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The Use of Monotone Norms in Epigraphical Analysis

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We introduce and study two operations on functions that may be useful on one hand for computing the distance to an epigraph and, on the other hand, for dealing with constraints in mathematical programming. Both of them involve the concept of monotone norms. A special attention is payed to convex functions.

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1. Introduction

Given an extended-real-valued function f on a normed space X, the problem of finding the distance from a point of the product space $X \times \mathbb{R}$ to the epigraph of f,

$$E(f) := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\} ,$$

arises in various situations.

Let us recall for instance that in variational convergence ([1] [3], [7], [28]...) a sequence (f_n) of extended-real-valued functions on a finite dimensional space $\mathbb{R}^p$ converges to f if and only if the sequence of the distance functions,

$$(x, r) \in \mathbb{R}^p \times \mathbb{R} \longmapsto \inf\{\delta((x, r), (z, s)) : (z, s) \in E(f_n)\} ,$$

pointwise converges to,

$$(x, r) \in \mathbb{R}^p \times \mathbb{R} \longmapsto \inf\{\delta((x, r), (z, s)) : (z, s) \in E(f)\} .$$

Here δ denotes an arbitrary norm on the finite dimensional space $\mathbb{R}^p \times \mathbb{R}$.

Another example is furnished by the proximal algorithm in convex optimization ([16], [21]). Suppose we have to minimize the lower semicontinuous proper convex function f on a Hilbert space H, with norm $\|\ \|$. Assume that f admits at least a global minimum :

$$\text{argmin } f := \{x \in H : f(x) = \inf_H f\} \neq \emptyset ,$$

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and let us choose a real number $r \leq \inf_H f$ and $x_0 \in H$.

Denoting by Π the projection of $H \times \mathbb{R}$ onto H and by $P_{E(f)}$ the hilbertian projection on the closed convex set $E(f)$ the sequence,

$$x_{n+1} = \Pi \circ P_{E(f)}\,(x_n, r)\;, \quad n \geq 0\;,$$

weakly converges to a global minimum of f and one has $\lim\limits_{n \to +\infty} f(x_n) = \inf\limits_H f$, (see [14], Corollary 2.1) .

Figure 1

Here the distance in the product space $H \times \mathbb{R}$ is given by the hilbertian norm,

$$(x, r) \in H \times \mathbb{R} \longmapsto \sqrt{||x||^2 + r^2}\;.$$

In the case when the function f is defined on an arbitrary normed space X, various norms on the product space $X \times \mathbb{R}$ may be useful for computing the distance to $E(f)$.

An elegant way for doing this is to consider a norm N on the space $\mathbb{R}^2$ which is <u>monotone</u> in the following sense (see for instance [4], [8], [12]),

$$(a, b) \leq (c, d) \Longrightarrow N(a, b) \leq N(c, d)\;,$$

for all nonnegative real numbers a, b, c, d, $\mathbb{R}^2$ being ordered by the product order,

$$(a,b) \leq (c,d) \Longleftrightarrow a \leq c \quad \text{and} \quad b \leq d\ .$$

We then consider the norm,

$$\mathcal{N}(x,r) = N(||x||, |r|)\ , \tag{1.1}$$

on the space $X \times \mathbb{R}$. In this way we obtain a multitude of equivalent norms on the product space $X \times \mathbb{R}$.

Let us mention some examples of monotone norms on $\mathbb{R}^2$.
For any real number $p > 1$, the norm N_p given by

$$N_p(r,s) = (|r|^p + |s|^p)^{\frac{1}{p}}$$

for any $(r,s) \in \mathbb{R}^2$, is monotone ; so is the norm N_∞ defined by

$$N_\infty(r,s) = \max(|r|, |s|) = |r| \vee |s|\ .$$

Of course, many other examples may be given by taking, for instance $N = \alpha N_p + \beta N_{p'}$, $N(r,s) = \alpha N(r,s) + \beta|s|$ $(\alpha > 0,\ \beta > 0,\ p, p' \in [1, +\infty])$... The great class of polyhedral monotone norms is also available !

We compute the distance to an epigraph with respect to the norm $\mathcal{N}$ defined by (1.1) in section 3. In doing so, we introduce a new operation on functions. This operation preserves the convexity and can be viewed as a generalized infimal convolution : we prove that many classical (variational and geometrical) properties of the infimal convolution can be extended, in some way, to this new operation.

Another operation on functions related to a monotone norm can be found in convex programming. Let us consider a constraint set C defined by n inequalities,

$$C = \{x \in X : g_i(x) \leq 0\ , \quad 1 \leq i \leq n\}\ ,$$

where $g_1, \cdots, g_n$ are real-valued convex fonctions on X. Given a monotone norm N on $\mathbb{R}^n$ we can write (see e.g [10] p.303)

$$C = \{x \in X : N(g_1^+(x), \cdots, g_n^+(x)) \leq 0\}\ ,$$

where g_i^+ is the nonnegative part of g_i . By setting,

$$\gamma(x) = N(g_1^+(x), \cdots, g_n^+(x))\ ,$$

we obtain a real-valued convex function γ on X such that,

$$C = \{x \in X : \gamma(x) \leq 0\}\ .$$

Then, the normal cone to C at a feasible point x, which is of particular interest when one has to minimize a convex function on C, very often coincides with the conical hull of the subdifferential of γ at x . So, it may be useful to compute such a subdifferential.

The functions like γ preserves the convexity of the components $g_1, \cdots, g_p$. This can be easily explain as follows : given nonnegative convex functions $f_1, \cdots, f_n$ on X, the function

$$x \longmapsto N(f_1(x), \cdots, f_n(x)) := N(f_1, \cdots, f_n)(x)$$

is obtained by composing the function

$$x \longmapsto (f_1(x), \cdots, f_n(x)) \tag{1.2}$$

with the norm N. As N is convex and nondecreasing on $\mathbb{R}^n_+$ and as (1.2) is convex with respect to the ordering cone $\mathbb{R}^n_+$, we have that $N(f_1, \cdots, f_n)$ is convex.

Another reason to make a study of $N(f_1, \cdots, f_n)$ is that it generalizes some usual operations as the sum $f_1 + \cdots + f_n$, the maximum $f_1 \vee \cdots \vee f_p$, the sum of order p, $p \geq 1$, $(f_1^p + \cdots f_n^p)^{\frac{1}{p}}$ (see [22]).

In this way we give a single proof for several results. This is done in section 4 in which we limit ourselves to the case n=2, for the results we obtain can obviously be extended to the case $n > 2$.

