol{\mathcal{P}} \subset CL(X)$, *$\boldsymbol{\mathcal{P}}$ covers X and $\boldsymbol{\mathcal{P}}$ contains all closed subsets of its elements;*

iii) *For every $A \in CL(X)$ and $P \in \boldsymbol{\mathcal{P}}$ with $P \neq \emptyset$ and $A \cap P = \emptyset$, there is $f \in \cup\boldsymbol{\mathcal{F}}$ such that $f \geq 0$ and $\sup_P f < \inf_A f$.*

Then the convergences are topological.

We do not prove the previous theorem, because in the next section we shall prove more: indeed, we shall characterize these topologies. Let us observe here that Assumption ii) is very strong in requiring that $\boldsymbol{\mathcal{P}}$ contains all closed subsets of its elements, while iii) is a sort of separation property: in particular, if $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{C}}$ and X is normal, then iii) holds even with $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{CL}}$ (see footnote [3] for the notation).

Another important question is when the topologies are admissible.

Proposition 3.4. *Suppose the families $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}} \subset CL(X)$ provide a topology, and moreover*

i) *$\boldsymbol{\mathcal{F}} \subset \boldsymbol{\mathcal{C}}$, and for every $x \in X$ and for every open set V in X with $x \in V$ there is $f \in \cup\boldsymbol{\mathcal{F}}$ such that $f(x) = 0$ and $\inf_{V^c} f > 0$,*

ii) *$\boldsymbol{\mathcal{P}}$ covers X.*

Then the topology $\tau(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}})$ is admissible.

Proof. We need to prove that $i : X \to CL(X)$ defined by $i(x) = \{x\}$ is a homeomorphism onto its image.

Let us show that i is an open mapping by showing that the image of an open subset V in X is

$$i(V) = i(X) \cap \bigcup_{x \in V} \mathcal{N}^-(x, \mathcal{F}_x, \frac{r_x}{2}, P_x),$$

where $\mathcal{F}_x \in \boldsymbol{\mathcal{F}}$ contains the function f_x such that $f_x(x) = 0$ and $\inf_{V^c} f_x = r_x > 0$, and P_x is an element of $\boldsymbol{\mathcal{P}}$ containing x.

The inclusion $i(V) \subset i(X) \cap \bigcup_{x \in V} \mathcal{N}^-(x, \mathcal{F}_x, \frac{r_x}{2}, P_x)$ is obvious. Conversely, let $\{y\} \in \mathcal{N}^-(z, \mathcal{F}_z, \frac{r_z}{2}, P_z)$ for some $z \in V$. Then $0 = f_z(z) = \inf_{w \in \{z\} \cap P_z} f_z(w) > f_z(y) - \frac{r_z}{2}$, hence $f_z(y) \leq \frac{r_z}{2}$ and $y \in V$.

For the continuity of i, it is not difficult to prove that

$$i^{-1}\{\mathcal{N}^+(A, \mathcal{F}, r, P)\} = P^c \cup \bigcap_{f \in \mathcal{F}} \{x : f(x) > \inf_A f - r\}$$

and

$$i^{-1}\{\mathcal{N}^-(A, \mathcal{F}, r, P)\} = \begin{cases} X & \text{if } A \cap P = \emptyset \\ \bigcap_{f \in \mathcal{F}} \{x : f(x) < \inf_{A \cap P} f + r\} & \text{otherwise} \end{cases}$$

□

We end this section with the presentation of a connection between our approach and the approach of [2] and [9, 10], where the hypertopologies are presented as weak topologies.

In the rest of the section, (X, d) is a metric space, $\boldsymbol{\mathcal{F}} \subset \boldsymbol{\mathcal{C}}$, and the family $\boldsymbol{\mathcal{P}}$ will be simply $\boldsymbol{\mathcal{P}}=\{X\}$. Remember that in this case, we shall indicate the topology by $\tau(\boldsymbol{\mathcal{F}})$ rather than by $\tau(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}})$.

There is a strict connection between the topology $\tau(\boldsymbol{\mathcal{F}})$ (for a suitable choice of $\boldsymbol{\mathcal{F}}$) and the weak topologies on $CL(X)$ induced by gap functionals of the form $d(E, \cdot)$ for $E \in \Omega$, Ω being a prescribed family of nonempty subsets of X. More precisely, the weak topology τ_{weak} induced on $CL(X)$ by the family of functionals $\{d(E, \cdot) : E \in \Omega\}$ is the topology $\tau(\boldsymbol{\mathcal{F}})$ when the elements of $\boldsymbol{\mathcal{F}}$ are all the sets $\{d(E_1, \cdot), \dots, d(E_n, \cdot)\}$ as $n \in \mathbb{N}^*$ and $E_i \in \Omega$. The lower and the upper parts of the weak topology τ_{weak} are exactly $\tau^-(\boldsymbol{\mathcal{F}})$ and $\tau^+(\boldsymbol{\mathcal{F}})$.

For example, if $\Omega = S = \{$ singleton subsets of $X\}$ and $\boldsymbol{\mathcal{DS}}= \{\{d(x_1, \cdot), \dots, d(x_n, \cdot)\}$: $n \in \mathbb{N}^*, x_i \in X\}$, then $\tau(\boldsymbol{\mathcal{DS}})=\tau_W$, $\tau^-(\boldsymbol{\mathcal{DS}}) = \tau_W^- = \tau_V^-$ and $\tau^+(\boldsymbol{\mathcal{DS}}) = \tau_W^+$. Another example is obtained when X is a reflexive Banach space.

Setting $\boldsymbol{\mathcal{DK}}_w = \{\{d(K_1, \cdot), \dots, d(K_n, \cdot)\} : n \in \mathbb{N}^*, K_i \in K_w(X)\}$, we have, on the set of the weakly closed subsets of X, $\tau(\boldsymbol{\mathcal{DK}}_w) = \tau_M$, $\tau^-(\boldsymbol{\mathcal{DK}}_w) = \tau_V^-$ and $\tau^+(\boldsymbol{\mathcal{DK}}_w) = \tau_M^+$ (cf. [2]).

More generally, define a family of functionals $\{F_i : i \in J\}$ on $CL(X)$ to be of *inf-type* if each F_i arises as inf of real valued lower bounded functions on X, i.e. if for every $i \in J$ there is a real valued lower bounded function f_i on X such that $F_i(A) = \inf_A f_i$ for every $A \in CL(X)$.

If $\{F_i : i \in J\}$ is a family of functionals on $CL(X)$ of inf-type, the weak topology on $CL(X)$ induced by $\{F_i : i \in J\}$ is the topology $\tau(\boldsymbol{\mathcal{F}})$ where the elements of $\boldsymbol{\mathcal{F}}$ are all finite sets $\{F_{i_1}, \dots, F_{i_n}\}$ as $n \in \mathbb{N}^*$ and $F_{i_h} \in \{F_i : i \in J\}$.

To conclude, as a consequence of Theorem 2.1 [2], we also have:

Theorem 3.5. *Let (X, d) be a metric space and let Ω be a class of nonempty closed subsets of X that is stable under enlargements (i.e. $clS_\varepsilon[E] \in \Omega$ for all $E \in \Omega$ and $\varepsilon > 0$) and that contains the singleton subsets of X. Let $\boldsymbol{\mathcal{D}}\Omega = \{\{d(E_1, \cdot), \dots, d(E_n, \cdot)\} : n \in \mathbb{N}^*, E_i \in \Omega\}$. Then the topology τ on $CL(X)$ having as a subbase all sets of the form $(E^c)^{++}$ where $E \in \Omega$, and V^-, V open in X, is the topology $\tau(\boldsymbol{\mathcal{D}}\Omega)$.*

Moreover $\tau^-(= \tau_V^-) = \tau^-(\boldsymbol{\mathcal{D}}\Omega)$ and $\tau^+ = \tau^+(\boldsymbol{\mathcal{D}}\Omega)$.

