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Integral Representations in Conuclear Cones

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We show that closed convex cones, having bounded order intervals (in particular weakly complete proper convex cones) in conuclear spaces, are generated by their extreme rays. An analogue of Choquet's theorem is obtained for these cones, as well as for the conuclear cones defined in this article. Well-capped cones are conuclear. The main tool is Choquet's notion of conical measure, of which we present the necessary properties here.

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Dedicated to Professor Gustave Choquet on the occasion of his 80th birthday.

Introduction

At the present time there are two entirely different methods known to obtain integral representations by extreme generators in convex cones : Choquet's theory of integral representations [8], [10], [11], [29] and certain generalizations [6], [15] on the one hand, and on the other hand the method, described more fully below, based on the nuclear spectral theorem of Maurin [26], or on the results of Berezanski [2], which followed the introduction of nuclear spaces into spectral theory by Gelfand and Kostjucenku [17], [18].

The present paper was motivated by the desire to accomplish a synthesis between these two approaches. In fact more than such a synthesis is obtained : namely a new theorem on the existence of extremals, independent of the Krein Milman theorem, and a corresponding integral representation theorem. This theorem is valid for a certain class of convex cones which we define in section 3, conuclear cones (having metrizable compact sets). The well-capped cones, defined by Choquet, are conuclear. But the closed convex cones, having bounded order intervals, in conuclear spaces, also happen to be conuclear cones.

Let Γ be a closed convex proper cone in a locally convex space F. Assume, for the purpose of this introduction, that the union $\mathrm{ext}(\Gamma)$ of the extreme rays is non trivial and that there exists a Suslin space, e.g. a second countable locally compact space T, and a continuous map $t \longmapsto e_t$ from T to $\mathrm{ext}(\Gamma)\backslash\{0\}$ such that each extreme ray of Γ contains e_t for precisely one value of t (this is an example of an admissible parametrization of the

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extreme rays, cf. 1.20). An integral representation of an element $f \in \Gamma$, as understood here, is then a decomposition $f = \int_T e_t dm(t)$, where m is a Radon measure on T.

Choquet's theorem states that if Γ has a compact metrizable hyperplane section or 'base' B, every element $f \in \Gamma$ has an integral representation $f = \int_T e\, dm(e)$, the integral extending over the set $T = B \cap \mathrm{ext}(\Gamma)$ of extreme points of B. Moreover, the measure is in each case uniquely determined by f if and only if B is a simplex i.e. Γ is a lattice with respect to its proper order : $f \leq g$ if $g - f \in \Gamma$. More generally, if Γ is the union of metrizable caps, compact convex sets $K \subset \Gamma$ such that $\Gamma \backslash K$ is convex, it is a consequence of Choquet's theorem that the elements of Γ still have integral representations. Classical integral representation theorems such as those of Bochner, Bernstein, etc. can be recovered in this way, and many others have been first obtained using Choquet theory (cf. [8], [11], [29]).

The second method applies essentially only to the important case of the integral representation of positive kernels on nuclear spaces. For instance, kernels invariant under some group action can be decomposed into extreme invariant kernels by interpreting the kernel as the reproducing kernel of a Hilbert space, diagonalising a maximal commutative C^*-algebra commuting with the group action, and using Mautner's theorem [14, ch2]. This method has been used by Maurin [25] and Schwartz [31]. An example of its use in a more complex situation can be found in the work of Borchers and Yngvason [3], and combined with Choquet theory, in Hegerfeldt [21]. This method has the drawback however that there is no statement as to uniqueness if the cone happens to be a lattice.

The main result of this paper, in its most practical form, is the theorem below, in which we make use of the following notation : If f belongs to the convex cone Γ, the set $I_f = \Gamma \cap (f - \Gamma)$ is the interval between 0 and f with respect to the proper order of Γ, and $\Gamma(f) = \bigcup_{\lambda \geq 0} I_{\lambda f}$, the set of elements of Γ majorized by a multiple of f, is the face generated by f in Γ. It is a convex subcone whose proper order equals the order induced on it by Γ. The cone Γ is a lattice if and only if $\Gamma(f)$ is a lattice for all $f \in \Gamma$.

Theorem. *Let F be a quasi-complete conuclear space. Let $\Gamma \subset F$ be a closed convex cone such that the order intervals $\Gamma \cap (f - \Gamma), f \in \Gamma$, are bounded subsets of the topological vector space F. Then :*

1. *Γ is the closed convex hull of its extreme rays.*
2. *If $t \longmapsto e_t, T \longrightarrow \mathrm{ext}(\Gamma)$ is an admissible parametrization of the extreme rays then :*

A) *For every $f \in \Gamma$ there exists a Radon measure m on the parameter space T such that $f = \int_T e_t dm(t)$.*

B) *The measure m is uniquely determined by f if and only if the face $\Gamma(f)$ generated by f is a lattice. In particular the representing measure is unique for every $f \in \Gamma$ if and only if Γ is a lattice.*

Remarks and examples.

There always exists an admissible parametrization of the extreme rays (cf. 1.20).

The strong dual of any barreled nuclear space is conuclear, e.g. the space of distributions $\mathcal{D}'(V)$ on a manifold V is conuclear. Every nuclear Fréchet space is conuclear ; for instance the space $C^\infty(V)$ is conuclear if V is countable at infinity (section 3 and [20]).

If Γ is a weakly complete proper convex cone in a locally convex space F, the order intervals $\Gamma \cap (f - \Gamma)$ are bounded (Choquet [8], prop. 30.10).

The condition that the order intervals be bounded in the topology of F is not enough to ensure the existence of extreme rays if the space F is not conuclear : for example the set of non-negative elements of $L^2[0,1]$ is a weakly complete cone, but it has no extreme rays.

On the other hand a closed convex proper cone, with unbounded order intervals, in a conuclear space, does not necessarily have any extreme rays. An example is the cone $C^\infty_+(\mathbb{R})$ of non negative functions in $C^\infty(\mathbb{R})$. Here every non zero $f \in C^\infty_+(\mathbb{R})$ can be decomposed as a sum $f = f_1 + f_2$, with $f_i \in C^\infty_+(\mathbb{R})$ not proportional to f (partition of unity), i.e. no $f \neq 0$ is extremal. The order interval I_f is unbounded in the C^∞-topology however, unless $f = 0$.

In the case of Bernstein's theorem on completely monotonic functions on $\mathbb{R}_+$ the cone $\Gamma = \{f \in C^\infty(0, +\infty) : (-1)^n f^{(n)} \geq 0 \ \forall\ n \geq 0\}$ does have order intervals which are bounded in the C^∞-topology. Since the extreme generators $e^{-tx}, t \geq 0$, are easy to determine (cf. [8] §32, [11]), Bernstein's theorem, according to which every $f \in \Gamma$ has an integral representation

$$f(x) = \int_0^{+\infty} e^{-tx} d\mu(t) \quad \forall\ x > 0$$

is an immediate consequence of the above theorem.

Any well-capped cone has bounded order intervals : if f belongs to a cap $K \subset \Gamma$ the interval I_f is contained in K, and so compact.

Let N be a barreled nuclear space. Then the cone Γ composed of separately continuous non-negative sesquilinear forms $K : N \times N \longrightarrow \mathbb{C}$, equipped with the topology of bi-bounded convergence, is the union of metrizable caps [35]. This makes the above theorem, particularly the uniqueness part, or theorem 5.3, available to cases where the nuclear spectral theorem applies.

An important example to which the theorem can be applied is the cone of positive definite distributions on a Lie group G :

$$\Gamma = \{T \in \mathcal{D}'(G) : T(\varphi * \tilde{\varphi}) \geq 0 \quad \forall\ \varphi \in \mathcal{C}^\infty_c(G)\}$$

or any closed convex subcone of Γ. Thanks to the factorization theorem of Dixmier and Malliavin [13], according to which every test function is a finite sum of convolution products of test functions, this cone has bounded order intervals. The subcone of central positive definite distributions is a lattice cone, which leads to a generalization of the Bochner-Schwartz theorem for unimodular Lie groups [12], [23], [35], [36].