2. More on monotone norms

The monotonicity and the continuity of N easily lead to the following property : for any families $(r_i)_{i \in I}$ and $(s_j)_{j \in J}$ of nonnegative real numbers bounded above, one has :

$$\begin{aligned} N(\sup_{i \in I} r_i, \sup_{j \in J} s_j) &= \sup_{(i,j) \in I \times J} N(r_i, s_j) \\ N(\inf_{i \in I} r_i, \inf_{j \in J} s_j) &= \inf_{(i,j) \in I \times J} N(r_i, s_j) \quad . \end{aligned} \tag{2.1}$$

The above relations remain valid for any families of nonnegative extended real numbers if one sets, for any a, b in $[0, +\infty]$:

$$N(a, b) = +\infty \quad \text{if} \quad a = +\infty \quad \text{or} \quad b = +\infty \; .$$

This quite natural convention will be systematically used in the sequel.

The concept of dual norm will be of particular importance in this paper. Let us recall that the dual norm N' of the norm N is defined on $\mathbb{R}^2$ by,

$$N'(\lambda, \mu) = \sup\{\lambda r + \mu s : (r, s) \in \mathbb{R}^2, \; N(r, s) \leq 1\} \; ,$$

for any $(\lambda, \mu) \in \mathbb{R}^2$. The dual norm of N' is the initial norm N.

We therefore have, for any $(r, s) \in \mathbb{R}^2$,

$$N(r, s) = \sup\{\lambda r + \mu s : (\lambda, \mu) \in B(N')\} \; , \tag{2.2}$$

where $B(N')$ denotes the closed unit ball of N'.

Clearly, N is monotone if and only if N' is monotone.

In such a case, the norm of a pair (r, s) of nonnegative real numbers, i.e. $(r, s) \in \mathbb{R}^2_+$, may be obtained by restricting the supremum in (2.2) to the nonnegative part $B_+(N')$ of the unit ball $B(N')$ of N' :

$$B_+(N') = B(N') \cap \mathbb{R}^2_+ \ .$$

We then have, for any $(r, s) \in \mathbb{R}^2_+$,

$$N(r, s) = \sup\{\lambda r + \mu s : (\lambda, \mu) \in B_+(N')\} \ .$$

Of course, by compactness of $B_+(N')$, the above supremum is attained :

$$N(r, s) = \max\{\lambda r + \mu s : (\lambda, \mu) \in B_+(N')\} \ , \quad \forall (r, s) \in \mathbb{R}^2_+ \quad . \tag{2.3}$$

The presence of the compact convex set $B_+(N')$ will allow us to apply a minimax theorem. On the other hand, let us note that each element of $B_+(N')$ is majorized (with respect to the product order on $\mathbb{R}^2_+$) by a maximal element, that is a $(\overline{\lambda}, \overline{\mu})$ in $B_+(N')$ such that,

$$(\{(\overline{\lambda}, \overline{\mu})\} + \mathbb{R}^2_+) \cap B_+(N') = \{(\overline{\lambda}, \overline{\mu})\} \ .$$

Consequently, denoting par $E_+(N')$ the set of maximal elements of $B_+(N')$, one also has,

$$N(r, s) = \max\{\lambda r + \mu s : (\lambda, \mu) \in E_+(N')\} \ .$$

Of course $E_+(N')$ is included in the nonnegative part $S_+(N')$ of the unit sphere of N' :

$$E_+(N') \subset S_+(N') := \{(\lambda, \mu) \in B_+(N') : N'(\lambda, \mu) = 1\} \ .$$

In fact, $E_+(N')$ is homeomorphic to a simplex ([15], Theorem 3.8). In particular, $E_+(N')$ is compact.

Clearly, the inclusion above may be strict ; let us consider for example the norms N_p ; in such a case one has $N'_p(\lambda, \mu) = N_q(\lambda, \mu)$ with $q = p/(p-1)$ if $p > 1$, $q = 1$ if $p = +\infty$, and $q = +\infty$ if $p = 1$. The maximal elements of $B_+(N_q)$ are given by,

$$E_+(N_q) = \begin{cases} S_+(N_q) & \text{if } q < +\infty \\ \{(1, 1)\} & \text{if } q = +\infty \ . \end{cases}$$

We shall frequently use the lemma below. It relates the fact that the operation $(r, s) \in \mathbb{R}^2_+ \longmapsto N(r, s)$ is strongly isotone in the sense of Moreau ([18] p.118) .

Lemma 2.1. *Let N be a monotone norm on $\mathbb{R}^2$. For any nonnegative real numbers r, s, t, one has the equivalences,*

a) $N(r, s) < t \iff \exists p > r \ , \quad \exists q > s \quad : N(p, q) = t$

b) $N(r, s) \leq t \iff \exists p \geq r \ , \quad \exists q \geq s \quad : N(p, q) = t \ .$

Proof. a) Assume that $N(r,s) < t$. By the continuity of N there exist $r' > r$ and $s' > s$ such that $0 < N(r',s') < t$. Let us set $\alpha = t/N(r',s')$, $\quad p = \alpha r'$, and $q = \alpha s'$; we then have $p > r$, $\quad q > s$, and $N(p,q) = t$.

Assume that $p > r$, $\quad q > s$, and $N(p,q) = t$, so that $t > 0$. Take $\lambda < 1$ such that $r < \lambda p$, $s < \lambda q$; we then have

$$N(r,s) \leq \lambda N(p,q) = \lambda t < t\ .$$

b) Assume that $N(r,s) \leq t$. If $N(r,s) = 0$ then $r = s = 0$ and it suffices to choose $p \geq 0$, $q \geq 0$ such that $N(p,q) = t$. If $N(r,s) > 0$ let us set $\alpha = t/N(r,s)$, $\quad p = \alpha r,\quad q = \alpha s$; we then have $p \geq r$, $\quad q \geq s, \quad N(p,q) = t$.

The other implication is trivial. □

3. Computing the distance to an epigraph : A new operation on nonnegative convex functionals

The various properties of monotone norms may be used for establishing some interesting formulas. We begin by computing the distance to an epigraph with respect to the norm $\mathcal{N}$ defined in (1.1). We denote by

$\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ the domain of an extended-real-valued function $f : X \to \overline{\mathbb{R}}$, and we set

$a_+ = \max(a, 0)$ for the nonnegative part of any $a \in \overline{\mathbb{R}}$.

Proposition 3.1. *The distance from any point $(x,s) \in X \times \mathbb{R}$ to the epigraph of the function $f : X \to \overline{\mathbb{R}}$, denoted by $\mathcal{N}((x,s), E(f))$, is given by :*

$$\mathcal{N}((x,s), E(f)) = \inf_{u \in X} \{N(||x-u||, (f(u)-s)_+)\ .$$

Proof. Because N is monotone and by the choice of the norm $\mathcal{N}$ in $X \times \mathbb{R}$, one has, with (2.1),

$$\begin{aligned}
\mathcal{N}((x,s), E(f)) &= \inf_{\substack{u \in X \\ r \in \mathbb{R}}} \{N(||x-u||, |r-s|) : f(u) \leq r\} = \\
&= \inf_{u \in \operatorname{dom} f} \ \inf_{r \geq f(u)} \{N(||x-u||, |r-s|)\} \\
&= \inf_{u \in \operatorname{dom} f} \{N(||x-u||, \inf_{r \geq f(u)} |r-s|)\} \qquad . \\
&= \inf_{u \in \operatorname{dom} f} \{N(||x-u||), (f(u)-s)_+\} \\
&= \inf_{u \in X} \{N(||x-u||, (f(u)-s)_+)\}
\end{aligned}$$

□

Let us assume that X is just a linear space.