In particular when $\Omega = CL_0(X)$ or $\Omega = CLB_0(X)$, setting

$\boldsymbol{\mathcal{DC}}= \{\{d(E_1, \cdot), \dots, d(E_n, \cdot)\} : n \in \mathbb{N}^*, E_i \in CL_0(X)\}$

and

$\boldsymbol{\mathcal{DB}}= \{\{d(E_1, \cdot), \dots, d(E_n, \cdot)\} : n \in \mathbb{N}^*, E_i \in CLB_0(X)\}$,

then we have

$\tau(\boldsymbol{\mathcal{DC}}) = \tau_p$, $\tau^-(\boldsymbol{\mathcal{DC}}) = \tau_p^- = \tau_V^-$, $\tau^+(\boldsymbol{\mathcal{DC}}) = \tau_p^+ = \tau_H^+$ (the proximal topology)

and

$\tau(\boldsymbol{\mathcal{DB}}) = \tau_{bp}$, $\tau^-(\boldsymbol{\mathcal{DB}}) = \tau^-_{bp} = \tau^-_V$, $\tau^+(\boldsymbol{\mathcal{DB}}) = \tau^+_{bp} = \tau^+_{bH}$ (the bounded proximal topology).

4. Connections to hit-and-miss topologies

The following theorems show how to relate, in a general fashion, topologies described by our method to hit-and-miss topologies. Recall that we always assume the space X to be Hausdorff. Moreover Assumptions (FU) and (PU) are supposed to be verified.

Let us start with a general theorem concerning lower topologies.

Theorem 4.1. *Let $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}}$ satisfy the following properties:*

i) $\boldsymbol{\mathcal{F}} \subset \boldsymbol{\mathcal{C}}$;

ii) $\boldsymbol{\mathcal{P}}$ *covers* X;

iii) *For every $x \in X$ and every open set V containing x there is $f \in \cup\, \boldsymbol{\mathcal{F}}$ such that $f \geq 0$ on X and $0 = f(x) < \inf_{V^c} f$.*

Then the convergence $c^-(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}})$ is topological, i.e. $\tau^-(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}})$ exists and $\tau^-(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}}) = \tau^-_V$, the lower Vietoris topology.

Proof. Notice first that in both topologies the unique neighborhood of the empty set is $CL(X)$. Therefore it is enough to compare the neighborhoods of a nonempty subset A of $CL(X)$.

Given $\emptyset \neq A \in CL(X)$, $\mathcal{F} = \{f_1, \dots, f_m\} \in \boldsymbol{\mathcal{F}}$, $r > 0$ and $P \in \boldsymbol{\mathcal{P}}$, we first show that there are open sets $V_1, \dots, V_n$ in X such that $A \in V_1^- \cap \dots \cap V_n^- \subset \mathcal{N}^-(A, \mathcal{F}, r, P)$. If $A \cap P = \emptyset$, then $\mathcal{N}^-(A, \mathcal{F}, r, P) \supset CL_0(X) = X^-$. If $A \cap P \neq \emptyset$, choose $a_i \in A \cap P$ so that $f_i(a_i) < \inf_{A\cap P} f_i + \frac{r}{2}$ for $i = 1, \dots, m$. For $i = 1, \dots, m$, let $V_i = \{x \in X : f_i(x) < f_i(a_i) + \frac{r}{2}\}$, a nonempty open set in X. As $a_i \in A \cap V_i$, then $A \in V_1^- \cap \dots \cap V_m^-$. Suppose $C \in V_1^- \cap \dots \cap V_m^-$: there are $c_1, \dots, c_m \in C$ so that $f_i(c_i) < \inf_{A\cap P} f_i + r$ for all i. Hence $\inf_C f_i - r < \inf_{A\cap P} f_i$, i.e. $V_1^- \cap \dots \cap V_m^- \subset \mathcal{N}^-(A, \mathcal{F}, r, P)$.

On the other hand, take $\emptyset \neq A \in CL(X)$ and open sets V_i such that $A \in V_1^- \cap \dots \cap V_n^-$. Let $a_i \in A \cap V_i$, $P_i \in \boldsymbol{\mathcal{P}}$ with $a_i \in P_i$, and $f_i \in \cup\, \boldsymbol{\mathcal{F}}$ with $f_i \geq 0$ on X and $0 = f_i(a_i) < r_i = \inf_{V_i^c} f_i$. Because of (PU) and (FU) we can select $P \in \boldsymbol{\mathcal{P}}$ and $\mathcal{F} \in \boldsymbol{\mathcal{F}}$ such that $P_1 \cup \dots \cup P_n \subset P$ and $\{f_1, \dots, f_n\} \subset \mathcal{F}$. Let $r = \min\{r_i : i = 1, \dots, n\}$: it is easy to show that $\mathcal{N}^-(A, \mathcal{F}, r, P) \subset V_1^- \cap \dots \cap V_n^-$. □

Theorem 4.2. *Let $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{F}}$ be with the following properties :*

i) $\boldsymbol{\mathcal{F}} \subset \boldsymbol{\mathcal{C}}$;

ii) *$\boldsymbol{\mathcal{P}} \subset CL(X)$ and $\boldsymbol{\mathcal{P}}$ contains all closed subsets of its elements;*

iii) *For every $A \in CL(X)$ and every $P \in \boldsymbol{\mathcal{P}}$ such that $A \cap P = \emptyset$ there is $f \in \cup\, \boldsymbol{\mathcal{F}}$ such that $\sup_P f < \inf_A f$.*

Then $\tau^+(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{P}}) = \tau^+(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{P}}) = \tau^+_{\boldsymbol{\mathcal{P}}}$, where $\tau^+_{\boldsymbol{\mathcal{P}}}$ is the topology with open base $(P^c)^+, P \in \boldsymbol{\mathcal{P}}$.

Remark 4.3. Condition ii) and (PU) imply $P_1 \cup P_2 \in \boldsymbol{\mathcal{P}}$, for every $P_1, P_2 \in \boldsymbol{\mathcal{P}}$. This in turn implies that $(P^c)^+, P \in \boldsymbol{\mathcal{P}}$ is an open base (and not only a subbase) for a topology.

Proof of the theorem
Let $A \in CL(X)$, $\{f_1, \ldots, f_n\} \in \boldsymbol{\mathcal{F}}$, $r > 0$, and $P \in \boldsymbol{\mathcal{P}}$. Let $E = \{x \in X : f_i(x) > \inf_A f_i - r/2$ for $i = 1, \ldots, n\}$, and let $P' = P \setminus E \in \boldsymbol{\mathcal{P}}$. Clearly $A \in (P'^c)^+$. Suppose $B \in (P'^c)^+$. Then $B \cap P \subset E$, so $\inf_{B\cap P} f_i > \inf_A f_i - r$ for all i, i.e. $B \in \mathcal{N}^+(A, \{f_1, \ldots, f_n\}, r, P)$. On the other hand, let $A \in CL(X)$ and $P \in \boldsymbol{\mathcal{P}}$ such that $A \in (P^c)^+$. We shall provide $P', r, \mathcal{F}$ such that $\mathcal{N}^+(A, \mathcal{F}, r, P') \subset (P^c)^+$. Any choice works if $P = \emptyset$. Suppose therefore $P \neq \emptyset$. By iii) there is $f \in \cup\, \boldsymbol{\mathcal{F}}$ with $\sup_P f < \inf_A f$. It is not difficult to prove that $\mathcal{N}^+(A, \mathcal{F}, r, P) \subset (P^c)^+$ for any $\mathcal{F} \in \boldsymbol{\mathcal{F}}$ with $f \in \boldsymbol{\mathcal{F}}$ and any $r > 0$ with $\inf_A f - r > \sup_P f$. This ends the proof. □

Remark 4.4.