An interesting example to which the theorem applies has been discussed by Wyss [40, 4.1]. The term "integral representation" can indeed be applied to this example, by the above theorem, because not only the cone is weakly complete and proper, but also the surrounding space is conuclear.

A further application can be found in [1].

Methods.

Since the proof of the theorem is long we give some indication here of the method used. We shall in fact prove a more general theorem (theorem 5.1), which at the same time comprises the case of a cone, not necessarily contained in a conuclear space, which is the union of its metrizable caps. Theorem 5.1 involves cones Γ, which we call conuclear cones, with the property that for every compact convex set $A \subset \Gamma$ containing 0, there exists another compact convex set B in Γ, containing A, such that A and the convex hull $\mathrm{co}(\Gamma \backslash B)$ are disjoint. It will turn out that closed convex cones having bounded order intervals in conuclear spaces have this property.

Cones which are the union of their caps (well-capped cones, cf. section 2) obviously also have this property, but an example due to Goullet de Rugy [19] shows that not all conuclear cones are well-capped (Example 3.7 below). As in the case of Choquet's theorem, which we obtain as a particular case, we have to impose metrizability on the compact sets. In the case of a conuclear space (as always assumed quasi-complete) all compact subsets are metrizable however.

The cones to which our theorem applies do not necessarily have a hyperplane section, even unbounded. The formulation of the theorem with the help of an admissible parametrization of the extreme rays, while practical for the above summary and in many applications, is neither esthetically satisfactory nor a good foundation for a proof of the theorem. Instead we make use of the conical measures, introduced by Choquet [7], [8] for the purpose of formulating representation theorems for cones. Rather than restricting attention to weakly complete cones however, and using maximal conical measures, we directly construct conical measures which are localizable on the extreme rays of a convex cone. In the case of a weakly complete proper convex cone in a conuclear space these are identical to the maximal conical measures used by Choquet. It will be seen that in the general case also our methods of proof are inspired by Choquet's methods.

To get the existence of extreme rays in a conuclear cone we shall not be able to apply the Krein Milman theorem (as in the case of well-capped cones). In fact at this writing it is unknown whether an arbitrary conuclear cone (one in which the compact subsets are not necessarily metrizable) has any extreme rays. Instead we use a method akin to, and inspired by, the method of Hervé [22] for proving the existence part of Choquet's theorem : if B is a convex compact metrizable set, and Φ is a strictly convex continuous function on B, any Radon measure m on B maximizing the integral $\int \Phi dm$ among all the Radon probability measures with resultant the given point $f \in B$, is concentrated on the set of extreme points of B.

A large part of the paper is therefore devoted to the properties of localizable conical measures (which for convenience we call conical integrals). This part may be of interest in itself however, as conical measures are used in connection with other subjects as well, e.g. in statistics (cf. [24]).

The notion of Radon measure on topological spaces following Bourbaki [4], Choquet [9] and Schwartz [32] is crucial to this paper. A summary of the facts needed about Radon measures can be found at the end of section 1.

I should like to thank D.H. Fremlin [16] for the proof of Lemma 1.9, in the case of an arbitrary locally convex space.

Related work.

This is not the first time a connection has been made between Choquet theory and the theory of nuclear spaces. Mokobodzki [27] has shown that certain normal cones in nuclear spaces are well-capped. Wittstock [39] has shown that in conuclear spaces convex compact sets are contained in simplices. Peters, in his paper [28] asks whether the cone of positive definite Bruhat distributions on a locally compact group is well-capped. In the case where the group is second countable the space of test functions has been shown by Bruhat to be nuclear and so the answer is positive by the results in [35] ; consequently the above theorem is also applicable.

The results in the present paper are not the most general possible. One can, following the example of Bourgin and Edgar [6], [15], generalize by replacing the compact metrizable sets by closed bounded Suslin sets having the Radon Nikodym property. But in the case of conuclear spaces no generality is gained by this (cf. Proposition 3.1), and the price would have been an appreciable lengthening of the proof (cf. [37]).

1. Conical integrals.

Let F be a locally convex Hausdorff space over $\mathbb{R}$, let F' be the set of continuous linear forms $\ell : F \longrightarrow \mathbb{R}$. Following Choquet [7], [8] define the space $h(F)$ to be the lattice generated by F' in $\mathbb{R}^F$. The elements $\varphi \in h(F)$, the Choquet test functions, can be written in the form

$$\varphi = \sup_i \ell_i - \sup_j \ell'_j \tag{1.1}$$

where (ℓ_i) and (ℓ'_j) are finite sequences of continuous linear forms. In particular the functions $\varphi \in h(F)$ are positive homogeneous of degree one :

$$\varphi(\lambda x) = \lambda\varphi(x) \qquad \forall\, x \in F,\ \forall\, \lambda \geq 0 \tag{1.2}$$

Let $h^+(F)$ denote the set of $\varphi \in h(F)$ such that $\varphi(x) \geq 0 \quad \forall\, x \in F$.

According to Choquet [7], [8] a **conical measure** is a linear form $\mu : h(F) \longrightarrow \mathbb{R}$, such that $\mu(\varphi) \geq 0$ for all $\varphi \in h^+(F)$. The set of all these positive linear forms obviously forms a convex cone in the algebraic dual of $h(F)$. Let it be denoted by $M^+(F)$.

Now let $F_* = \{x \in F : x \neq 0\}$, and let $\mathfrak{B}(F_*)$ be the set of Borel subsets of F_*. Let $m : \mathfrak{B}(F_*) \longrightarrow [0, +\infty]$ be a measure, i.e. a countably additive set function, having the following two properties :

$$m(A) = \sup_{K \subset A} m(K) \qquad \forall\, A \in \mathfrak{B}(F_*) \quad (K \text{ compact in } F_*) \tag{1.3}$$

$$\int |\ell(x)| dm(x) < +\infty \qquad \forall\, \ell \in F' \tag{1.4}$$

For each $\ell \in F'$ consider the set $O = \{x \in F : |\ell(x)| > 1\}$. Then O is an open subset of F_* and $m(O) < +\infty$. Thus, since every $x \in F_*$ belongs to a set of this type, m is locally finite on F_*. Consequently, being inner regular and locally finite, m is a Radon measure (cf. the Note on Radon measures at the end of this section). The condition (1.4) implies that the functions $\varphi \in h(F)$ are m-summable.

Definition 1.1. A conical measure μ is said to be **localizable** if there exists a measure m, having the properties (1.3) and (1.4) such that :

$$\mu(\varphi) = \int \varphi dm \qquad \forall\, \varphi \in h(F) \tag{1.5}$$

In this case μ is said to be **localized** in m. If m is concentrated on a set S, μ is **localizable** on S. We shall use the term **conical integral** instead of "localizable conical measure". The set of conical integrals will be denoted by $M^+_{loc}(F)$.

Remark. Every conical measure localizable on a compact subset $K \subset F$ (cf. Choquet [8, 30.4]) is localizable according to the above definition, namely on $K \cap F_*$ (any mass at 0 can be removed without changing the integrals (1.5)).

Remark. There are infinitely many different Radon measures in which a given conical integral $\mu \neq 0$ can be localized. For instance by (1.2) we have $\mu(\varphi) = \int \varphi dm = \frac{1}{\lambda}\int \varphi(\lambda x) dm(x)$ for all $\lambda > 0$. If m is bounded one may make use of this invariance to replace m by a probability measure.

1.1. Localization on sections

Definition 1.2. A section is a subset $S \subset F_*$ meeting each ray in at most one point : $x \neq 0,\ \alpha > 0,\ \beta > 0,\ \alpha x \in S,\ \beta x \in S \Longrightarrow \alpha = \beta$.

Thus, for example, a non homogeneous hyperplane $\{x : \ell(x) = 1\}$ in F is a section, as is the set of points at distance $r > 0$ from the origin with respect to a norm on F.

Our next objective is to prove :

1. That a conical integral has at most one localization on a given section S.
2. There exists a section on which every conical integral can be localized.

Let us first prove the following approximation theorem, a variant of the lattice form of the Stone-Weierstrass theorem.