Limiting ourselves to the nonnegative extended-real-valued functionals, the expression appearing in Proposition 3.1 is a special formulation of the following operation : to each $f, g : X \to [0, +\infty]$ we associate the functional, denoted by $f \underset{N}{\square} g$, defined for any $x \in X$ by,

$$(f \underset{N}{\square} g)(x) = \inf_{u \in X} N(f(x-u), g(u)) \ . \tag{3.1}$$

Let us give some examples of the above operation. By taking $N = N_1$ we recover the infimal convolution, or epigraphical sum $f \square g$ ([17], [2],...)

$$(f \square g)(x) = \inf_{u \in X} \{f(x-u) + g(u)\} = (f \underset{N_1}{\square} g)(x) \ .$$

In the case where $N = N_\infty$ we get the quasi-infimal convolution, or level sum $f \triangle g$ ([20], [26], [27], [23]),

$$(f \triangle g)(x) = \inf_{u \in X} \max(f(x-u), g(u)) = (f \underset{N_\infty}{\square} g)(x) \ .$$

For $N = N_p$, $p \in]1, +\infty[$, we obtain the inverse sum of order p ([22]),

$$\inf_{u \in X} (f^p(x-u) + g^p(u))^{\frac{1}{p}} = (f \underset{N_p}{\square} g)(x) \ .$$

It is remarkable that, without any more assumptions, the operations $(f, g) \longmapsto f \underset{N}{\square} g$ enjoys interesting properties.

From a variational view point we have :

Proposition 3.2. *For any extended-nonnegative-valued functionals f, g on X :*

$$\inf_X (f \underset{N}{\square} g) = N(\inf_X f, \inf_X g) \ .$$

Proof.

$$\begin{aligned} \inf_X (f \underset{N}{\square} g) &= \inf_{x \in X} \quad \inf_{u \in X} N(f(x-u), g(u)) \\ &= \inf_{u \in X} \quad \inf_{x \in X} N(f(x-u), g(u)) \qquad \text{(see (2.1))} \\ &= \inf_{u \in X} N(\inf_X f, g(u)) \\ &= N(\inf_X f, \inf_X g) \end{aligned}$$

□

To obtain an epigraphical description of $f \underset{N}{\square} g$ let us introduce a binary operation on $X \times \mathbb{R}$. For any (u, r), (v, s) in $X \times \mathbb{R}_+$ we define,

$$(u, r) \underset{N}{\oplus} (v, s) = (u + v, N(r, s)) \ . \tag{3.2}$$

Remark that $\underset{N}{\oplus}$ is commutative iff the norm N satisfies $N(r,s) = N(s,r)$ for any $r, s \in \mathbb{R}_+ = [0, +\infty[$, and that $\underset{N}{\oplus}$ is associative iff, for any $r, s, t \in \mathbb{R}_+$,

$$N(N(r,s),t) = N(r, N(s,t)) \ .$$

These two properties hold for the norms N_p, $p \in [1, +\infty]$.

The definition (3.2) may be extended to arbitrary subsets $\mathfrak{A}, \mathfrak{B}$ of $X \times \mathbb{R}$ by setting,

$$\mathfrak{A} \underset{N}{\oplus} \mathfrak{B} = \bigcup_{\substack{(u,r)\in\mathfrak{A} \\ (v,s)\in\mathfrak{B}}} (u,r) \oplus (v,s) \ . \tag{3.3}$$

We then have, denoting by,

$$E_s h = \{(x,r) \in X \times \mathbb{R} : h(x) < r\} \ ,$$

the strict epigraph of a function $h : X \to \overline{\mathbb{R}}$:

Proposition 3.3. *For any nonnegative extended-real-valued functionals f, g on X, the following formula holds :*

$$E_s(f \underset{N}{\square} g) = E_s f \underset{N}{\oplus} E_s g \ .$$

Proof. An element (x,r) of $X \times \mathbb{R}$ belongs to $E_s(f \underset{N}{\square} g)$ iff there exists $u \in X$ such that

$$N(f(x-u), g(u)) < r \ .$$

By Lemma 2.1, this amounts to the existence of nonnegative real numbers s, t such that,

$$f(x-u) < s \ , \ g(u) < t \ , \ N(s,t) = r \ .$$

In other words,

$$(x,r) = (x-u, s) \underset{N}{\oplus} (u,t) \ ,$$

with $(x-u, s) \in E_s f$ and $(u,t) \in E_s g$.

By (3.3) we have, equivalently,

$$(x,r) \in E_s f \underset{N}{\oplus} E_s g \ .$$

$\square$

By taking the projection onto the X axis in the above formula we get the relation,

$$\operatorname{dom} f \underset{N}{\square} g = \operatorname{dom} f + \operatorname{dom} g \ ,$$

which can also be obtained directly (the sum in the right hand member is taken in Minkowski sense).

A description of $f \underset{N}{\square} g$ in terms of level sets is also available.

To this end we set, for any $h : X \to \overline{\mathbb{R}}$ and any real number r,

$$\{h < t\} = \{x \in X : h(x) < t\}$$
$$\{h \le t\} = \{x \in X : h(x) \le t\} .$$

Proposition 3.4. *For any nonnegative extended-real-valued functionals f, g on X, and any $t \ge 0$, one has the formula :*

$$\{f \underset{N}{\square} g < t\} = \bigcup_{\substack{r \ge 0, s \ge 0 \\ N(r,s)=t}} \{f < r\} + \{g < s\} .$$

Proof. An element $x \in X$ belongs to $\{f \underset{N}{\square} g < t\}$ iff there exists $u \in X$ such that

$$N(f(x-u),\ g(u)) < t .$$

By Lemma 2.1, this amounts to the existence of $r \ge 0$, $s \ge 0$ such that

$$f(x-u) < r,\ g(u) < s, \quad N(r,s) = t .$$

By writting x under the form $(x-u)+u$ we have, equivalently,

$$x \in \bigcup_{\substack{r \ge 0, s \ge 0 \\ N(r,s)=t}} \{f < r\} + \{g < s\} .$$

$\square$

Remark 3.5. In the case when $N = N_1$ (resp. $N = N_\infty$), Proposition 3.3 (resp. Proposition 3.4) gives the well known formula

$$E_s(f \square g) = E_s(f) + E_s(g)$$
$$(\text{ resp. } \{f \triangle g < t\} = \{f < t\} + \{g < t\}) .$$

Formulas involving the epigraphs (resp. level sets) instead of strict epigraphs (resp. strict level sets) can be obtained in the case when the infimum in definition (3.1) is attained. To this end, we need the notion of exactness.