1) If $\boldsymbol{\mathcal{R}}$ is equivalent to $\boldsymbol{\mathcal{P}}$, then $\tau^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{R}}) = \tau^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}}) = \tau^+_{\boldsymbol{\mathcal{P}}}$ by Proposition 3.4. Note that given $\boldsymbol{\mathcal{R}} \subset CL(X)$ there exists and is unique the family $\boldsymbol{\mathcal{P}}$ that satisfies ii) and is equivalent to $\boldsymbol{\mathcal{R}}$.
2) The proof shows that if $\tau^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}})$ exists and $\boldsymbol{\mathcal{P}}$ satisfies i) then $\tau^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{R}}) = \tau^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}}) \preceq \tau^+_{\boldsymbol{\mathcal{P}}}$ for all $\boldsymbol{\mathcal{R}} \sim \boldsymbol{\mathcal{P}}$, whereas if $\boldsymbol{\mathcal{F}}$ satisfies ii), then $\tau^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{R}}) = \tau^+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}}) \succeq \tau^+_{\boldsymbol{\mathcal{P}}}$ for all $\boldsymbol{\mathcal{R}} \sim \boldsymbol{\mathcal{P}}$.

When (X, d) is a metric space, as we already noticed, upper topologies can be defined by a subbase consisting of the sets $CL(X)$ and $(E^c)^{++}$, when E ranges among the elements of a fixed family of subsets in X.

We shall now characterize these upper topologies via families of finite sets of gap functions on X and families of finite sets of uniformly continuous functions on X whose sublevel sets are in a fixed family of subsets of X. More precisely, if Ω is a family of nonempty subsets of X, we shall consider

$\boldsymbol{\mathcal{D}}\Omega = \{\{d(E_1, \cdot), \ldots, d(E_n, \cdot)\} : n \in \mathbb{N}^*, E_i \in \Omega\}$

and

$\boldsymbol{\mathcal{UC}}\Omega = \{\{f_1, \ldots, f_n\} : n \in \mathbb{N}^*, f_i : X \to \mathbb{R}$ are uniformly continuous and inf-$\Omega\}$,

where $f : X \to \mathbb{R}$ is said to be *inf-*Ω if $\{x \in X : f(x) \leq r\} \in \Omega \cup \{\emptyset\}$ for all $r > 0$. In the particular case when $\Omega = CL_0(X)$, for simplicity we shall indicate the previous family by $\boldsymbol{\mathcal{UC}}$.

Theorem 4.5. *Let (X, d) be a metric space and let Ω, $\boldsymbol{\mathcal{P}} \subset CL(X)$ be families of subsets of X with the following properties:*

i) *$\boldsymbol{\mathcal{P}}$ contains all closed subsets of its elements;*
ii) *$\boldsymbol{\mathcal{P}}$ contains some ε-enlargement of its elements, i.e. for all $P \in \boldsymbol{\mathcal{P}}$ there is $\varepsilon > 0$ such that $B_\varepsilon[P] \in \boldsymbol{\mathcal{P}}$;*
iii) *Ω satisfies (PU) and $\boldsymbol{\mathcal{P}} \setminus \{\emptyset\} \subset \Omega$.*

Then $\tau^+(\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}})$ and $\tau^+(\boldsymbol{\mathcal{UC}},\boldsymbol{\mathcal{P}})$ exist and $\tau^+(\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}}) = \tau^+(\boldsymbol{\mathcal{UC}},\boldsymbol{\mathcal{P}}) = \tau^{++}_{\boldsymbol{\mathcal{P}}}$, the topology with subbase $(P^c)^{++}, P \in \boldsymbol{\mathcal{P}}$.

Proof. Let $\mathcal{N}_A^+$ be the filter of neighborhoods of $A \in CL(X)$ in $\tau_{\boldsymbol{\mathcal{P}}}^{++}$. We shall show that $\mathcal{N}_A^+ \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}}) \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{UC}},\boldsymbol{\mathcal{P}}) \preceq \mathcal{N}_A^+$ for all $A \in CL(X)$.

For every $P_1,\ldots,P_n \in \boldsymbol{\mathcal{P}}$ such that $A \in (P_1^c)^{++} \cap \ldots \cap (P_n^c)^{++}$, choose $r > 0$ such that $S_r[A] \subset P_i^c$ for all $i = 1,\ldots,n$, and let $\varepsilon_i > 0$ such that $B_{\varepsilon_i}[P_i] \in \boldsymbol{\mathcal{P}}$. Because of i) we can suppose $\varepsilon_i < \frac{r}{2}$. By (PU) for $\boldsymbol{\mathcal{P}}$ there is $P \in \boldsymbol{\mathcal{P}}$ such that $B_{\varepsilon_1}[P_1] \cup \ldots \cup B_{\varepsilon_n}[P_n] \subset P$. Observe that we can assume $P_i \neq \emptyset$ for all i, so from iii) $\{d(P_1,\cdot),\ldots,d(P_n,\cdot)\} \in \boldsymbol{\mathcal{D}}\Omega$. Let us now prove that $\mathcal{N}^+(A,\{d(P_1,\cdot),\ldots,d(P_n,\cdot)\},\frac{r}{2},P) \subset (P_1^c)^{++} \cap \ldots \cap (P_n^c)^{++}$. If $A \neq \emptyset$ and $B \in \mathcal{N}^+(A,\{d(P_1,\cdot),\ldots,d(P_n,\cdot)\},\frac{r}{2},P)$, then for $i = 1,\ldots,n$, $d(P_i, B \cap B_{\varepsilon_i}[P_i]) \geq d(P_i, B \cap P) > d(P_i, A) - \frac{r}{2} \geq d(S_r[A]^c, A) - \frac{r}{2} \geq \frac{r}{2}$. Hence $B \cap B_{\varepsilon_i}[P_i] \subset S_{\frac{r}{2}}[P_i]^c$. Thus $B \cap S_{\varepsilon_i}[P_i] \subset S_{\frac{r}{2}}[P_i]^c \cap S_{\frac{r}{2}}[P_i] = \emptyset$. If $A = \emptyset$, we have the same result. In fact $B \in (P^c)^+$ implies $B \cap S_{\varepsilon_i}[P_i] = \emptyset$ for all i as well. Therefore $\mathcal{N}_A^+ \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}})$.

$\mathcal{N}^+(A,\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}}) \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{UC}},\boldsymbol{\mathcal{P}})$ because $\boldsymbol{\mathcal{D}}\Omega \subset \boldsymbol{\mathcal{UC}}$.

Finally $\mathcal{N}^+(A,\boldsymbol{\mathcal{UC}},\boldsymbol{\mathcal{P}}) \preceq \mathcal{N}_A^+$. Suppose first $A \neq \emptyset$: for every $\{f_1,\ldots,f_n\} \in \boldsymbol{\mathcal{UC}}$, $r > 0$ and $P \in \boldsymbol{\mathcal{P}}$, choose $\varepsilon > 0$ such that $| f_i(x) - f_i(y) | < \frac{r}{2}$, for $i = 1,\ldots,n$, provided $d(x,y) < \varepsilon$. Let $Q = P \setminus S_\varepsilon[A]$. If $B \cap Q = \emptyset$, then $B \cap P \subset S_\varepsilon[A]$, so $\inf_{B\cap P} f_i > \inf_A f_i - r$. Indeed if $x \in B \cap P$, there is $a_x \in A$ such that $d(x,a_x) < \varepsilon$, hence $f_i(x) > f_i(a_x) - \frac{r}{2} \geq \inf_A f_i - \frac{r}{2}$. Therefore $A \in (Q^c)^{++} \subset (Q^c)^+ \subset \mathcal{N}^+ (A,\{f_1,\ldots,f_n\},r,P)$. If $A = \emptyset$, then $\emptyset \in (P^c)^{++} \subset (P^c)^+ \subset \mathcal{N}^+(\emptyset,\{f_1,\ldots,f_n\},r,P)$. □

Remark 4.6.

1) The proof shows that if $\tau^+(\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}})$ and $\tau^+(\boldsymbol{\mathcal{UC}},\boldsymbol{\mathcal{P}})$ exist and $\boldsymbol{\mathcal{P}}$ satisfies i), then $\tau^+(\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}}) \preceq \tau^+(\boldsymbol{\mathcal{UC}},\boldsymbol{\mathcal{P}}) \preceq \tau_{\boldsymbol{\mathcal{P}}}^{++}$.
2) The theorem cannot be directly applied to the family $\boldsymbol{\mathcal{P}}$ of the compact subsets of X because of ii), but from the proof one can see that the result holds for this family too, by using that $(P^c)^+ = (P^c)^{++}$ for every compact set P.