Theorem 1.3. *Let X be a completely regular Hausdorff space (containing at least two points). Let m be a Radon measure on X. Let $\mathcal{H}$ be a subspace of $\mathcal{L}^1(X, m)$ consisting of* continuous *functions. Assume moreover that :*

a) *$\mathcal{H}$ is a sublattice : $\varphi, \psi \in \mathcal{H} \Longrightarrow \sup(\varphi, \psi) \in \mathcal{H}$, and*

b) $\forall\, x, y \in X, x \neq y, \forall\, \alpha, \beta \in \mathbb{R}, \exists\, \varphi \in \mathcal{H}$ *such that* $\varphi(x) = \alpha$ *and* $\varphi(y) = \beta$.

Then $\mathcal{H}$ *is dense in* $\mathcal{L}^1(X, m)$.

The proof essentially consists in replacing X by a large compact subset K and keeping control of what happens outside K. Denote $\mathcal{H}^+ = \{h \in \mathcal{H} : h(x) \geq 0 \quad \forall\, x \in X\}$:

Lemma 1.4. *For every compact set* $K \subset X, \epsilon > 0$ *and number* $M \geq 0$ *there is a function* $h \in \mathcal{H}^+$ *such that* $h(x) \geq M$ *for all* $x \in K$, *and* $\int_{K^c} hdm \leq \epsilon$.

Proof. Let $x \in K$ and $y \in K^c = X \backslash K$, then there exists a function $\varphi_{x,y} \in \mathcal{H}$ such that $\varphi_{x,y}(x) > M$ and $\varphi_{x,y}(y) < 0$. Thus there exists a neighborhood V_x of x such that $\varphi_{x,y}(z) > M$ for all $z \in V_x$. If $K \subset V_{x_1} \cup \ldots \cup V_{x_n}$, let $\varphi_y = \sup(\varphi_{x_1,y}, \ldots, \varphi_{x_n,y})$. Then $\varphi_y(x) \geq M$ for all $x \in K$, and $\varphi_y(y) < 0$. Let φ_{y_o} be such a function. Now choose a compact set $H \subset K^c$ such that $\int_{K^c \backslash H} \varphi_{y_o}^+ dm \leq \epsilon$. For every point $y \in H$ there exists a function φ_y as previously defined, such that $\varphi_y(y) < 0$, and consequently $\varphi_y(z) < 0$ for z in some neighborhood W_y of y. If $H \subset W_{y_1} \cup \ldots \cup W_{y_m}$, let $\varphi = \inf(\varphi_{y_o}, \varphi_{y_1}, \ldots, \varphi_{y_m})$. This function then has the following properties :

i) $\varphi(x) \geq M \quad \forall\, x \in K$,

ii) $\varphi(y) \leq 0 \quad \forall\, y \in H$,

iii) $\varphi \leq \varphi_{y_o}$.

Consequently $\int_{K^c} \varphi^+ dm = \int_{K^c \backslash H} \varphi^+ dm \leq \int_{K^c \backslash H} \varphi_{y_o}^+ dm \leq \epsilon$. Thus the function $h = \varphi^+$ satisfies the requirement. □

Lemma 1.5. *Let* $\Phi \in \mathcal{L}^1(X, m; \mathbb{R})$ *be a continuous* m*-integrable function. Then there exists* $\psi \in \mathcal{H}$ *such that* $\int_X |\Phi - \psi| dm \leq 3\epsilon$.

Proof. Choose a compact set $K \subset X$, with $m(K) > 0$, such that $\int_{K^c} |\Phi| dm \leq \epsilon$ and put $M = \max_{x \in K} |\Phi(x)|$. Let $h \in \mathcal{H}^+$ be chosen so that $h(x) \geq M$ on K and $\int_{K^c} hdm \leq \epsilon$ (lemma 1). By the approximation theorem of Stone-Weierstrass [5, ch. 10, § 4 no 1, Cor. prop. 2] there exists $\varphi \in \mathcal{H}$ such that $|\Phi(x) - \varphi(x)| \leq \epsilon / m(K)$ for all $x \in K$. Let $\psi = \sup(-h, \inf(\varphi, h))$. Then $\psi \in \mathcal{H}$ and $|\Phi(x) - \psi(x)| \leq \epsilon / m(K)$ for all $x \in K$, while $|\psi(x)| \leq h(x)$ for all $x \in X$. Consequently we have

$$\int_X |\Phi - \psi| dm = \int_K |\Phi - \psi| dm + \int_{K^c} |\Phi| dm + \int_{K^c} |\psi| dm \leq 3\epsilon.$$

□

Proof of the theorem. It is well known that in the case of a completely regular space the subspace of continuous integrable functions is dense in $\mathcal{L}^1(X, m; \mathbb{R})$. With lemma 1.5 this completes the proof. □

The following theorem was proved by Choquet [8, II, p. 192] in the case of a compact section S.

Theorem 1.6. *Let $S \subset F_*$ be a section. Let m_1 and m_2 be localizations on S of a conical measure μ. Then $m_1 = m_2$.*

Proof. Let $m = m_1 + m_2$. Then m is concentrated on S. Let $\mathcal{H}$ be the set of restrictions $\varphi|_S$, with $\varphi \in h(F)$. Then $\mathcal{H}$ satisfies the hypotheses of theorem 1.3 : if $x \in S$ and $y \in S$ are non proportional there exists $\ell \in F'$ such that $\ell(x) = \alpha$ and $\ell(y) = \beta$. If $y = \lambda x$ we must have $\lambda < 0$ according to the hypothesis on S. Then if $\ell_1 \in F'$ is such that $\ell_1(x) = 1$, we have $\ell_1(y) < 0, \ell_1^+(x) = 1$ and $\ell_1^+(y) = 0$. Similarly there exists $\ell_2 \in F'$ such that $\ell_2^+(x) = 0$ and $\ell_2^+(y) = 1$. Then $\varphi = \alpha\ell_1^+ + \beta\ell_2^+$ has the required property. Consequently $\mathcal{H}$ is dense in $\mathcal{L}^1(S, m)$. The continuous linear forms $\varphi \longmapsto \int \varphi dm_i$, $i = 1, 2$, coincide on $\mathcal{H}$, and are therefore identical. It follows that $m_1 = m_2$. □

Theorem 1.7. *There exists a section $S \subset F_*$ such that every conical integral can be localized on S.*

The proof will result from the following two lemmas :

Lemma 1.8. *Let m be an arbitrary localization of μ. Let $p : F \longmapsto [0, +\infty]$ be a function with the following properties.*

a) *p is positive homogeneous of degree 1 : $p(\lambda x) = \lambda p(x) \quad \forall \lambda \geq 0 \, \forall x \in F$.*

b) *p is m-measurable.*

c. $0 < p(x) < +\infty$ m-almost everywhere.

Then μ is localizable on the section $S = \{x \in F : p(x) = 1\}$.

Proof. Let $X = \{x \in F_* : 0 < p(x) < +\infty\}$. For $x \in X$ let $\rho(x) = x/p(x)$. Then ρ is defined m-almost everywhere and m-measurable (cf. Note on Radon measures, below). Let $\widetilde{m} = \rho(pm)$, the image of the measure pm under ρ. Then $\widetilde{m}$ is a Radon measure, concentrated on S, localizing μ. Note that pm may not be locally finite. Since the result is crucial we give the details : First let us check that $\widetilde{m}$ satisfies the condition (1.3) : Let $\lambda < \widetilde{m}(A) = \int_{\rho^{-1}(A)} pdm$. Then there exists a compact set $K \subset \rho^{-1}(A) \subset X$ such that $\lambda < \int_K pdm$. By Lusin's theorem we may assume that the restriction $\rho|_K$ is continuous. Let $H = \rho(K)$. Then H is a compact subset of A and $K \subset \rho^{-1}(H)$. Thus $\lambda < \widetilde{m}(H) = \int_{\rho^{-1}(H)} pdm$, which proves that $\widetilde{m}$ satisfies the condition (1.3). Let $\varphi \in h^+(F)$. Then $\varphi(\rho(x))p(x) = \varphi(x)$ for m almost all x. Therefore $\int \varphi d\widetilde{m} = \int \varphi(\rho(x))p(x)dm(x) = \int \varphi(x)dm(x) = \mu(\varphi)$. Putting $\varphi = |\ell|$ we see that $\widetilde{m}$ also satisfies the condition (1.4). In particular, $\widetilde{m}$ is locally finite. Thus, every element φ in $h(F)$ being the difference of functions $\varphi^\pm$ in $h^+(F)$, $\widetilde{m}$ is a Radon measure localizing μ. Finally, it has to be shown that $\widetilde{m}$ is concentrated on S. Since $\rho(x) \in S$ for all $x \in X$, it is sufficient to show that S is $\widetilde{m}$-measurable. Let $C \subset F_*$ be a compact subset. Then $\widetilde{m}(C) < +\infty$. Thus, as in the proof of condition (1.3), given $\epsilon > 0$ we can find a compact set $H = \rho(K) \subset C$ such that $\widetilde{m}(C) - \widetilde{m}(H) \leq \epsilon$. But then $H \subset S \cap C \subset C$, and so $S \cap C$ belongs to the Lebesgue completion of $\mathfrak{B}(F_*)$ with respect to $\widetilde{m}$. The compact set C being arbitrary, S is $\widetilde{m}$-measurable. □