Definition 3.6. The operation $f \underset{N}{\square} g$ is said to be exact at a given point $x \in \operatorname{dom} f \underset{N}{\square} g$ if there exists $u \in X$ such that,

$$(f \underset{N}{\square} g)(x) = N(f(x-u),\ g(u)) ;$$

$f \underset{N}{\square} g$ is said to be exact if it is exact at each point of X.

With the same arguments as in the proof of proposition 3.3. and 3.4. we get :

Proposition 3.7. *Assume that* $f \underset{N}{\square} g$ *is exact. Then,*

$$E(f \underset{N}{\square} g) = E(f) \underset{N}{\oplus} E(g) \ ,$$

and, for any $t \geq 0$,

$$\{f \underset{N}{\square} g \leq t\} = \bigcup_{\substack{r \geq 0, s \geq 0 \\ N(r,s)=t}} \{f \leq t\} + \{g \leq t\} \ .$$

We also provide a formula for the set of approximate minimizers of $f \underset{N}{\square} g$. Let us recall that the set of ε-minimizers of a functional $h : X \to [0, +\infty]$, which is proper (i.e. dom $h \neq \emptyset$) is defined, for any $\varepsilon \geq 0$, by

$$\varepsilon - \text{ argmin } h = \{x \in X : h(x) \leq \inf_X h + \varepsilon\} \ .$$

For $\varepsilon > 0$ the set ε-argmin h is never void, and for $\varepsilon = 0$ we recover the set of exact minimizers :

$$0 - \text{ argmin } h = \text{ argmin } h \ .$$

As a first estimation we have :

Proposition 3.8. *For any proper nonnegative extended-real-valued functionals* f_1, f_2 *on* X*, and any* $\varepsilon_1 \geq 0$*,* $\varepsilon_2 \geq 0$*, we have,*

$$\varepsilon_1 - \text{ argmin } f_1 \ + \ \varepsilon_2 - \text{ argmin } f_2 \ \subset \ N(\varepsilon_1, \varepsilon_2) - \text{ argmin } f_1 \underset{N}{\square} f_2 \ .$$

Proof. Let us take $x_i \in \varepsilon_i -$ argmin f_i , $i = 1, 2,$ and consider $x = x_1 + x_2$.
We then have :

$$\begin{aligned}(f_1 \underset{N}{\square} f_2)(x) \leq N(f_1(x_1), f_2(x_2)) &\leq N(\varepsilon_1 + \inf_X f_1, \ \varepsilon_2 + \inf_X f_2) \\ &\leq N(\varepsilon_1, \varepsilon_2) + N(\inf_X f_1, \inf_X f_2) \\ &\leq N(\varepsilon_1, \varepsilon_2) + \inf_X (f_1 \underset{N}{\square} f_2) \ .\end{aligned}$$

□

Proposition below gives an exact formula :

Proposition 3.9. *For any proper nonnegative extended-real-valued functionals* f, g*, one has, for all* $\varepsilon > 0$*,*

$$\varepsilon - \text{ argmin } f \underset{N}{\square} g = \bigcap_{\delta > \varepsilon} \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ N(\varepsilon_1 + \alpha, \varepsilon_2 + \beta) = N(\alpha, \beta) + \delta}} \varepsilon_1 - \text{ argmin } f \ + \ \varepsilon_2 - \text{ argmin } g \ ,$$

where $\alpha = \inf_X f$ *and* $\beta = \inf_X g$.

Proof. Let us set $h = f \underset{N}{\square} g$, and $\gamma = \inf_X h$. By proposition 3.2., we have $\gamma = N(\alpha, \beta)$. On the other hand,

$$\varepsilon - \text{ argmin } h = \{h \leq \gamma + \varepsilon\} = \bigcap_{\delta > \varepsilon} \{h < \gamma + \delta\} \, .$$

Now, by proposition 3.4,

$$\{h < \gamma + \delta\} \subset \bigcup_{\substack{r \geq 0, s \geq 0 \\ N(r,s) = \gamma + \delta}} \{f \leq r\} + \{g \leq s\} \, .$$

But, for each $r \geq 0$,

$$\{f \leq r\} = \{f \leq \alpha + (r - \alpha)\} = \begin{cases} (r - \alpha) - \text{ argmin } f & \text{if } r \geq \alpha \\ \emptyset & \text{if } r < \alpha \, , \end{cases}$$

and idem for $\{g \leq s\}$. We deduce that,

$$\{h < \gamma + \delta\} \subset \bigcup_{\substack{r \geq \alpha, \ s \geq \beta \\ N(r,s) = \gamma + \delta}} (r - \alpha) - \text{ argmin } f \ + \ (s - \beta) \ - \text{ argmin } g \, .$$

Conversely, let us take x in the right hand side above. We then have $x = x_1 + x_2$ with $f(x_1) \leq r$, $\quad g(x_2) \leq s$, hence,

$$h(x) \leq N(f(x_1), \ g(x_2)) \leq N(r, s) = \gamma + \delta \qquad \text{so that}$$

$$\bigcup_{\substack{r \geq \alpha, \ s \geq \beta \\ N(r,s) = \gamma + \delta}} (r - \alpha) - \text{ argmin } f \ + \ (s - \beta) - \text{ argmin } g \ \subset \ \{h \leq \gamma + \delta\} \, .$$

The result follows by setting $\varepsilon_1 = r - \alpha$, $\quad \varepsilon_2 = s - \beta$, and by taking the intersection for all $\delta \in]\varepsilon, +\infty[$. $\square$

In the case of exactness we get similarly :

Proposition 3.10. *Assume that $f \underset{N}{\square} g$ is proper and exact. For any $\varepsilon \geq 0$ we then have,*

$$\varepsilon - \text{ argmin } f \underset{N}{\square} g = \bigcup_{\substack{\varepsilon_1 \geq 0, \ \varepsilon_2 \geq 0 \\ N(\varepsilon_1 + \alpha, \ \varepsilon_2 + \beta) = N(\alpha, \beta) + \varepsilon}} \varepsilon_1 - \text{ argmin } f \ + \ \varepsilon_2 - \text{ argmin } g \, ,$$

with $\alpha = \inf_X f$, $\quad \beta = \inf_X g$.

When $f \underset{N}{\square} g$ is exact, Proposition 3.10 says in particular that the set of exact minimizers of $f \underset{N}{\square} g$ is obtained by taking an union indexed by the set,

$$I_{\alpha,\beta} = \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ : N(\varepsilon_1 + \alpha, \ \varepsilon_2 + \beta) = N(\alpha, \beta)\} \, .$$

It is important to quote that if (α, β) is a maximal element, for the product order on $\mathbb{R}^2_+$, of the set,

$$\{(r, s) \in \mathbb{R}^2_+ : r \geq \alpha,\ s \geq \beta\ , \quad N(r, s) = N(\alpha, \beta)\}\ ,$$

then $I_{\alpha,\beta}$ reduces to the singleton $\{(\alpha, \beta)\}$.