Theorem 4.7. *Let (X,d) be a metric space and let Ω and $\boldsymbol{\mathcal{P}}$ be families of subsets of X with the following properties:*

i) *Ω is closed under finite unions and stable under enlargements, i.e. $clS_\varepsilon[E] \in \Omega$ for all $E \in \Omega$ and $\varepsilon > 0$;*
ii) *$\boldsymbol{\mathcal{P}}$ contains all closed subsets of its elements;*
iii) *$\boldsymbol{\mathcal{P}}$ contains some ε-enlargement of its elements;*
iv) *$\Omega \subset \boldsymbol{\mathcal{P}}$.*

Then $\tau^+(\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}})$ and $\tau^+(\boldsymbol{\mathcal{UC}}\Omega,\boldsymbol{\mathcal{P}})$ exist and $\tau^+(\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}}) = \tau^+(\boldsymbol{\mathcal{UC}}\Omega,\boldsymbol{\mathcal{P}}) = \tau^+(\boldsymbol{\mathcal{UC}}\Omega) = \tau_\Omega^{++}$, the topology having as an open base $CL(X)$ and $(E^c)^{++}, E \in \Omega$.

Proof. Let $\mathcal{N}_A^+$ be the filter of neighborhoods of $A \in CL(X)$ in τ_Ω^{++}. We shall show that $\mathcal{N}_A^+ \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}}) \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{UC}}\Omega,\boldsymbol{\mathcal{P}}) \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{UC}}\Omega,\{X\}) \preceq \mathcal{N}_A^+$ for all $A \in CL(X)$.

For every $E \in \Omega$ with $A \in (E^c)^{++}$, let $r > 0$ such that $S_r[A] \subset E^c$ and $0 < \varepsilon < \frac{r}{2}$ such that $B_\varepsilon[E] \in \boldsymbol{\mathcal{P}}$. Then $\mathcal{N}^+(A,\{d(E,\cdot)\},\frac{r}{2},B_\varepsilon[E]) \subset (E^c)^{++}$ because, if $B \in$

$\mathcal{N}^+(A, \{d(E,\cdot)\}, \frac{r}{2}, B_\varepsilon[E])$, then $d(E, B\cap B_\varepsilon[E]) > d(E,A) - \frac{r}{2} \geq d(S_r[A]^c, A) - \frac{r}{2} \geq \frac{r}{2}$. Hence $B\cap S_\varepsilon[E] \subset S_{\frac{r}{2}}[E] \cap (S_{\frac{r}{2}}[E])^c = \emptyset$, i.e. $B \in (E^c)^{++}$ (and $(B_\varepsilon[E]^c)^+ \subset (E^c)^{++}$ if $A=\emptyset$). Thus $\mathcal{N}_A^+ \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}})$.

$\mathcal{N}^+(A,\boldsymbol{\mathcal{D}}\Omega,\boldsymbol{\mathcal{P}}) \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{UC}}\Omega,\boldsymbol{\mathcal{P}})$ because $\boldsymbol{\mathcal{D}}\Omega \subset \boldsymbol{\mathcal{UC}}\Omega$. In fact, for every $E\in\Omega$, $d(E,\cdot)$ is uniformly continuous on X and $\{x\in X : d(E,x)\leq r\} = B_r[E] \in \Omega$ because of i). $\mathcal{N}^+(A,\boldsymbol{\mathcal{UC}}\Omega,\boldsymbol{\mathcal{P}}) \preceq \mathcal{N}^+(A,\boldsymbol{\mathcal{UC}}\Omega,\{X\})$ is always true.

Finally, $\mathcal{N}^+(A,\boldsymbol{\mathcal{UC}}\Omega,\{X\}) \preceq \mathcal{N}_A^+$: for every $\{f_1,\ldots,f_n\}\in\boldsymbol{\mathcal{UC}}\Omega$ and $r>0$, consider $E = \cup_{i=1}^n \{x : f_i(x) \leq \inf_A f_i - \frac{r}{2}\}$. $E\in\Omega$ because f_i is inf-Ω for all i and Ω is closed under finite unions. Moreover $A \in (E^c)^{++}$: if $\varepsilon>0$ is so that $d(x,y)<\varepsilon$ implies $\mid f_i(x) - f_i(y)\mid < \frac{r}{2}$ for all i, then $S_\varepsilon[A] \subset E^c$. Furthermore $(E^c)^+ \subset \mathcal{N}^+(A,\{f_1,\ldots,f_n\},r)$, because $B\in (E^c)^+$, then $f_i(b) > \inf_A f_i - \frac{r}{2}$ for all $b\in B$ and all i, so $\inf_B f_i > \inf_A f_i - \frac{r}{2}$, i.e. $B\in\mathcal{N}^+(A,\{f_1,\ldots,f_n\},r)$. □

5. Connections.

In this section we show that the general definition of hypertopologies given in Section 2 allows us to recover known hypertopologies by selecting suitable families $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}}$.

We first list the families $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}}$ we use.

Families $\boldsymbol{\mathcal{F}}$ of sets of real-valued lower-bounded functions on X:

$\boldsymbol{\mathcal{C}}=\{\{f_1,\ldots,f_n\} : n\in\mathbb{N}^*,\ f_i : X\to\mathbb{R}$ continuous $\}$,

and when (X,d) is a metric space

$\boldsymbol{\mathcal{UC}}=\{\{f_1,\ldots,f_n\} : n\in\mathbb{N}^*,\ f_i : X\to\mathbb{R}$ uniformly continuous $\}$,

$\boldsymbol{\mathcal{UCB}}=\{\{f_1,\ldots,f_n\} : n\in\mathbb{N}^*,\ f_i : X\to\mathbb{R}$ uniformly continuous and inf- $CLB(X)\}$,

$\boldsymbol{\mathcal{DC}}=\{\{d(E_1,\cdot),\ldots,d(E_n,\cdot)\} : n\in\mathbb{N}^*, E_i\in CL_0(X)\}$,

$\boldsymbol{\mathcal{DB}}=\{\{d(E_1,\cdot),\ldots,d(E_n,\cdot)\} : n\in\mathbb{N}^*, E_i\in CLB_0(X)\}$,

$\boldsymbol{\mathcal{DS}}=\{\{d(x_1,\cdot),\ldots,d(x_n,\cdot)\} : n\in\mathbb{N}^*, x_i\in X\}$,

$\boldsymbol{\mathcal{D}}=\{\{d(x,\cdot) : x\in X\}\}$.

When X is a normed space we also consider

$\boldsymbol{\mathcal{UCK}}_w\boldsymbol{\mathcal{C}}=\{\{f_1,\ldots,f_n\} : n\in\mathbb{N}^*, f_i : X\to\mathbb{R}$ uniformly continuous and inf-$K_wC(X)\}$,

$\boldsymbol{\mathcal{UCK}}_w=\{\{f_1,\ldots,f_n\} : n\in\mathbb{N}^*, f_i : X\to\mathbb{R}$ uniformly continuous and inf-$K_w(X)\}$,

$\boldsymbol{\mathcal{DK}}_w=\{\{d(E_1,\cdot),\ldots,d(E_n,\cdot)\} : n\in\mathbb{N}^*, E_i\in K_{w,0}(X)\}$,

$\boldsymbol{\mathcal{DK}}_w\boldsymbol{\mathcal{C}}=\{\{d(E_1,\cdot),\ldots,d(E_n,\cdot)\} : n\in\mathbb{N}^*, E_i\in K_wC_0(X)\}$,

$\boldsymbol{\mathcal{DBC}}$=$\{\{d(E_1,\cdot),\ldots,d(E_n,\cdot)\} : n \in \mathbb{N}^*, E_i \in CBC_0(X)\}$.