Lemma 1.9. *There exists a universally measurable function $p : F_* \longrightarrow (0, +\infty)$ which is positive homogeneous of degree 1.*

Proof. We first give a proof in the particular case, where F is metrizable or separated by a countable subset of the dual. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of continuous seminorms such that for all $x \in F_*$ there exists n for which $q_n(x) > 0$. (If F is separated by the sequence $(\ell_n)_{n\in\mathbb{N}}$ in F' one can take $q_n(x) = |\ell_n(x)|$). Let $F_n = \{x \in F : q_n(x) > 0\}$. Let $\Gamma_1 = F_1$, and $\Gamma_{n+1} = F_{n+1} \setminus \bigcup_{i=1}^{n} F_i$. Then the sets Γ_n are pairwise disjoint cones covering F_*. Let $p(x) = q_n(x)$ if $x \in \Gamma_n$, i.e. if n is the first index for which $q_n(x) > 0$. Then p is a Borel function, positive homogeneous of degree 1, and $0 < p(x) < +\infty$ for all $x \in F_*$.

Therefore in this case theorem 1.7 is proved.

For the generalization to arbitrary spaces I am indebted to D.H. Fremlin [16]. The proof proceeds by transfinite induction : Let $(q_i)_{i\in I}$ be a family of continuous seminorms such that for each $x \in F_*$ there exists $i \in I$ with $q_i(x) > 0$ (e.g. a fundamental system of continuous seminorms). Let the set I be well-ordered. If λ is the corresponding ordinal number we may assume $I = [0, \lambda)$. If $0 \le \kappa \le \lambda$, let $p_\kappa(x) = q_i(x)$ if $i \in [0, \kappa)$ is the smallest index in $[0, \kappa)$ for which $q_i(x) > 0$, and $p_\kappa(x) = 0$ if there is no such index. Let m be a bounded Radon measure on F_*. Then it follows by induction on κ that p_κ is m-measurable for all $\kappa \le \lambda$. Clearly p_κ is m-measurable if κ is countable. Assume that p_ξ is m-measurable for all ordinals $\xi < \kappa$. If $\kappa = \xi + 1$ we have $p_\kappa(x) = q_\xi(x)$ if $p_\xi(x) = 0$, while $p_\kappa(x) = p_\xi(x)$ if $p_\xi(x) > 0$. Thus p_κ is m-measurable. Next assume that $\kappa = \sup_n \xi_n$, $(\xi_n)_{n\in\mathbb{N}}$ being a sequence of ordinals smaller than κ. Then $p_\kappa = \sup_n p_{\xi_n}$, and so p_κ is m-measurable. Finally, if κ is not the supremum of a sequence of smaller ordinals, let

$$H_\xi = \{x : p_\xi(x) = 0\} = \{x : q_\eta(x) = 0 \quad \forall\, \eta < \xi\}$$

Then these sets are closed, they decrease as ξ increases, and their intersection for $\xi < \kappa$ is H_κ. Consequently we have $m(H_\kappa) = \inf_{\xi<\kappa} m(H_\xi)$. Let $\xi_n < \kappa$ be such that $m(H_{\xi_n}) \le m(H_\kappa) + 1/n$, and let $\xi = \sup_n \xi_n$. Then $m(H_\kappa) = m(H_\xi)$, while by the hypothesis $\xi < \kappa$. But $p_\kappa(x) = p_\xi(x)$ for all x not belonging to $H_\xi \setminus H_\kappa$, i.e. for m-almost all x. Thus p_κ is m- measurable. This proves that p_λ is universally measurable. As we have $p_\lambda(x) > 0$ for all $x \in F_*$, the function $p = p_\lambda$ has the required properties.

Thus theorem 1.7 is valid for an arbitrary locally convex Hausdorff space. □

1.2. Integration.

Theorem 1.10. *Let μ be a conical integral, and let m_1 and m_2 be localizations of μ.*

1. *Let $\Phi : F_* \longrightarrow \overline{\mathbb{R}}$ be positive homogeneous of degree $\alpha \in \mathbb{R}$ i.e. : $\Phi(\lambda x) = \lambda^\alpha \Phi(x)$ $\forall\, \lambda > 0$, $\forall\, x \in F_*$. Then Φ is m_1-measurable (resp. m_1-almost everywhere equal to 0) if and only if Φ is m_2-measurable (resp. m_2-a.e. equal to 0).*

2. *Let $\Phi : F_* \longrightarrow \overline{\mathbb{R}}$ be homogeneous of degree 1. Then Φ is m_1-summable if and only if Φ is m_2-summable, and in that case one has $\int \Phi \, dm_1 = \int \Phi \, dm_2$. The same equality holds if $\Phi : F_* \longrightarrow [0, +\infty]$ is homogeneous of degree 1 and measurable.*

Definition 1.11. The function Φ will be said to be μ-measurable if it is homogeneous of degree α for some $\alpha \in \mathbb{R}$, and measurable with respect to some (hence any) localization m of μ. Similarly Φ is μ-summable if Φ is homogeneous of degree 1, and m-summable. If Φ is μ-summable or non-negative, homogeneous of degree 1 and μ-measurable, the integral of Φ with respect to μ is defined by :

$$\int \Phi(x) d\mu(x) = \int \Phi(x) dm(x) \tag{1.6}$$

The notation $\mu(\Phi)$ will also be used to denote the integral.

Proof of 1.10. Let $p : F_* \longrightarrow (0, +\infty)$ be positive homogeneous of degree 1 and universally measurable. Let $S = \{x : p(x) = 1\}$ be the corresponding section, let $\rho(x) = x/p(x)$ and let $\widetilde{m}_i = \rho(pm_i)$. Then $\widetilde{m}_1$ and $\widetilde{m}_2$ being localizations of μ on S, one has $\widetilde{m}_1 = \widetilde{m}_2$ by theorem 1.6. Thus it suffices to prove the theorem for m and $\widetilde{m}$.

In case 1. a function Φ is $\widetilde{m}$-measurable (resp. $\widetilde{m}$-negligible) if and only if $(\Phi \circ \rho)p$ is m-measurable (resp. m-negligible). Since $\Phi(\rho(x))p(x) = p(x)^{1-\alpha}\Phi(x)$ this is clearly equivalent to Φ being m-measurable (resp. m-negligible). Similarly, in case 2, since $\Phi(\rho(x))p(x) = \Phi(x)$ we have, if $\Phi \geq 0, \int \Phi d\widetilde{m} = \int \Phi dm \leq +\infty$. In particular if $\Phi : X \longrightarrow \overline{\mathbb{R}}, \int |\Phi| d\widetilde{m} = \int |\Phi| dm \leq +\infty$ which shows that Φ is m-summable if and only if Φ is $\widetilde{m}$-summable. The equalities of the integrals in that case is obtained similarly, or by decomposing Φ in its positive and negative parts Φ^+ and Φ^-, which are also homogeneous of degree 1 and μ-summable.