This is for instance the case when $N = N_p$ with $p \in [1, +\infty[$, and when $N = N_\infty$ if, besides, $\alpha = \beta$. In this way we recover some results of [23] [27] .

Another important fact is that the operation $\underset{N}{\Box}$ preserves the convexity. (Recall that $h : X \to [0, +\infty]$ is said to be convex if for any $u, v \in X$ and any $t \in]0, 1[$ one has $h(tu + (1 - t)v) \leq th(u) + (1 - t)h(v)$).

Proposition 3.11. *Let f, g be nonnegative extended-real-valued convex functionals. Then $f \underset{N}{\Box} g$ is convex.*

Proof. The monotonicity of N ensures that the function $(u, x) \in X \times X \longmapsto N(f(x - u), g(u))$ is convex. Therefore $f \underset{N}{\Box} g$ is a marginal function of a convex function of two variables. Hence $f \underset{N}{\Box} g$ is convex. $\Box$

It is also possible to establish a general formula for the convex approximate subdifferential of $f \underset{N}{\Box} g$. To this end we take for X a Hausdorff topological locally convex space (ℓ.c.s.) with dual X^*. Let us recall that the ε-subdifferential, $\varepsilon \geq 0$, of a convex function $h : X \to \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} h$ is defined by ([5], [9]...) :

$$\partial_\varepsilon h(x) = \{y \in X^* : \forall u \in X : h(u) - h(x) \geq < u - x, y > -\varepsilon\}\ .$$

For computing the approximate subdifferential of $f \underset{N}{\Box} g$ we shall use the Legendre-Fenchel transformation : at each convex function $h : X \to \mathbb{R} \cup \{+\infty\}$ is associated its Fenchel conjugate, which is defined on X^* by,

$$h^*(y) = \sup_{x \in X} (< x, y > -f(x))\ ,$$

for any $y \in X^*$.

We then have classically,

$$\partial_\varepsilon h(x) = \{h^* - < x, \cdot > + h(x) \leq \varepsilon\}\ , \tag{3.4}$$

for any $\varepsilon \geq 0$.

In what follows $\mathcal{C}^+_0(X)$ will denote the set of proper nonnegative extended-real-valued convex functionals on X. We adopt the following convention, valid for any $h : X \to [0, +\infty]$:

$$(0h)(x) = 0h(x) = \begin{cases} 0 & \text{if } x \in \operatorname{dom} h \\ +\infty & \text{if } x \notin \operatorname{dom} h \end{cases} \tag{3.5}$$

Proposition 3.12. *Assume that* $f, g \in \mathcal{C}_0^+(X)$. *Then,*

$$(f \underset{N}{\square} g)^* = \min_{(\lambda,\mu)\in E_+(N')} \{(\lambda f)^* + (\mu g)^*\} .$$

Proof. For any $y \in X^*$ we easily get,

$$(f \underset{N}{\square} g)^*(y) = \sup_{(u,x)\in D} (< x, y > -N(f(x-u), g(u))) ,$$

where D is the convex set defined as follows,

$$D = \{(u,x) \in X \times X : x - u \in \text{ dom } f \text{ and } u \in \text{ dom } g\} .$$

By using the expression of $N(f(x-u), g(u))$ in terms of the dual norm we obtain,

$$(f \underset{N}{\square} g)^*(y) = \sup_{(u,x)\in D} \inf_{(\lambda,\mu)\in B_+(N')} (< x, y) > -\lambda f(x-u) - \mu g(u)) .$$

By the minimax theorem ([24] Thm 4.2) one can write,

$$\begin{aligned}(f \underset{N}{\square} g)^*(y) &= \inf_{(\lambda,\mu)\in B_+(N')} \sup_{(u,x)\in D} (< x, y > -\lambda f(x-u) - \mu g(u)) \\ &= \inf_{(\lambda,\mu)\in B_+(N')} ((\lambda f)^*(y) + (\mu g)^*(y)) .\end{aligned}$$

We now observe that the above infimum is attained as $B_+(N')$ is compact and, due to the convention (3.5), the functional $(\lambda,\mu) \in B_+(N') \longmapsto (\lambda f)^*(y) + (\mu g)^*(y)$ is lower semicontinuous (see e.g. [17] p.52).

Taking into account the fact that the application,

$$(\lambda,\mu) \in B_+(N') \longmapsto (\lambda f)^* + (\mu g)^* \in \overline{\mathbb{R}}^{X^*} ,$$

is nonincreasing and, on the other hand, the fact that each $(\lambda,\mu) \in B_+(N')$ is majorized by a maximal element of $B_+(N')$, the minimum in(λ,μ) can be taken over the set $E_+(N')$ of maximal elements of $B_+(N')$. $\square$

Let us observe that for $N = N_1$, the formula in Proposition 3.12 reduces to the well known relation

$$(f \square g)^* = f^* + g^* .$$

In the case when $N = N_\infty$ we get,

$$(f \triangle g)^* = \min\{(\lambda f)^* + (\mu g)^* : \lambda \geq 0\ ,\ \mu \geq 0\ ,\ \lambda + \mu = 1\} ,$$

a formula quoted in [23], Proposition 4.1.

We are now in position to compute the approximate subdifferential of $f \underset{N}{\square} g$.

Proposition 3.13. *Let f and g be two functions in $\mathcal{C}_0^+(X)$, and let u, v, x in X such that $u + v = x$, $u \in \mathrm{dom} f$, $v \in \mathrm{dom} g$. For any $\varepsilon \geq 0$ we then have :*

$$\partial_\varepsilon(f \underset{N}{\square} g)(x) = \bigcup_{(\lambda,\mu)\in E_+(N')} \bigcup_{\substack{\varepsilon_1\geq 0,\, \varepsilon_2\geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon+\lambda f(u)+\mu g(v)-(f \underset{N}{\square} g)(x)}} \partial_{\varepsilon_1}(\lambda f)(u) \cap \partial_{\varepsilon_2}(\mu g)(v) .$$

Proof. By using (3.4) and Proposition 3.12, we have,

$$\begin{aligned}\partial_\varepsilon(f \underset{N}{\square} g)(x) &= \bigcup_{(\lambda,\mu)\in E_+(N')} \{(\lambda f)^* + (\mu g)^* + (f \underset{N}{\square} g)(x) - < x, \cdot > \leq \varepsilon\} \\ &= \bigcup_{(\lambda,\mu)\in E_+(N')} \{[(\lambda f)^* + \lambda f(u) - < u, \cdot >] + [(\mu g)^* + \mu g(v) - < v, \cdot >] \\ &\leq \varepsilon + \lambda f(u) + \mu g(v) - (f \underset{N}{\square} g)(x)\}\end{aligned}$$