$\boldsymbol{\mathcal{DCC}}$=$\{\{d(E_1,\cdot),\ldots,d(E_n,\cdot)\} : n \in \mathbb{N}^*, E_i \in CC_0(X)\}$.

where

$CC(X) = \{$ all finite unions of closed convex subsets of $X\}$

$CBC(X) = \{$ all finite unions of closed bounded and convex subsets of $X\}$

and

$K_wC(X) = \{$ all finite unions of weakly compact convex subsets of $X\}$.

Observe that all the families $\boldsymbol{\mathcal{F}}$ we consider but $\boldsymbol{\mathcal{D}}$ consist of finite sets of functions on X, and that $\boldsymbol{\mathcal{D}}$ consists of a single element. In particular, (FU) is always satisfied.

Families $\boldsymbol{\mathcal{P}}$ of subsets of X:

$\{X\} = \{$the singleton $X\}(\sim \boldsymbol{\mathcal{CL}})$,

$\boldsymbol{\mathcal{CLB}}$= $\{$the closed and bounded subsets of $X\}$,

$\boldsymbol{\mathcal{K}} = \{$the compact subsets of $X\}$,

and, when X is a normed space,

$\boldsymbol{\mathcal{K}}_w = \{$the weakly compact subsets of $X\}$.

All these families $\boldsymbol{\mathcal{P}}$ are closed under finite unions of their elements, hence they satisfy (PU).

We start by describing hypertopologies whose lower parts are usually strictly finer than the lower Vietoris topology.

Lemma 5.1.
For every $A \in CL(X)$, $S_\varepsilon[A] = \{y \in X : d(x,y) > d(A,x) - \varepsilon \;\; \forall x \in X\}$.

Proof. The proof is easy and is left to the reader. Observe only that in the case $A = \emptyset$, then $S_\varepsilon[A]$ is empty for each $\varepsilon > 0$ if d is unbounded, while this is not always the case if the distance is bounded. □

Corollary 5.2.
$\tau_H = \tau(\boldsymbol{\mathcal{D}},\{X\})$.
The same holds for lower and upper parts separately.

Corollary 5.3.
$\tau_{bH} = \tau(\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{CLB}})$.
The same holds for lower and upper parts separately.

We now turn our attention to topologies that can be described by families of finite subsets of functions.

As a consequence of Theorem 4.2. we immediately have:

$\tau^+(\mathcal{C})=\tau_V^+$, when X is normal;

$\tau^+(\mathcal{C},\mathcal{K}) = \tau_F^+$, when X is completely regular;

$\tau^+(\mathcal{C},\mathcal{CLB}) = \tau_{bV}^+$, when X is metric.

From Theorem 4.1 we have:
$\tau^-(\mathcal{C})=\tau^-(\mathcal{C},\mathcal{K}) = \tau^-(\mathcal{C},\mathcal{CLB}) = \tau_V^-$ if X is completely regular.

Consequently, we have:

Proposition 5.4.
If X is a normal space, then $\tau(\mathcal{C}) = \tau_V$;
If X is completely regular, then $\tau(\mathcal{C},\mathcal{K}) = \tau_F$;
If X is a metric space, then $\tau(\mathcal{C},\mathcal{CLB}) = \tau_{bV}$.

From Theorem 4.5. we get $\tau^+(\mathcal{UC}) = \tau^+(\mathcal{DC}) = \tau_p^+$ and $\tau^+(\mathcal{UC},\mathcal{CLB}) = \tau^+(\mathcal{DC},\mathcal{CLB}) = \tau_{bp}^+$.

Remark 4.6, 2) after the same theorem gives $\tau^+(\mathcal{UC},\mathcal{K}) = \tau^+(\mathcal{DC},\mathcal{K}) = \tau_F^+$.

Since $\{d(x,\cdot) : x \in X\} \subset \cup \mathcal{F}$, for $\mathcal{F}=\mathcal{UC}$ or $\mathcal{F}=\mathcal{DC}$, Theorem 4.1 implies $\tau^-(\mathcal{UC}) = \tau^-(\mathcal{DC}) = \tau^-(\mathcal{UC},\mathcal{CLB}) = \tau^-(\mathcal{DC},\mathcal{CLB}) = \tau^-(\mathcal{UC},\mathcal{K}) = \tau^-(\mathcal{DC},\mathcal{K}) = \tau_V^-$.

Therefore the following proposition holds:

Proposition 5.5. *If (X,d) is a metric space, then*
$\tau(\mathcal{UC}) = \tau(\mathcal{DC}) = \tau_p$;
$\tau(\mathcal{UC},\mathcal{CLB}) = \tau(\mathcal{DC},\mathcal{CLB}) = \tau_{bp}$;
$\tau(\mathcal{UC},\mathcal{K}) = \tau(\mathcal{DC},\mathcal{K}) = \tau_F$.

Remark 5.6. Since $\mathcal{DC} \subset \mathcal{UC}$, by Proposition 3.1., we have that the results of Proposition 5.5 hold for any $\mathcal{F}$ with $\mathcal{DC} \subset \mathcal{F} \subset \mathcal{UC}$.

We have already noticed that $\tau^+(\mathcal{DC},\mathcal{K}) = \tau_F^+$. As a matter of fact, for all families $\mathcal{F}$ we are considering, we have $\tau^+(\mathcal{F},\mathcal{K}) = \tau_F^+$. A general result is the following:

Theorem 5.7. *Let Ω be a family of subsets of (X,d) such that $S = \{$ singletons of $X\} \subset \Omega \subset CL_0(X)$. Then we have $\tau^+(\mathcal{D}\Omega,\mathcal{K}) = \tau_F^+$.*

Proof. Since $S \subset \Omega \subset CL_0(X)$, we have $\mathcal{DS} \subset \mathcal{D}\Omega \subset \mathcal{DC}$, so for all $A \in CL(X)$ $\mathcal{N}^+(A,\mathcal{DS},\mathcal{K}) \preceq \mathcal{N}^+(A,\mathcal{D}\Omega,\mathcal{K}) \preceq \mathcal{N}^+(A,\mathcal{DC},\mathcal{K}) =\mathcal{N}_A^+$, where $\mathcal{N}_A^+$ is the filter of neighborhoods of A in τ_F^+.

We only need to show that $\mathcal{N}_A^+ \preceq \mathcal{N}^+(A,\mathcal{DS},\mathcal{K})$. Let K be a nonempty compact subset of X such that $A \in (K^c)^+$ and let $r > 0$ be such that $d(A,K) = r$. Choose $x_1,\dots,x_n \in K$ such that $K \subset \cup_{i=1}^n B(x_i,\frac{r}{2})$. Then $\mathcal{N}^+(A,\{d(x_1,\cdot),\dots,d(x_n,\cdot)\},\frac{r}{2},K) \subset (K^c)^+$. In fact, if $d(x_i, B \cap K) > d(x_i, A) - \frac{r}{2} \geq d(K, A) - \frac{r}{2} = \frac{r}{2}$ for all i, then $B \cap K = \emptyset$; otherwise for $b \in B \cap K$, choose $j \in \{1, \dots, n\}$ with $b \in B(x_j, \frac{r}{2})$: then $d(x_j, B \cap K) \leq d(x_j, b) < \frac{r}{2}$, a contradiction. □

From Theorem 5.7. the following characterization for the upper Fell topology follows

$\tau^+(\mathcal{DS},\mathcal{K}) = \tau^+(\mathcal{DB},\mathcal{K}) = \tau^+(\mathcal{DC},\mathcal{K}) = \tau_F^+$ for (X, d) metric;

if X is normed, $\tau^+(\mathcal{DBC},\mathcal{K}) = \tau^+(\mathcal{DK}_w,\mathcal{K}) = \tau^+(\mathcal{DK}_w\mathcal{C},\mathcal{K}) = \tau_F^+$.

Since $\tau^-(\mathcal{DS},\mathcal{K}) = \tau^-(\mathcal{DB},\mathcal{K}) = \tau^-(\mathcal{DC},\mathcal{K}) = \tau_V^-$ and,

for X normed, $\tau^-(\mathcal{DBC},\mathcal{K}) = \tau_V^-$ according to Theorem 4.1., we have:

Proposition 5.8. *If (X, d) is a metric space, then*
$\tau(\mathcal{DS},\mathcal{K}) = \tau(\mathcal{DB},\mathcal{K}) = \tau(\mathcal{DC},\mathcal{K}) = \tau_F$;
if X is a normed space, then $\tau(\mathcal{DBC},\mathcal{K}) = \tau(\mathcal{DK}_w,\mathcal{K}) = \tau(\mathcal{DK}_w\mathcal{C},\mathcal{K}) = \tau_F$.