Remark 1.12. Let $\Gamma \subset F$ be a **cone** i.e. a set such that $\lambda\Gamma \subset \Gamma$ for all $\lambda \geq 0$. Then the indicator function is homogeneous of degree 0, and so it makes sense to say that Γ is μ-measurable (resp. μ-negligible) if it is m-measurable (resp. m-negligible) with respect to some localization m of μ.

Let $\mathcal{L}^1(\mu)$ denote the linear space of functions $\Phi : F_* \longrightarrow \mathbb{R}$ which are positive homogeneous of degree 1, and μ-summable, equipped with the seminorm $N_1 : \Phi \longmapsto \mu(|\Phi|)$. The quotient of $\mathcal{L}^1(\mu)$ by the subspace of null functions is as usual denoted $L^1(\mu)$. It is a Banach space. In fact we have :

Theorem 1.13. *Let μ be a conical integral. Then*

1. *$\mathcal{L}^1(\mu)$ is complete.*
2. *The space $h(F)$ is a dense subspace of $\mathcal{L}^1(\mu)$.*

Proof. 1. Let m be a localization of μ on a section S. Let $\Phi_n \in \mathcal{L}^1(\mu)$ be such that $\sum_n \mu(|\Phi_n|) < +\infty$. Let $\Phi(x) = \sum_n \Phi_n(x)$ if $\sum_n |\Phi_n(x)| < +\infty$, and let $\Phi(x) = 0$ otherwise.

Then Φ is positive homogeneous of degree 1, and $\Phi = \sum_n \Phi_n$ in the space $\mathcal{L}^1(m)$, and so in $\mathcal{L}^1(\mu)$. This implies that $\mathcal{L}^1(\mu)$ is complete.

2. Let S be a section on which μ can be localized in a measure m (theorem 1.7). Then, as shown in the proof of theorem 1.6, the set $\mathcal{H}$ of restrictions $\varphi|_S$, with $\varphi \in h(F)$, satisfies the conditions of theorem 1.3. The space $\mathcal{H}$ being dense in $\mathcal{L}^1(S, m)$, $h(F)$ is dense in $\mathcal{L}^1(\mu)$. □

Note that since $\mu(||\Phi| - |\varphi||) \leq \mu(|\Phi - \varphi|)$, the set $h^+(F)$ is also dense in $\mathcal{L}^1_+(\mu)$.

Let us determine the dual of $L^1(\mu)$: Let $\mathcal{L}^\infty(\mu)$ be the space of functions

$\Psi : F_* \longrightarrow \mathbb{R}$ which are positive homogeneous of degree 0, i.e. constant on rays, μ-measurable, and such that there exists a number $M \in \mathbb{R}$ such that $|\Psi(x)| \leq M$ μ-almost everywhere (i.e. m-almost everywhere for any localization m of μ). Let $N_\infty(\Psi)$ denote the smallest number M with this property. The quotient of the space $\mathcal{L}^\infty(\mu)$ by the subspace of functions equal to 0 μ-a.e. is denoted by $L^\infty(\mu)$.

Theorem 1.14. *Let $\Phi \in \mathcal{L}^1(\mu), \Psi \in \mathcal{L}^\infty(\mu)$. Then $\Phi\Psi \in \mathcal{L}^1(\mu)$ and*

$$|\int \Phi\Psi d\mu| \leq N_1(\Phi)N_\infty(\Psi) \tag{1.7}$$

In particular Ψ defines a continuous linear form on $L^1(\mu)$ which only depends on the class of Ψ in $L^\infty(\mu)$. The corresponding map from $L^\infty(\mu)$ to $L^1(\mu)'$ is an isometric isomorphism of the first space onto the second : i.e. $L^1(\mu)' = L^\infty(\mu)$.

Proof. $\Phi\Psi$ being homogeneous of degree 1 the first assertion is obvious. Let m be a localization of μ on a section S which meets every ray in F_*. Then the map $\Psi \longmapsto \Psi|_S$ is a linear isomorphism of $\mathcal{L}^\infty(\mu)$ onto $\mathcal{L}^\infty(S, m)$. Thus the assertion results from the corresponding theorem for Radon measures.

In particular, if Γ is a μ-measurable cone one defines the integral of $\Phi \in \mathcal{L}^1(\mu)$ over Γ :

$$\int_\Gamma \Phi d\mu = \int 1_\Gamma \Phi \, d\mu \tag{1.8}$$

Remark. If $1 < p < +\infty$ one can similarly define the space $\mathcal{L}^p(\mu)$ to be the space of functions $\Phi : F_* \longrightarrow \mathbb{R}$ which are positive homogeneous of degree 1/p, μ-measurable, and such that $\int |\Phi|^p d\mu < +\infty$. The space $L^p(\mu)$ is a Banach space whose dual is isomorphic to $L^{p'}(\mu)$, if $1/p' + 1/p = 1$. Theorems 1.13 and 1.14 allow us to prove the following fact however, relevant to our subject :

Theorem 1.15. *Let μ be a conical integral, i.e. a localizable conical measure, and let ν be a conical measure such that $0 \leq \nu \leq \mu$ i.e. :*

$$0 \leq \nu(\varphi) \leq \mu(\varphi) \qquad \forall\, \varphi \in h^+(F) \tag{1.9}$$

Then ν is localizable. Moreover, if m is a localization of μ there exists a localization n of ν such that $0 \leq n \leq m$.

Proof. For $\varphi \in h(F)$ we have :

$$|\nu(\varphi)| \leq \nu(|\varphi|) \leq \mu(|\varphi|) \tag{1.10}$$

Thus by theorem 1.13 ν extends to a continuous linear form on $\mathcal{L}^1(\mu)$. By theorem 1.14 there exists a function $\Psi \in \mathcal{L}^\infty(\mu)$ such that :

$$\nu(\varphi) = \int \varphi \Psi d\mu \qquad \forall\, \varphi \in h(F) \tag{1.11}$$

Now since $h^+(F)$ is dense in $\mathcal{L}^1_+(\mu)$, the inequalities (1.9) imply that $0 \leq \Psi(x) \leq 1$

μ a.e., and so we may assume $0 \leq \Psi \leq 1$. If μ is localized in m we have, by definition of the integral in (1.11) that $\nu(\varphi) = \int \varphi \Psi dm$. This means that ν is localized in the measure $n = \Psi m$, and $0 \leq n \leq m$. □

1.3. Admissible sections.

In connection with integral representations one is interested in conical integrals which are concentrated on cones :

If μ is localizable on Γ, i.e. some localization of μ is concentrated on Γ, it is a consequence of theorem 1.10 (remark 1.12) that every localization of μ is concentrated on Γ. In this case the conical integral μ will be said to be itself **concentrated** on Γ. More intrinsically, Γ_* denoting $\Gamma \backslash \{0\}$: μ is concentrated on Γ if the cone $F \backslash \Gamma_*$ is μ-negligible.

Obviously, if μ is concentrated on Γ we have $\mu(\varphi) \geq 0$ for every $\varphi \in h(F)$ such that $\varphi(x) \geq 0$ for all $x \in \Gamma$, i.e. μ is carried by Γ.

Conversely it can be shown that if Γ is a weakly closed cone (in particular if Γ is a closed convex cone) a conical integral μ carried by Γ is concentrated on Γ, [34].

A subset S of Γ_* such that each ray in Γ meets S in precisely one point will be called a **section** of Γ. One can then define the **gauge** of S, as the function $p_S : \Gamma \longrightarrow [0, +\infty)$, positive homogeneous of degree 1, which is equal to 1 on S. It then follows that $S = \{x \in \Gamma : p_S(x) = 1\}$.

Definition 1.16. The section S of Γ will be called **admissible** if the function $p_S : \Gamma \longrightarrow [0, +\infty)$ is **universally measurable**.

For instance a **hyperplane section** (where p_S is the restriction to Γ of a continuous linear form) is an admissible section. Note however that even proper closed convex cones do not in general have hyperplane sections (e.g. $\mathbb{R}^{\mathbb{N}}_+$ in $\mathbb{R}^{\mathbb{N}}$).