Now, by the classical Fenchel inequality, the functionals in the brackets are nonnegative, so that,

$$\partial_\varepsilon(f \underset{N}{\square} g)(x) = \bigcup_{(\lambda,\mu)\in E_+(N')} \bigcup_{\substack{\varepsilon_1\geq 0,\, \varepsilon_2\geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon+\lambda f(u)+\mu g(v)-(f \underset{N}{\square} g)(x)}} A(\lambda, \varepsilon_1, u) \cap B(\mu, \varepsilon_2, v) ,$$

with $A(\lambda, \varepsilon_1, u) := \{(\lambda f)^* + \lambda f(u) - < u, \cdot > \leq \varepsilon_1\}$ and $B(\mu, \varepsilon_2, v) := \{(\mu g)^* + \mu g(v) - \langle v, \cdot \rangle \leq \varepsilon_2\}$,
and the result follows from (3.4). $\square$

Corollary 3.14. *Let $f, g \in \mathcal{C}_0^+(X)$; assume that $f \underset{N}{\square} g$ is finite and exact at a point x, and consider $u \in \mathrm{dom} f$, $v \in \mathrm{dom} g$, such that $u + v = x$ and $(f \underset{N}{\square} g)(x) = N(f(u), g(v))$. We then have,*

$$\partial(f \underset{N}{\square} g)(x) = \bigcup_{\substack{(\lambda,\mu)\in E_+(N') \\ \lambda f(u)+\mu g(v)=N(f(u),g(v))}} \partial(\lambda f)(u) \cap \partial(\mu g)(v) .$$

Proof. We apply Proposition 3.13 by taking $\varepsilon = 0$. Then, the formula follows from the fact that, for any $(\lambda, \mu) \in E_+(N')$, one has $\lambda f(u) + \mu g(v) - N(f(u), g(v) \leq 0$. $\square$

Let us see what Corollary 3.14 says for $N = N_1$. The union set in the above formula is then reduced to the singleton $\{(1, 1)\}$ and we get,

$$\partial(f \square g)(x) = \partial f(u) \cap \partial g(v) ,$$

a classical formula which can be found for instance in [13] p.368.

With additional continuity requirement we obtain :

Corollary 3.15. *Let f , g , u , v , x be as in* corollary 3.14. *Assume moreover that f (resp. g) is continuous at u (resp. v). Then,*

$$\partial(f \underset{N}{\square} g)(x) = \bigcup_{\substack{(\lambda,\mu)\in E_+(N') \\ \lambda f(u)+\mu g(v)=N(f(u),\,g(v))}} \lambda\partial f(u) \cap \mu\partial g(v) \ .$$

Proof. We apply corollary 3.14. On one hand u belongs to the interior of the domain of f. Hence, due to (3.5), $\partial(0f)(u) = \{0\}$. In the same way : $\partial(0g)(v) = \{0\}$. On the other hand, for any $\lambda > 0$ (resp. $\mu > 0$) one has $\partial(\lambda f)(u) = \lambda\partial f(u)$ (resp. $\partial(\mu g)(v) = \mu\partial g(v)$). □

Assume that $f(u) \neq 0$ or $g(v) \neq 0$. Applying the above formula for $p \in]1, +\infty[$ we have, with $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

$$\partial(f \underset{N_p}{\square} g)(x) = \Big(\frac{f^q(u)}{f^q(u) + g^q(v)}\Big)^{\frac{1}{p}} \partial f(u) \cap \Big(\frac{g^q(v)}{f^q(u) + g^q(v)}\Big)^{\frac{1}{p}} \partial g(v) \ .$$

If $p = 1$ and $f(u) = g(v)$ we obtain,

$$\partial(f \triangle g)(x) = \bigcup_{\substack{\lambda\geq 0,\mu\geq 0 \\ \lambda+\mu=1}} \lambda\partial f(u) \cap \mu\partial g(v) \ ,$$

a formula established in [23], Proposition 4.2, by using a totally different method.

A simpler formula arises in the situation we describe below ; it is an easy consequence of Corollary 3.15.

Corollary 3.16. *Let f, g, u, v, x be as above. Assume, moreover, that $f(u) = g(v) = 0$. Then,*

$$\partial(f \underset{N}{\square} g)(x) = \bigcup_{(\lambda,\,\mu)\in E_+(N')} \lambda\partial f(u) \cap \mu\partial g(v) \ .$$

4. An intermediate operation.

In the definition of $(f \underset{N}{\square} g)(x)$ appears the function,

$$u \in X \longmapsto N(f(x-u)\ ,\ g(u)) \ ,$$

which deserves a special treatment for the reasons we have explained in the introduction. For any nonnegative extended-real-valued functionals f , g on X we consider the functional $N(f, g)$ defined on X by,

$$(N(f, g))(x) = N(f(x)\ ,\ g(x)) \ ,$$

for any $x \in X$.

We easily obtain the general relations,

$$\operatorname{dom} N(f,g) = \operatorname{dom} f \cap \operatorname{dom} g \ ,$$
$$\inf_X N(f,g) \geq N(\inf_X f \ , \ \inf_X g) \ .$$

Let us give a description of the level sets of $N(f,g)$.

Proposition 4.1. *For any f , $g : X \to [0,+\infty]$, and $t \in \mathbb{R}_+$, we have :*

$$\{N(f,g) \leq t\} \ = \bigcup_{\substack{r \geq 0 \ , \ s \geq 0 \\ N(r,s)=t}} \{f \leq r\} \ \cap \ \{g \leq s\} \ .$$

Proof. By Lemma 2.1 the inequality $N(f(x) \ , \ g(x)) \leq t$ amounts to the existence of r and s in $\mathbb{R}_+$ such that $f(x) \leq r$, $g(x) \leq s$ and $N(r,s) = t$, which gives the above formula. □

The approximate minimizers of $N(f,g)$ may be obtained as follows :

Proposition 4.2. *Let f , g be two nonnegative proper functionals on X , and let $\alpha = \inf_X f$, $\beta = \inf_X g$, $\gamma = \inf_X N(f,g)$. Then, for any $\varepsilon \geq 0$,*

$$\varepsilon - \operatorname{argmin} \ N(f,g) = \bigcup_{\substack{\varepsilon_1 \geq 0 \ , \ \varepsilon_2 \geq 0 \\ N(\alpha+\varepsilon_1 \ , \ \beta+\varepsilon_2)=\gamma+\varepsilon}} \varepsilon_1 \ - \operatorname{argmin} \ f \ \cap \ \varepsilon_2 \ - \operatorname{argmin} \ g \ .$$

Proof. By using Proposition 4.1 we get,

$$\begin{aligned} \varepsilon\text{- argmin } N(f,g) &= \{N(f,g) \leq \gamma + \varepsilon\} \\ &= \bigcup_{\substack{r \geq 0 \ , \ s \geq 0 \\ N(r,s)=\gamma+\varepsilon}} \{f \leq r\} \ \cap \ \{g \leq s\} \\ &= \bigcup_{\substack{r \geq \alpha \ , \ s \geq \beta \\ N(r,s)=\gamma+\varepsilon}} (r-\alpha) \ - \operatorname{argmin} \ f \ \cap \ (s-\beta) \ - \operatorname{argmin} \ g \ . \end{aligned}$$

The result follows by setting $r - \alpha = \varepsilon_1$, $s - p = \varepsilon_2$. □

We now give a formula for the exact subdifferential of $N(f,g)$.