In order to complete the discussion of the topologies obtained when $\mathcal{P} = \mathcal{K}$, we have to analyze $\tau^\pm(\mathcal{UCB},\mathcal{K})$ and $\tau^\pm(\mathcal{D},\mathcal{K})$.

Proposition 5.9. *If (X, d) is a metric space, then*
$\tau^+(\mathcal{UCB},\mathcal{K}) = \tau_F^+$ and $\tau^-(\mathcal{UCB},\mathcal{K}) = \tau_V^-$.

Proof. For every $A \in CL(X)$, let $\mathcal{N}_A^+$ be the filter of neighborhoods of A in τ_F^+. Since $\mathcal{UCB} \subset \mathcal{UC}$, we have $\mathcal{N}^+(A,\mathcal{UCB},\mathcal{K}) \preceq \mathcal{N}^+(A,\mathcal{UC},\mathcal{K}) = \mathcal{N}_A^+$. For every nonempty compact set K with $A \in (K^c)^+$, let $d(A, K) = r > 0$. Hence $\mathcal{N}^+(A, \{d(K, \cdot)\}, \frac{r}{2}, K) \subset (K^c)^+$, because, if $B \in \mathcal{N}^+(A, \{d(K, \cdot), \frac{r}{2}, K)$, then $d(K, B \cap K) > d(A, K) - \frac{r}{2} = \frac{r}{2}$. $d(K, B \cap K) = 0$ if $B \cap K \neq \emptyset$, so we must have $B \cap K = \emptyset$. Since $\{d(K, \cdot)\} \in \mathcal{UCB}$, we thus get $\mathcal{N}_A^+ \preceq \mathcal{N}^+(A,\mathcal{UCB},\mathcal{K})$, which gives $\tau^+(\mathcal{UCB},\mathcal{K}) = \tau_F^+$
The result for the lower parts follows from Theorem 4.1. □

Proposition 5.10. *If (X, d) is a metric space, then*
$\tau^+(\mathcal{D},\mathcal{K}) = \tau_F^+$ and $\tau^-(\mathcal{D},\mathcal{K}) = \tau_V^-$.

Proof. For every $A \in CL(X)$, let $\mathcal{N}_A^+$ be the filter of neighborhoods of A in τ_F^+. As a consequence of Lemma 5.1, a base for the filter $\mathcal{N}^+(A,\mathcal{D},\mathcal{K})$ consists of the sets $\{B \in CL(X) : B \cap K \subset S_r[A]\}$, as $K \in K(X)$ and $r > 0$. Let $\mathcal{U}(A) = \{B \in CL(X) : B \cap K \subset S_r[A]\}$ for fixed $K \in K(X)$ and $r > 0$, and let $P = K \setminus S_r[A]$. Then P is compact and $A \in (P^c)^{++} = (P^c)^+ \subset \mathcal{U}(A)$ because if $B \cap P = \emptyset$, then $B \cap K \subset S_r[A]$. Therefore $\mathcal{N}^+(A,\mathcal{D},\mathcal{K}) \preceq \mathcal{N}_A^+$. Conversely $\mathcal{N}_A^+ = \mathcal{N}^+(A,\mathcal{DS},\mathcal{K}) \preceq \mathcal{N}^+(A,\mathcal{D},\mathcal{K})$ because $\mathcal{DS} \preceq \mathcal{D}$, since every element of $\mathcal{DS}$ is contained in the unique element $\{d(x, \cdot) : x \in X\}$ of $\mathcal{D}$. Thus $\tau^+(\mathcal{D},\mathcal{K}) = \tau_F^+$. For every $A \in CL(X)$, let $\mathcal{N}_A^-$ be the filter of neighborhoods of A in τ_V^-. A base for the filter $\mathcal{N}^-(A,\mathcal{D},\mathcal{K})$ consists of the sets $\{B \in CL(X) : A \cap K \subset S_r[B]\}$, as $K \in K(X)$ and $r > 0$. Let $\mathcal{V}(A) = \{B \in CL(X) : A \cap K \subset S_r[B]\}$ for fixed $K \in K(X)$

and $r > 0$. If $A \cap K = \emptyset$, then $\mathcal{V}(A) = CL(X) \in \mathcal{N}_A^-$. Suppose then $A \cap K \neq \emptyset$ and choose $x_1, \dots, x_n \in A \cap K$ such that $A \cap K \subset \cup_{i=1}^n B(x_i, \frac{r}{2})$. Clearly, $A \in \cap_{i=1}^n B(x_i, \frac{r}{2})^-$. Moreover $\cap_{i=1}^n B(x_i, \frac{r}{2})^- \subset \mathcal{V}(A)$, because if $B \in \cap_{i=1}^n B(x_i, \frac{r}{2})^-$, then there are $y_i \in B$ such that $d(x_i, y_i) < \frac{r}{2}$ for all i. Hence $S_r[B] \supset S_r[\{y_1, \dots, y_n\}] \supset A \cap K$. Therefore $\mathcal{N}^-(A,\mathcal{D},\mathcal{K}) \preceq \mathcal{N}_A^-$. Conversely, $\mathcal{N}_A^- = \mathcal{N}^-(A,\mathcal{DS},\mathcal{K}) \preceq \mathcal{N}^-(A,\mathcal{D},\mathcal{K})$ because $\mathcal{DS} \preceq \mathcal{D}$. Thus $\tau^-(\mathcal{D},\mathcal{K}) = \tau_V^-$. □

Corollaries 5.2. and 5.3. and Proposition 5.10. yield

Proposition 5.11. *If (X, d) is a metric space, then*
$\tau(\mathcal{D}) = \tau_H$;
$\tau(\mathcal{D},\mathcal{CLB}) = \tau_{bH}$;
$\tau(\mathcal{D},\mathcal{K}) = \tau_F$.

Theorem 4.7. allows us to analyze $\tau^+(\mathcal{DB},\mathcal{P})$ and $\tau^+(\mathcal{UCB},\mathcal{P})$ when $\mathcal{P}=\mathcal{CL}$ or $\mathcal{P}=\mathcal{CLB}$: namely

$\tau^+(\mathcal{DB}) = \tau^+(\mathcal{UCB}) = \tau_{bp}^+$ (with the choice $\mathcal{P}=\mathcal{CL}$, $\Omega = CLB_0(X)$),

and

$\tau^+(\mathcal{DB},\mathcal{CLB}) = \tau^+(\mathcal{UCB},\mathcal{CLB}) = \tau_{bp}^+$ (with the choice $\mathcal{P}=\mathcal{CLB}$, $\Omega = CLB_0(X)$).

Again, Theorem 4.1. gives:

$\tau^-(\mathcal{DB}) = \tau^-(\mathcal{UCB}) = \tau^-(\mathcal{DB},\mathcal{CLB}) = \tau^-(\mathcal{UCB},\mathcal{CLB}) = \tau_V^-$.

Therefore, with Propositions 5.8. and 5.9., we have

Proposition 5.12. *If (X, d) is a metric space, then*
$\tau(\mathcal{UCB}) = \tau(\mathcal{DB}) = \tau_{bp}$;
$\tau(\mathcal{UCB},\mathcal{CLB}) = \tau(\mathcal{DB},\mathcal{CLB}) = \tau_{bp}$;
$\tau(\mathcal{UCB},\mathcal{K}) = \tau(\mathcal{DB},\mathcal{K}) = \tau_F$.

Remark 5.13. Since $\mathcal{DB} \subset \mathcal{UCB}$, we have that the results of Proposition 5.12 hold for any $\mathcal{F}$ with $\mathcal{DB} \subset \mathcal{F} \subset \mathcal{UCB}$.