The section constructed for the benefit of theorem 1.7 is an admissible section of F_*. We therefore have as a consequence of theorem 1.7 :

Corollary 1.17. *Every cone Γ has an admissible section.*

Proof. The restriction to Γ of a universally measurable function is a universally measurable function on Γ (whatever the topological properties of Γ). It suffices therefore to take the intersection of Γ with an admissible section of F_*. □

Theorem 1.18. *Let Γ be a cone and let S be an admissible section of Γ. Let μ be a conical integral which is concentrated on Γ. Then μ has a unique localization on S.*

Proof. The uniqueness is a consequence of theorem 1.6. Existence : Let m be a localization of μ. Then m is concentrated on Γ, hence $0 < p_S(x) < +\infty$ m a.e.. Let $\rho(x) = x/p_S(x)$. By lemma 1.8 μ is localizable on $S = \{x \in \Gamma : p_S(x) = 1\}$ in the measure $\widetilde{m} = \rho(p_S m)$.

To obtain practical criteria yielding admissible sections recall that a topological Hausdorff space is called a Suslin space if it is the continuous image of a Polish space. Such a space is of course separable. Borel subsets of Suslin spaces are Suslin. Schwartz [32] has shown that practically every separable space encountered in analysis is Suslin (the only exception seems to be $\mathbb{R}^{\mathbb{R}}$). In particular any nuclear space which is a strict inductive limit of a sequence of Fréchet spaces (necessarily separable), is a Suslin space and its strong dual is also a Suslin space [32, p.115]. □

Theorem 1.19.

1. *Let $S \subset \Gamma$ be a Suslin section of Γ (i.e. a section of Γ which is a Suslin space). Then S is an admissible section.*
2. *A cone Γ has a Suslin section if and only if Γ is itself a Suslin space. In particular, every closed cone in a locally convex Suslin space has a Suslin section.*

As a consequence we see that a Borel section of a Suslin cone is admissible. On the other hand there seems to be no particular reason for a Borel section of an arbitrary cone to be admissible.

Proof of 1.19. Let p_S be the gauge of S. Let $\pi : \mathbb{R}_+ \times \Gamma \longrightarrow \Gamma$ be the function defined by $\pi(\lambda, x) = \lambda x$. Then for every number $\alpha \geq 0$, the set

$$\{x \in \Gamma : p_S(x) \leq \alpha\} = \bigcup_{0 \leq \lambda \leq \alpha} \lambda S = \pi([0, \alpha] \times S) \tag{1.12}$$

is the continuous image of the Suslin space $[0, \alpha] \times S$ and so is a Suslin subspace of Γ. Thus it is a universally measurable subset of Γ [32, p. 124]. If Γ has a Suslin section $S, \Gamma = \pi([0, +\infty) \times S)$ is itself a Suslin space. Conversely, if Γ is a Suslin space, there exists a countable family $(\ell_n)_{n \in \mathbb{N}}$ of continuous linear forms separating the points of Γ [32, p.105]. Let $p(x) = |\ell_n(x)|$ if n is the smallest index for which $|\ell_n(x)|$ is positive. Then $S = \{x \in \Gamma : p(x) = 1\}$ is a Borel section of Γ, and therefore a Suslin section. □

To translate the theorem on integral representations below into the more usual concrete terms we introduce the following definition :

Definition 1.20. An admissible parametrization of the cone Γ is a one-to-one continuous map $t \longmapsto \gamma(t)$ from a Hausdorff space T to Γ such that the image $S = \mathrm{Im}(\gamma)$ is an admissible section and the inverse map $S \longrightarrow T$ is universally measurable.

(Trivial example : $T = S$ an admissible section, γ the identity).

Theorem 1.21. *Let μ be a conical integral concentrated on Γ. Let γ be an admissible parametrization of Γ. Then there exists a unique Radon measure m on T such that :*

$$\mu(\varphi) = \int_T \varphi(\gamma(t))dm(t) \tag{1.13}$$

for all $\varphi \in h^+(F)$. Conversely, if m is a Radon measure on T such that

$$\int_T |\ell(\gamma(t))|dm(t) < +\infty \qquad \forall\, \ell \in F' \tag{1.14}$$

formula (1.13) *defines a conical integral which is localizable on Γ.*

Proof. Let n be a localization of μ on $S = \mathrm{Im}(\gamma)$, and let m be the image of n under γ^{-1}. Then γ being continuous, m is locally finite. Moreover m is the only measure whose image under γ is n. Thus n being uniquely determined by μ, so is m. For the converse observe that by (1.14) the image n of m under γ satisfies condition (1.4), and so is locally finite. The regularity condition (1.3) follows from the continuity of γ. □

Example 1.22. Let T be a Suslin space. Then if $\gamma : T \longrightarrow \Gamma\backslash\{0\}$ is continuous, one-to-one, and such that every ray of Γ meets the image $\mathrm{Im}(\gamma)$ in one point, γ is an admissible parametrization of Γ.

Indeed $S = \mathrm{Im}(\gamma)$ is then a Suslin section of Γ, and the inverse map is universally measurable by von Neumann's selection theorem ([32] p.127).

Note that every Polish space, and in particular every locally compact space having a countable basis of open sets, is Suslin (cf. Introduction). Also the set $\mathcal{E}(\mathcal{K})$ of extreme points of a compact convex metrizable set K is a G_δ in K, so $\mathcal{E}(\mathcal{K})$ is Polish ([29], prop. 1.3).

1.4. Note on Radon measures.

A positive Radon measure m on a Hausdorff space X is a positive measure on the Borel sets of X which is locally finite (the open sets of finite measure cover X) and inner regular with respect to compact sets. Since we make use of positive measures only we refer to positive Radon measures simply as Radon measures.

Choquet [9] has characterized Radon measures as functions of a compact set which are increasing, sub-additive, additive for disjoint sets and continuous on the right. This characterization is in the case of Hausdorff spaces almost as useful as the Riesz Markov theorem in the case of locally compact spaces.

The integration theory used in this article for m, following [32] and [4], is the standard theory for the measure space $(X, \mathfrak{B}^m, m)$, where $\mathfrak{B}^m$ is the set of m-measurable subsets of X i.e. the set of subsets $A \subset X$ such that for every compact subset K of $X, A \cap K$ is in the Lebesgue completion with respect to m of the Borel σ-algebra. For $A \in \mathfrak{B}^m$ $m(A)$ is defined by inner regularity as in (1.3). The Radon measure m is concentrated on a set $S \subset X$ if S is m-measurable and $m(X \backslash S) = 0$. By Lusin's theorem a function $\Phi : X \longrightarrow \mathbb{R}$ is measurable with respect to $\mathfrak{B}^m$ if and only if for every compact $K \subset X$, and $\varepsilon > 0$ there exists a compact subset $K' \subset K$, such that $m(K \backslash K') \leq \varepsilon$ and the restriction of Φ to K' is continuous. For functions Φ to more general topological spaces this is taken as a definition of m-measurability.

If n is a Radon measure on an arbitrary topological subspace $S \subset X$, its image under the inclusion map is a Radon measure m on X which is concentrated on S, and n is the restriction of m to S. Thus one need not distinguish between measures on S and measures on X concentrated on S (cf. [32, pp. 36,37]).

The integral of a non-negative m-measurable function $\Phi \geq 0$ has the property : $\int \Phi dm = \sup\limits_K \int_K \Phi dm$. We use the term m-**summable**, to denote the m-measurable functions $\Phi : X \longrightarrow \mathbb{R}$ such that $\int |\Phi| dm < +\infty$. These functions are called essentially m-integrable by Bourbaki [4].

If m is moderate, i.e. if X is the union of a countable set of open subsets of finite measure, m is then outer regular as well, and the terms m-summable and m-integrable are synonymous.

If there exists a sequence $(\ell_n)_{n \in \mathbb{N}}$ in F' separating the points of F, every measure m on F_* which satisfies condition (1.4) is moderate. The open sets $O_{n,k} = \{x \in F_* \ : \ |\ell_n(x)| > 1/k\}$ then form a countable covering of F_* by open sets of finite m-measure.