Theorem 4.3. *Let f and g be two nonnegative extended-real-valued convex functionals on X. Assume that f and g are finite and continuous at a given point $x \in X$. Then,*

$$\partial N(f,g)(x) = \bigcup_{\substack{(\lambda,\mu) \in E_+(N') \\ \lambda f(x)+\mu g(x)=N(f(x) \ , \ g(x))}} \lambda \partial f(x) + \mu \partial g(x) \ .$$

Proof. The function $N(f,g)$ coincides with the pointwise supremum over the compact set $E_+(N')$ of the functionals $\lambda f+\mu g$. On the other hand, there exists an open neighborhood U of x where f and g are finite and continuous. Hence the function $(\lambda,\mu,u)\in E_+(N')\times U\to\lambda f(u)+\mu g(u)$ is finite and continuous. We then have (see [25] or [13] Thm 6.4.9),

$$\partial N(f,g)(x)=\ \overline{\mathrm{co}}\bigcup_{\substack{(\lambda,\mu)\in E_+(N')\\ \lambda f(x)+\mu g(x)=N(f(x)\ ,\ g(x))}}\partial(\lambda f+\mu g)(x)\ ,$$

where $\overline{\mathrm{co}}$ denotes the weak* closure of the convex hull. Now, by [19] Theorem 3, $\partial(\lambda f+\mu g)(x)=\lambda\partial f(x)+\mu\partial g(x)$. Moreover, as $\partial f(x)$ and $\partial g(x)$ are weak* compact and as $E_+(N')$ is compact, the set $E:=\bigcup_{\substack{(\lambda,\mu)\in E_+(N')\\ \lambda f(x)+\mu g(x)=N(f(x)\ ,\ g(x))}}\lambda\partial f(x)+\mu\partial g(x)$ is weak* compact. On the other hand, it can be easily proved that $\{(\lambda,\mu)\in E_+(N'):\lambda f(x)+\mu g(x)=N(f(x),g(x))\}$ is convex. Applying the lemma 2.4 in [22], we infer that E is convex too. □

When dealing with nonnegative convex functions f and g vanishing at x, the above formula may be simplified :

Corollary 4.4. *Let f and g be two nonnegative extended-real-valued convex functionals on X and let $x\in X$ be such that $f(x)=g(x)=0$.*
Assume moreover that f and g are continuous at x. Then,

$$\partial N(f,g)(x)=\bigcup_{(\lambda,\mu)\in E_+(N')}\lambda\partial f(x)+\mu\partial g(x)\ .$$

To compute the Fenchel conjugate of the function $N(f,g)$ we need to introduce some notations. Given a function $\psi:X^*\to\mathbb{R}\cup\{+\infty\}$ and a positive real number λ we set, for any $y\in X^*$,

$$\psi_\lambda(y)=\lambda\psi(\frac{y}{\lambda})\ .$$

When ψ is convex, so is the function of two variables,

$$(\lambda,y)\in\,]0,+\infty[\,\times X^*\longmapsto\psi_\lambda(y)\ .$$

Let us consider two functions f and g in $\Gamma_0^+(X)$, the set of nonnegative proper convex lower semicontinuous functions on X, and, denoting by $B_+^*(N')=\{(\lambda,\mu)\in B_+(N'):\lambda>0\ ,\ \mu>0\}$ the positive part of the unit ball of N' , let us define,

$$\varphi=\inf_{(\lambda,\mu)\in B_+^*(N')}(f^*)_\lambda\ \square\ (g^*)_\mu\ , \tag{4.1}$$

or, more explicitly, for any $y \in X^*$,

$$\varphi(y) = \inf \{ \lambda f^*(\frac{y-v}{\lambda}) + \mu g^*(\frac{v}{\mu}) : (\lambda, \mu) \in B_+^*(N') , v \in X^* \} .$$

We observe that φ is a marginal function of a convex function of four variables : v , y , λ , μ. Hence φ is convex. On the other hand, as $(\lambda f)^* = (f^*)_\lambda$ and $(\mu g)^* = (g^*)_\mu$, one easily sees that the mapping,

$$(\lambda, \mu) \in B_+^*(N') \longmapsto (f^*)_\lambda \,\square\, (g^*)_\mu ,$$

is nonincreasing with respect to the product order of $\mathbb{R}^2$.

It follows that in (4.1) one may replace $B_+^*(N')$ by the nonvoid subset of its maximal elements. This subset is also given by,

$$E_+^*(N') = \{(\lambda, \mu) \in E_+(N') : \lambda > 0 , \mu > 0\} .$$

We are now in position to state the result about the Fenchel conjugate of the convex lower semicontinuous nonnegative proper function $N(f, g)$.

Theorem 4.5. *Let f , g be two functions in $\Gamma_0^+(X)$ such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *The Fenchel conjugate of $N(f, g)$ coincides with the weak* lower semicontinuous hull $\overline{\varphi}$ of the convex function φ defined in (4.1).*

Proof. As a matter of computation, one easily obtains that

$$\varphi^*(x) = \sup_{(\lambda,\mu)\in E_+^*(N')} \lambda f(x) + \mu g(x) ,$$

for any $x \in X$. In the other words,

$$\varphi^* = N(f, g) .$$

Hence $(N(f, g))^*$ coincides with the biconjugate of φ. As $N(f, g) \in \Gamma_0^+(X)$, we also have $\varphi^{**} = (N(f, g))^* \in \Gamma_0^+(X^*)$. Therefore $(N(f, g))^*$ coincides with the lower semicontinuous hull $\overline{\varphi}$ of φ. $\square$

By using Theorem 4.5 we are going to establish a general formula on the ε-subdifferential of $N(f, g)$.