Let us now consider the topologies generated by $\mathcal{DS}$.

We already know from Theorem 5.7. and from the discussion preceding Proposition 5.8. that $\tau^+(\mathcal{DS},\mathcal{K}) = \tau_F^+$ and $\tau^-(\mathcal{DS},\mathcal{K}) = \tau_V^-$. $\tau^+(\mathcal{DS}) = \tau_W^+$ and $\tau^-(\mathcal{DS}) = \tau_V^- = \tau_W^-$ immediately follow from the definition. Moreover Theorem 4.1. implies that $\tau^-(\mathcal{DS},\mathcal{CLB}) = \tau_V^-$. It remains to analyze $\tau^+(\mathcal{DS},\mathcal{CLB})$.

Proposition 5.14.
$\tau^+(\mathcal{DS},\mathcal{CLB}) = \tau_W^+$.

Proof. For every $A \in CL(X), x_1, \ldots, x_n \in X$ and $r > 0$, $\{B \in CL(X) : d(x_i, B) > d(x_i, A) - r \ \forall i = 1, \ldots, n\} \supset \{B \in CL(X) : d(x_i, B \cap E) > d(x_i, A) - r \ \forall i = 1, \ldots, n\}$ $= \mathcal{N}^+(A, \{d(x_1, \cdot), \ldots, d(x_n, \cdot)\}, r, E)$, where $E = \cup_{i=1}^n C(x_i, s_i) \in CLB(X)$, with $s_i = d(x_i, A) - \frac{r}{2}$ and

$$C(x_i, s_i) = \begin{cases} \{y \in X : d(x_i, y) \leq s_i\} & \text{if } s_i \geq 0 \\ \emptyset & \text{if } s_i < 0 \end{cases}$$

In fact, if $d(x_i, B \cap E) > d(x_i, A) - r$, also $d(x_i, B) > d(x_i, A) - r$ for all i: if $B = \emptyset$ the property is true, whereas if $B \neq \emptyset$ at least one between $B \setminus E$ and $B \cap E$ is nonempty. If $b \in B \setminus E$, then $d(b, x_i) > s_i = d(x_i, A) - \frac{r}{2}$ for all i, so $d(B \setminus E, x_i) \geq d(x_i, A) - \frac{r}{2} > d(x_i, A) - r$. Thus $d(B, x_i) = \min\{d(B \setminus E, x_i), d(B \cap E, x_i)\} > d(x_i, A) - r$ for all i. The converse is obvious because $\tau^+(\boldsymbol{\mathcal{DS}}) = \tau_W^+$. □

We therefore have:

Proposition 5.15. *If (X, d) is a metric space, then*
$\tau(\boldsymbol{\mathcal{DS}}) = \tau(\boldsymbol{\mathcal{DS}}, \boldsymbol{\mathcal{CLB}}) = \tau_W$;
$\tau(\boldsymbol{\mathcal{DS}}, \boldsymbol{\mathcal{K}}) = \tau_F$.

When X is a normed space we have seen that $\tau^+(\boldsymbol{\mathcal{DBC}}, \boldsymbol{\mathcal{K}}) = \tau_F^+$. From Theorem 4.1. we have $\tau^-(\boldsymbol{\mathcal{DBC}}) = \tau^-(\boldsymbol{\mathcal{DBC}}, \boldsymbol{\mathcal{CLB}}) = \tau^-(\boldsymbol{\mathcal{DBC}}, \boldsymbol{\mathcal{K}}) = \tau_V^-$. From Theorem 4.7. we have $\tau^+(\boldsymbol{\mathcal{DBC}}, \boldsymbol{\mathcal{CLB}}) = \tau_S^+$ (the slice topology), and from the definition, also $\tau^+(\boldsymbol{\mathcal{DBC}}) = \tau_S^+$. A similar argument applies to the linear topology

Proposition 5.16. *If X is a normed space, then*
$\tau(\boldsymbol{\mathcal{DCC}}) = \tau_l$;
$\tau(\boldsymbol{\mathcal{DBC}}) = \tau(\boldsymbol{\mathcal{DBC}}, \boldsymbol{\mathcal{CLB}}) = \tau_S$;
$\tau(\boldsymbol{\mathcal{DBC}}, \boldsymbol{\mathcal{K}}) = \tau_F$.

The characterization of the upper Mosco topology on the weakly closed subsets of a reflexive Banach space X is obtained from Theorems 4.5. and 4.7. In fact, the weak lower semicontinuity of the norm implies that τ_M^+ has open basis $(K^c)^{++}$ when K ranges among the weakly compact subsets of X.
Chosing $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{K}}_w$ and $\Omega \in \{CL_0(X), CLB_0(X), K_{w,0}(X)\}$, we have from Theorem 4.5.
$\tau^+(\boldsymbol{\mathcal{UC}}, \boldsymbol{\mathcal{K}}_w) = \tau^+(\boldsymbol{\mathcal{DC}}, \boldsymbol{\mathcal{K}}_w) = \tau^+(\boldsymbol{\mathcal{DK}}_w, \boldsymbol{\mathcal{K}}_w) = \tau_M^+$.
Choosing $\boldsymbol{\mathcal{P}} \in \{\boldsymbol{\mathcal{CL}}, \boldsymbol{\mathcal{CLB}}, \boldsymbol{\mathcal{K}}_w\}$ and $\Omega = K_{w,0}(X)$, we have from Theorem 4.7.
$\tau^+(\boldsymbol{\mathcal{DK}}_w) = \tau^+(\boldsymbol{\mathcal{DK}}_w, \boldsymbol{\mathcal{CLB}}) = \tau^+(\boldsymbol{\mathcal{DK}}_w, \boldsymbol{\mathcal{K}}_w) = \tau_M^+$ and
$\tau^+(\boldsymbol{\mathcal{UCK}}_w) = \tau^+(\boldsymbol{\mathcal{UCK}}_w, \boldsymbol{\mathcal{CLB}}) = \tau^+(\boldsymbol{\mathcal{UCK}}_w, \boldsymbol{\mathcal{K}}_w) = \tau_M^+$.

Reflexivity of the space X also implies that τ_M^+ can be described by the open subbase $(K^c)^{++}$ for K weakly compact and convex.

Hence Theorem 4.7. with $\mathcal{P} = \mathcal{K}_w$ and $\Omega = K_{w,0}C(X)$ gives

$\tau^+(\mathcal{DK}_w\mathcal{C},\mathcal{K}_w) = \tau^+(\mathcal{UCK}_w\mathcal{C},\mathcal{K}_w) = \tau^+(\mathcal{UCK}_w\mathcal{C}) = \tau_M^+$.

According to Theorem 4.1. and Proposition 3.1. we therefore have

Proposition 5.17. *If X is a reflexive Banach space, then on the closed and convex subsets of X*

$\tau(\mathcal{F},\mathcal{K}_w) = \tau_M$ *for any* $\mathcal{F}$ *with* $\mathcal{K}_w\mathcal{C} \subset \mathcal{F} \subset \mathcal{DC}$ *or* $\mathcal{UCK}_w\mathcal{C} \subset \mathcal{F} \subset \mathcal{UC}$;

$\tau(\mathcal{UCK}_w) = \tau(\mathcal{UCK}_w,\mathcal{CLB}) = \tau_M$;

$\tau(\mathcal{DK}_w) = \tau(\mathcal{DK}_w,\mathcal{CLB}) = \tau_M$.

The results so far obtained are collected in the following table.