There exist conuclear spaces which are not countably separated, for instance the dual of $\mathbb{R}^I, I$ uncountable, so we do not make any assumption as to the space being countably separated. All spaces of practical importance seem to be countably separated however.

2. Integral representations.

Recall from [7] that if μ is a conical measure on F, the **resultant** $r(\mu)$ of μ is the point f, belonging to the weak completion of F (the algebraic dual of F') such that

$$\ell(f) = \mu(\ell) \qquad \forall\, \ell \in F' \tag{2.1}$$

If μ is a conical integral it follows that for every localization m of μ,

$$\ell(f) = \int \ell(x) dm(x) \qquad \forall\, \ell \in F' \tag{2.2}$$

Relation (2.2), which expresses that f is the resultant of the measure m, will be abbreviated as follows :

$$f = \int x dm(x) \tag{2.3}$$

The identity map $x \to x$ being homogeneous of degree 1 one may also write :

$$f = \int x d\mu(x) \tag{2.4}$$

Let $\mathcal{M}^+(F)$ denote the set of conical integrals μ such that $r(\mu)$ belongs to F :

$$\mathcal{M}^+(F) = \{\mu \in M^+_{loc}(F) : r(\mu) \in F\} \tag{2.5}$$

If $\Gamma \subset F$ is a cone let $\mathcal{M}^+(\Gamma)$ denote the set of $\mu \in \mathcal{M}^+(F)$ such that μ is concentrated on Γ.

Observe that $\mathcal{M}^+(\Gamma)$ is a convex subcone of $M^+(F)$: if μ and ν have localizations m and n concentrated on Γ, $\nu + \mu$ is localized in $n + m$ and so concentrated on Γ. Also $r(\nu + \mu) = r(\nu) + r(\mu)$ belongs to F.

We shall make use of the notation :

$$\Gamma' = \{\ell \in F' : \ell(x) \geq 0 \quad \forall\, x \in \Gamma\} \tag{2.6}$$

Proposition 2.1. *If Γ is a closed convex cone we have*

$$r(\mu) \in \Gamma \quad \forall\, \mu \in \mathcal{M}^+(\Gamma) \tag{2.7}$$

Proof. By (2.2) $\ell(r(\mu)) \geq 0$ for all $\ell \in \Gamma'$. But by the Hahn-Banach separation theorem one has $\Gamma = \{x \in F : \ell(x) \geq 0 \quad \forall\, \ell \in \Gamma'\}$. □

For the remaining part of the paper Γ will be a closed convex cone of F which is proper : i.e. $\Gamma \cap -\Gamma = \{0\}$. One then defines an order relation in F by putting :

$$f \leq g \Longleftrightarrow g - f \in \Gamma \tag{2.8}$$

In particular $\Gamma = \{f \in F : f \geq 0\}$.

The cone Γ is said to be a **lattice** if any two members of Γ have a smallest common majorant with respect to this order relation.

An **extreme generator** of Γ is a point $f \in \Gamma$ such that $0 \leq g \leq f$ implies that $g = \lambda f$ for some number $\lambda \geq 0$. Let $\mathrm{ext}(\Gamma)$ denote the set of extreme generators. It is obviously a subcone of Γ. Alternatively :

$$\mathrm{ext}(\Gamma) = \{f \in \Gamma : f = f_1 + f_2, f_i \in \Gamma \Longrightarrow f_i = \lambda_i f\} \tag{2.9}$$

If $\mathrm{ext}(\Gamma) = \{0\}$ Γ is said to have no extreme generator. Note that $\mathcal{M}^+(\mathrm{ext}(\Gamma)) \subset \mathcal{M}^+(\Gamma)$. Consequently we have $r(\mu) \in \Gamma$ for all $\mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$.

Definition 2.2. Let Γ be a closed convex proper cone in F, and let $f \in \Gamma$. An **integral representation** of f by means of extreme generators is a conical integral $\mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$ such that $r(\mu) = f$.

Thus the point $f \in \Gamma$ has a (unique) integral representation by means of extreme generators if there exists $\mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$ (resp. if there exists a unique
$\mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma)))$ such that $r(\mu) = f$.

Let us verify that this definition agrees with the usual concept of "integral representation" :

In the first place, assume Γ has a hyperplane section, a base :

$$B = \{x \in \Gamma : \ell_o(x) = 1\} \tag{2.10}$$

where $\ell_o \in F'$ is such that $\ell_o(x) > 0$ for all $x \in \Gamma_*$. Then $B \cap \mathrm{ext}(\Gamma) = \mathcal{E}(B)$, the set of extreme points of B, is an admissible section of $\mathrm{ext}(\Gamma)$, and so every $\mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$ has a unique localization m on $\mathcal{E}(B)$. Thus $f \in \Gamma$ has a (unique) integral representation if and only if there exists a (unique) Radon measure m on $\mathcal{E}(B)$, satisfying condition (1.4), such that $f = \int x dm(x)$.

Next assume $\gamma : t \longmapsto e_t$ is an admissible parametrization of $\mathrm{ext}(\Gamma)$ defined on T. Then f has a (unique) integral representation if and only if there exists a (unique) Radon measure on T satisfying condition (1.14) such that $\ell(f) = \int \ell(e_t) dm(t)$ for all $\ell \in F'$, i.e. $f = \int e_t dm(t)$. Thus our definition is simply a precise rendering of the accepted notion.

The problem is to know for which convex cones Γ every element $f \in \Gamma$ has a (unique) integral representation. Precisely : for which convex cones is the map $r : \mathcal{M}^+(\mathrm{ext}(\Gamma)) \to \Gamma$ surjective (respectively bijective).

To summarize the main results known today we shall say that a closed convex cone Γ has the integral representation property (I.R.P.) if :

a) For every closed convex subcone $\Gamma_1 \subset \Gamma$, the map $r : \mathcal{M}^+(\mathrm{ext}(\Gamma_1)) \to \Gamma_1$ is onto.

b) The map $r : \mathcal{M}^+(\mathrm{ext}(\Gamma_1)) \longrightarrow \Gamma_1$ is bijective if and only if Γ_1 is a lattice (with respect to its proper order).

Then Choquet's theorem affirms that if Γ has a compact metrizable base, Γ has the I.R.P. .

The theorem of Edgar and Bourgin [6], [15] says that if F is a separable Banach space, and Γ has a bounded base having the Radon Nikodym property, then Γ has the I.R.P. . Their results and theorem 5.1 below have a common generalization in which the compact metrizable sets are replaced by closed bounded Suslin sets having the Radon-Nikodym property [37].

On the other hand, Choquet's results on weakly complete cones show that every closed convex proper subcone of $\mathbb{R}^{\mathbb{N}}$ has the I.R.P. . More generally, every weakly complete cone which is the union of metrizable caps, in particular every weakly complete weakly metrizable proper convex cone, has the I.R.P. (cf. [8, §30]).

Recall that a **cap** of a cone Γ is a convex compact set $K \subset \Gamma$ such that $\Gamma \backslash K$ is convex. The extreme points of such a cap lie on the extreme rays of Γ, i.e. $\mathcal{E}(K) \subset \mathrm{ext}(\Gamma)$. Thus if

Γ is a closed convex cone, not necessarily weakly complete, which is the union of its metrizable caps, Choquet's theorem immediately implies that the map $r : \mathcal{M}^+(\mathrm{ext}(\Gamma)) \longrightarrow \Gamma$ is onto. We'll show that such a cone actually has the I.R.P., i.e. also the uniqueness theorem holds, even if Γ is not weakly complete.

3. Conuclear spaces and conuclear cones.

Recall the definition of a conuclear space (cf. [20], [30], [32]) : Let F be a locally convex Hausdorff space, which, to avoid trivial complications, is assumed to be quasi-complete (i.e. the closed bounded sets are complete).

Let $\mathfrak{S}$ be a set of closed, bounded, convex, symmetric subsets of F. With every set $A \in \mathfrak{S}$ one associates the space $F_A = \bigcup_{\lambda \geq 0} \lambda A$, equipped with the norm whose unit ball is A. Then F_A is a Banach space continuously included in F. If $B \in \mathfrak{S}$ and $A \subset B$ one has continuous inclusions $F_A \hookrightarrow F_B \hookrightarrow F$.