Theorem 4.6. *Let $f, g \in \Gamma_0^+(X)$, $h = N(f, g)$, $x \in$* dom $f \cap$ dom g. *Then, for any $\varepsilon > 0$,*

$$\partial_\varepsilon h(x) = \overline{\bigcup_{(\lambda,\mu)\in E_+^*(N')} \; \bigcup_{\substack{\varepsilon_1 \geq 0 ,\, \varepsilon_2 \geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon+\lambda f(x)+\mu g(x)-h(x)}} \lambda \partial_{\frac{\varepsilon_1}{\lambda}} f(x) + \mu\, \partial_{\frac{\varepsilon_2}{\mu}} g(x)} .$$

Proof. By a mere translation one may assume that $x = 0$. We then have,

$$\partial_\varepsilon h(0) = \{h^* + h(0) \le \varepsilon\} ,$$

or, from Theorem 4.5,

$$\partial_\varepsilon h(0) = \{\overline{\varphi + h(0)} \le \varepsilon\} .$$

Observe that the infimum of the function $y \in X^* \longmapsto \varphi(y) + h(0)$ coincides with $-(\varphi + h(0))^*(0) = -\varphi^*(0) + h(0) = 0$.
Hence, see for instance [20] Theorem 7.6,

$$\partial_\varepsilon h(0) = \overline{\{\varphi + h(0) < \varepsilon\}} .$$

Now, for any $y \in X^*$, the following lines are equivalent :

$$y \in \{\varphi + h(0) < \varepsilon\}$$

$$\exists(\lambda,\mu) \in E_+^*(N') \; , \; \exists(v,w) \in (X^*)^2 : \; y = \; v + w \quad \text{and} \quad (f^*)_\lambda(v) + (g^*)_\mu(w) \; < \; \varepsilon - h(0)$$

$$\exists(\lambda,\mu) \in E_+^*(N') : \lambda \; [\; f^*(\frac{v}{\lambda}) + f(0) \;] + \mu \; [g^*(\frac{w}{\mu}) + g(0)] \; < \; \varepsilon + \lambda f(0) + \mu g(0) - h(0) \; .$$

The two brackets above being nonnegative by Fenchel inequality, the last line amounts to the existence of $\varepsilon_1 \ge 0$ and $\varepsilon_2 \ge 0$ such that,

$$f^*(\frac{v}{\lambda}) + f(0) \; < \; \frac{\varepsilon_1}{\lambda} \; , \; g^*(\frac{w}{\mu}) + g(0) \; < \; \frac{\varepsilon_2}{\mu} \; , \; \varepsilon_1 + \varepsilon_2 = \varepsilon + \lambda f(0) + \mu g(0) - h(0) \; .$$

In particular (see (3.4)) : $v \in \lambda \partial_{\frac{\varepsilon_1}{\lambda}} f(0)$ and $w \in \mu \partial_{\frac{\varepsilon_2}{\mu}} g(0)$.

Therefore, we have proved the inclusion (recall that $\partial_\varepsilon h(0)$ is weak* closed) :

$$\partial_\varepsilon h(0) \subset \overline{\bigcup_{(\lambda,\mu)\in E_+^*(N')} \; \bigcup_{\substack{\varepsilon_1 \ge 0 \, , \, \varepsilon_2 \ge 0 \\ \varepsilon_1+\varepsilon_2 = \varepsilon + \lambda f(0) + \mu g(0) - h(0)}} \lambda \partial_{\frac{\varepsilon_1}{\lambda}} f(0) + \mu \partial_{\frac{\varepsilon_2}{\mu}} g(0)} \; .$$

Conversely, let us take $(\lambda,\mu) \in E_+^*(N')$, $\varepsilon_1 \ge 0$, $\varepsilon_2 \ge 0$ such that $\varepsilon_1 + \varepsilon_2 = \varepsilon + \lambda f(0) + \mu g(0) - h(0)$, $v \in \partial_{\frac{\varepsilon_1}{\lambda}} f(0)$, $w \in \partial_{\frac{\varepsilon_2}{\mu}} g(0)$. We have to show that $y := \lambda v + \mu w$ belongs to $\partial_\varepsilon h(0)$. From Theorem 4.4 we have,

$$\begin{aligned} h^*(y) = \overline{\varphi}(y) \le \varphi(y) &\le (f^*)_\lambda \;\square\; (g^*)_\mu(y) \\ &\le (f^*)_\lambda(\lambda v) + (g^*)_\mu(\mu w) \\ &= \lambda f^*(v) + \mu g^*(w) \; . \end{aligned}$$

As $v \in \partial_{\frac{\varepsilon_1}{\lambda}} f(0)$ and $w \in \partial_{\frac{\varepsilon_2}{\mu}} g(0)$, we then have,

$$h^*(y) \leq \lambda \left[\frac{\varepsilon_1}{\lambda} - f(0) \right] + \mu \left[\frac{\varepsilon_2}{\mu} - g(0) \right] = \varepsilon - h(0) ,$$

hence, by (3.4), $y \in \partial_\varepsilon h(0)$. □

We now give some applications of Theorem 4.6 and corollary 4.4 ; we always suppose that f and g belong to $\Gamma_0^+(X)$, $x \in \operatorname{dom} f \cap \operatorname{dom} g$, and $\varepsilon > 0$.

1) For $N = N_1$, one has $N' = N\infty$ so that $E_+^*(N') = \{(1,1)\}$. Here $N(f,g)$ is nothing but the sum $f + g$ and we get,

$$\partial_\varepsilon (f+g)(x) = \overline{\bigcup_{\substack{\varepsilon_1 \geq 0 \, , \, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)} ,$$

a formula quoted in [11] Theorem 3.2.

2) For $N = N_\infty$, one has $N' = N_1$ and $E_+^*(N') = \{(\lambda,\mu) : \ \lambda > 0 \ , \ \mu > 0 \ , \ \lambda + \mu = 1\}$. In the present case, $N(f,g)$ coincides with the pointwise supremum $f \vee g$ of f and g and we have, denoting by $S^* = \{(\lambda,\mu) : \lambda > 0 \ , \ \mu > 0 \ , \ \lambda + \mu = 1\}$ the positive part of the simplex S,

$$\partial_\varepsilon (f \ \vee \ g)(x) = \overline{\bigcup_{(\lambda,\mu)\in S^*} \ \bigcup_{\substack{\varepsilon_1 \geq 0 \, , \, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \lambda f(x) + \mu g(x) - (f \vee g)(x)}} \lambda \partial_{\frac{\varepsilon_1}{\lambda}} f(x) + \mu \partial_{\frac{\varepsilon_2}{\mu}} g(x)} .$$

a formula quoted in [11] Theorem 2.2.

3) Let $p \in]1,+\infty[$ and let $N = N_p$. Then $N' = N_q$ $(\frac{1}{p} + \frac{1}{q} = 1)$ and $E_+^*(N') = \{(\lambda,\mu) :$ $\lambda > 0$,
$\mu > 0$, $\lambda^q + \mu^q = 1\}$. Here $N(f,g)$ represents the direct sum of order p . Assuming f and g continuous at the point x, with $f(x) = g(x) = 0$, we have, by corollary 4.4,

$$\partial (f^p + g^p)^{\frac{1}{p}}(x) = \bigcup_{(\alpha,\beta)\in S} \alpha^{\frac{1}{q}} \, \partial f(x) + \beta^{\frac{1}{q}} \, \partial g(x) .$$

In such a case, the subdifferential of the direct sum of order p coincides with the direct sum of order q of the subdifferentials.

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