Topology	Functions' family	Sets' family	Space
Fell	$\mathcal{C}$	$\mathcal{K}$	*completely regular*
	$\mathcal{DS} \subset \mathcal{D}\Omega \subset \mathcal{DC}$	"	*metric*
	$\mathcal{C} \subset \mathcal{F} \subset \mathcal{UC}$	"	"
	$\mathcal{DB} \subset \mathcal{F} \subset \mathcal{UCB}$	"	"
	$\mathcal{D}$	"	"
Wijsman	$\mathcal{DS}$	$\{\mathrm{x}\}$ *or* $\mathcal{CLB}$	*metric*
b-Proximal	$\mathcal{DC} \subset \mathcal{F} \subset \mathcal{UC}$	$\mathcal{CLB}$	*metric*
	$\mathcal{DB} \subset \mathcal{F} \subset \mathcal{UCB}$	$\{\mathrm{x}\}$ *or* $\mathcal{CLB}$	"
Proximal	$\mathcal{DC} \subset \mathcal{F} \subset \mathcal{UC}$	$\{\mathrm{x}\}$	*metric*
b-Vietoris	$\mathcal{C}$	$\mathcal{CLB}$	*metric*
Vietoris	$\mathcal{C}$	$\{\mathrm{x}\}$	*normal*
b-Hausdorff	$\mathcal{D}$	$\mathcal{CLB}$	*metric*
Hausdorff	$\mathcal{D}$	$\{\mathrm{x}\}$	*metric*
Slice	$\mathcal{DBC}$	$\{\mathrm{x}\}$ *or* $\mathcal{CLB}$	*normed*
Linear	$\mathcal{DCC}$	$\{\mathrm{x}\}$	*normed*
Mosco	$\mathcal{DK}_w\mathcal{C} \subset \mathcal{F} \subset \mathcal{DC}$	$\mathcal{K}_w$	*reflexive Banach*
	$\mathcal{UCK}_w\mathcal{C} \subset \mathcal{F} \subset \mathcal{UC}$	$\mathcal{K}_w$	"
	$\mathcal{UCK}_w$	$\{\mathrm{x}\}$ *or* $\mathcal{CLB}$ *or* $\mathcal{K}_w$	"
	$\mathcal{DK}_w$	$\{\mathrm{x}\}$ *or* $\mathcal{CLB}$ *or* $\mathcal{K}_w$	"

6. Conclusions.

In Section 5 we have explicitely determined some hypertopologies by selecting particular families $\mathcal{F}$ and $\mathcal{P}$. An immediate observation is the large arbitrariety in the choice of $\mathcal{F}$ and the relatively small arbitrariety in the choice of $\mathcal{P}$. In fact, the global behaviour of the hypertopology is determined by $\mathcal{F}$, whereas $\mathcal{P}$ plays the role of localizing element.

The limits dictated by the choice of $\mathcal{P}$ can be seen for instance in the following example. As we have already remarked, the families $\mathcal{P}$={ bounded sets }, $\mathcal{P}'$={ open bounded sets},

$\boldsymbol{\mathcal{P}}''$={ closed bounded sets } and $\boldsymbol{\mathcal{P}}'''$={ balls } are all equivalent. This makes impossible to recover hypertopologies like the ball and the ball proximal by using the same kind of techniques that yield to the bounded Vietoris and the proximal topologies (Theorems 4.2. and 4.5.). The question of the possibility of including ball and ball proximal topologies in our scheme is still open.

Because of the definition of the upper parts of the ball and the ball proximal topologies, the first tempting family $\boldsymbol{\mathcal{F}}$ one can choose for trying to recover them is $\boldsymbol{\mathcal{D}}$**Ball**=$\{\{d(E_1,\cdot),\ldots, d(E_n,\cdot)\} : n \in \mathbb{N}^*, E_i$ is a finite union of balls }. However the topology $\tau^+(\boldsymbol{\mathcal{D}}\mathbf{Ball})$ is in general strictly finer than the upper part of the ball proximal topology. Notice that, according to Theorems 4.1. and 5.7., we have $\tau^-(\boldsymbol{\mathcal{D}}\mathbf{Ball},\boldsymbol{\mathcal{P}})= \tau_V^-$ for any $\boldsymbol{\mathcal{P}}\subset CL(X)$ that covers X, and $\tau^+(\boldsymbol{\mathcal{D}}\mathbf{Ball}, \boldsymbol{\mathcal{K}})=\tau_F^+$.

On the other hand, the large arbitrariety in the choice of $\boldsymbol{\mathcal{F}}$ makes this unified method of describing hypertopologies a very powerful tool for defining new hypertopologies.

One can start with a problem, select the families $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}}$ that seem more reasonable in the context, and then construct the hypertopology. Or, more abstractly, one can select a family $\boldsymbol{\mathcal{F}}$ consisting of functions that are naturally attached to a space, and then study the various hypertopologies obtained with the different choices of $\boldsymbol{\mathcal{P}}$. For example, the choice of $\boldsymbol{\mathcal{C}}$ for a topological space or the choice of $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{D}}\Omega$ for a metric space are natural, since continuous functions and gap functions are the most natural real-valued functions on those spaces. On a normed linear space, the most natural functions to consider are the continuous linear functions.

To conclude, a comment on the assumption that all functions here considered are lower bounded. This is a reasonable assumption, and suitable to avoid technicalities in the definitions of $\mathcal{N}^+(A,\mathcal{F},r,P)$ and $\mathcal{N}^-(A,\mathcal{F},r,P)$.

However, in some instances, we should like better to avoid such an assumption. For instance, when working in the convex case, we would maybe like to deal with linear functionals. In such a case, a generalization of the definition of $\mathcal{N}^+(A,\mathcal{F},r,P)$ and $\mathcal{N}^-(A,\mathcal{F},r,P)$ could be the following.

For every $A \in CL(X)$ and for every $\mathcal{F}$, set $\mathcal{F}_A = \{f \in \mathcal{F} : \inf_A f > -\infty\}$ and let:

$$\mathcal{N}^+(A,\mathcal{F},r,P) = \begin{cases} \mathcal{N}^+(A,\mathcal{F}_A,r,P) & \text{if } \mathcal{F}_A \neq \emptyset \\ CL(X) & \text{otherwise} \end{cases}$$

$$\mathcal{N}^-(A,\mathcal{F},r,P) = \begin{cases} \mathcal{N}^-(A,\mathcal{F}_A,r,P) & \text{if } \mathcal{F}_A \neq \emptyset \\ CL(X) & \text{otherwise} \end{cases}$$

The interested reader can check that our theorems in Sections 3, 4, 5 still hold with these definitions. Moreover, we can for instance prove:

Theorem 6.1. *Let X be a normed linear space. Then*

i) $\tau^+(\boldsymbol{\mathcal{B}}) = \tau_{hs}^+$, *the topology generated by the sets $H_{f,s}^+$, where $H_{f,s} = \{x \in X : f(x) > s\}$, $s \in \mathbb{R}$, is the half upper space determined by the linear map f;*

ii) $\tau^+(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{K}}_w) = \tau_M^+$ *on the closed and convex subsets of* X*, when* X *is a reflexive Banach space.*

Proof. The first part of the theorem comes immediately from the inclusion

$$\mathcal{N}^+(A,\mathcal{F},r) \subset \bigcap_{f\in\mathcal{F}} H^+_{f,r_f} \subset \mathcal{N}^+(A,\mathcal{F},r+\varepsilon)$$

where $\emptyset \neq A \in CL(X)$, $\mathcal{F} \in \boldsymbol{\mathcal{B}}$, $r_f = inf_A f - r$ and $\varepsilon > 0$.

For the second part of the theorem, we appeal to the proof of Theorem 4.2.: the set $E = \{x \in X : f_i(x) > \inf_A f_i - r/2 \text{ for } i = 1,\dots,n\}$, in that proof is weakly open for $\{f_1,\dots,f_n\} \in \boldsymbol{\mathcal{B}}$. Therefore we can conclude $\tau^+(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{K}}_w) \preceq \tau_M^+$.

For the opposite relation, note that in the assumption of reflexivity of X, τ_M^+ is generated by the sets $(K^c)^{++}$ for K weakly compact and convex. Suppose then $\emptyset \neq A \in (P^c)^+$ for some weakly compact P. Hence, we can find a weakly compact and convex K such that $A \in (K^c)^{++} \subset (P^c)^+$. Let $\varepsilon > 0$ such that $B_\varepsilon[K] \cap A = \emptyset$, and let f be a continuous linear function on X such that $\inf_A f > \sup_K f + r$, then $\mathcal{N}^+(A,\{f\},r,K) \subset (K^c)^+ \subset (P^c)^+$.

□

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