The space F is $\mathfrak{S}$-conuclear if for every set $A \in \mathfrak{S}$, there exists $B \in \mathfrak{S}$, such that $A \subset B$, and such that the inclusion $F_A \hookrightarrow F_B$ is a nuclear map (cf. [32] p.222, 227). If $\mathfrak{S}$ is the set of all closed convex bounded symmetric sets of F, the space F is simply called a **conuclear** space.

Proposition 3.1. *In a quasi-complete conuclear space every closed bounded set A is compact and metrizable.*

Proof. The closed convex hull of $A \cup (-A)$ being bounded we may assume that A is closed convex and symmetric. Then there exists a closed bounded symmetric set B containing A such that the inclusion map $F_A \hookrightarrow F_B$ is nuclear. But then, nuclear maps being compact, A is relatively compact in the Banach space F_B. Being closed in F, A is closed and so compact in F_B, hence compact metrizable in F. □

Most conuclear spaces are duals of nuclear spaces. In fact a space is conuclear, quasi-complete, and barreled if and only if it is the strong dual of a quasi-complete barreled nuclear space. The dual of a nuclear Fréchet space being nuclear, and nuclear Fréchet spaces being reflexive, every nuclear Fréchet space is conuclear (cf.[20], [32]).

One can give an equivalent definition of conuclear spaces as follows :

Proposition 3.2. *The space F is $\mathfrak{S}$-conuclear if for every set $A \in \mathfrak{S}$, there exists $B \in \mathfrak{S}$, with $A \subset B$, such that the inclusion $F_A \hookrightarrow F_B$ is an absolutely summing map.*

Proof. Nuclear maps being absolutely summing, and the product of two absolutely summing maps being nuclear, this is clear (cf. [30] 3.3.5). □

If B is a closed convex bounded set in F, containing 0, let :

$$p_B(x) = \inf\{\lambda \geq 0 : x \in \lambda B\} \tag{3.1}$$

Then $p_B : F \to [0, +\infty]$ is subadditive, positive homogeneous of degree 1, and one has :

$$B = \{x \in F : p_B(x) \leq 1\} \tag{3.2}$$

If B is symmetric $F_B = \{x \in F : p_B(x) < +\infty\}$ and the restriction of p_B to F_B is the norm of F_B.

Proposition 3.3. *The inclusion map $F_A \hookrightarrow F_B$ is absolutely summing if and only if there exists a constant $M \geq 0$ with the property that for every finite family $(x_i)_{i\in I}$ in F_A one has*

$$\sum_{i\in J} x_i \in A \quad \forall\, J \subset I \Longrightarrow \sum_{i\in I} p_B(x_i) \leq M. \tag{3.3}$$

(which implies $\sum\limits_{i\in I} p_{MB}(x_i) \leq 1$).

Proof. More generally, let F_A and F_B be normed spaces over $\mathbb{R}$ with unit ball A and B respectively, and let $u : F_A \longrightarrow F_B$ be a linear map. Denote A° the unit ball of the dual $(F_A)'$. Then u is absolutely summing if there exists M such that for every finite family $(x_i)_{i\in I}$ in F_A one has :

$$\sum_{i\in I} | < x_i, x' > | \leq 1 \quad \forall\, x' \in A^\circ \Longrightarrow \sum_{i\in I} ||u(x_i)|| \leq M \tag{3.4}$$

Now the condition $\sum\limits_{i\in I} | < x_i, x' > | \leq 1 \quad \forall\, x' \in A^\circ$ implies $| < \sum\limits_{i\in J} x_i, x' > | \leq 1 \quad \forall\, x' \in A^\circ$, hence $\sum\limits_{i\in J} x_i \in A$, for all $J \subset I$. Conversely the condition $\sum\limits_{i\in J} x_i \in A$, for all $J \subset I$, implies $\sum\limits_{i\in I} | < x_i, x' > | \leq 2 \quad \forall\, x' \in A^\circ$; it suffices to distinguish the indices $i \in I$ such that $< x_i, x' >> 0$ and $< x_i, x' >\leq 0$. The conclusion follows. □

One can now define conuclear cones in analogy with this : Let Γ be a convex cone in a locally convex Hausdorff space F. Let $\mathrm{co}(S)$ denote the convex hull of a set $S \subset F$.

Proposition 3.4. *Let A, B be closed convex bounded sets in Γ, containing 0. Then the following properties of the pair (A, B) are equivalent :*

1. *For every finite family $(x_i)_{i\in I}$ of elements in Γ one has :*

$$\sum_{i\in I} x_i \in A \Longrightarrow \sum_{i\in I} p_B(x_i) \leq 1 \tag{3.5}$$

2. $A \cap \mathrm{co}(\Gamma \backslash B) = \emptyset$.

Notation : If (A, B) satisfies these equivalent conditions we write $A << B$. Note that $A << B$ implies $A \subset B$.

Proof of the equivalence : 1. $\Longrightarrow$ 2. Let (A, B) satisfy 1. and let us assume on the contrary that $A \cap \mathrm{co}(\Gamma\backslash B) \neq \emptyset$. Let $y_i \in \Gamma\backslash B, \sum\limits_i \alpha_i y_i \in A, \alpha_i \geq 0$, and $\sum\limits_i \alpha_i = 1$. Then $\sum\limits_i \alpha_i p_B(y_i) = \sum\limits_i p_B(\alpha_i y_i) \leq 1$, which is a contradiction since $p_B(y_i) > 1$ for all i.

2. $\Longrightarrow$ 1. Let $x_i \in \Gamma$ be such that $\sum_i x_i \in A, x_i \neq 0$ for all i, and assume to the contrary that $\sum_i p_B(x_i) > 1$. Then there are numbers α_i such that $0 < \alpha_i < p_B(x_i)$ and $\sum_i \alpha_i = 1$. Let $y_i = x_i/\alpha_i$. Then $p_B(y_i) > 1$ so $y_i \in \Gamma \backslash B$, and $\sum_i x_i = \sum_i \alpha_i y_i \in \mathrm{co}(\Gamma\backslash B)$, in contradiction to the assumption. □

Let $\mathfrak{S}$ be a set of closed bounded convex subsets of Γ containing 0. It is convenient to assume that $\mathfrak{S}$ satisfies the following two conditions :

$$\begin{aligned} A \in \mathfrak{S} \Longrightarrow \lambda A \in \mathfrak{S} \quad \forall\, \lambda \geq 0, \\ \Gamma = \cup\{A : A \in \mathfrak{S}\}. \end{aligned} \tag{3.6}$$

Definition 3.5. The cone Γ will be called $\mathfrak{S}$-conuclear if Γ is the union of the sets belonging to $\mathfrak{S}$ and if for every $A \in \mathfrak{S}$ there exists $B \in \mathfrak{S}$ such that $A << B$.

If $\mathfrak{S}$ is the set of all compact convex subsets of Γ containing 0, we shall say that Γ is *cc*-conuclear.

Example 3.6. Let Γ be the union of its caps (resp. its metrizable caps). Let $\mathcal{K}$ (resp. $\mathcal{K}_o$) denote the set of caps (resp. metrizable caps) of Γ. Then Γ is $\mathcal{K}$-conuclear (resp. $\mathcal{K}_o$-conuclear).

In fact a cap is a convex compact subset $A \subset \Gamma$ containing 0 such that $\Gamma\backslash A$ is convex. Thus $A << A$ by the second characterization in 3.4. This can also be seen by the first characterization, because it is known that the gauge p_A of a cap is additive.

Example 3.7. Let $F = C([0,1])' = M([0,1])$ be the space of real Radon measures on $[0,1]$, equipped with the weak * topology. Let δ be the Dirac measure at 0 (or any other point). Let $K = \{\nu - \delta : ||\nu|| \leq 1\}$ and let $\Gamma = \bigcup_{\lambda \geq 0} \lambda K$. Then Γ is a closed convex cc-conuclear cone in $M([0,1])$ which is not the union of its caps.

The fact that it is closed is seen as follows : every element in Γ can be written in the form $\lambda(\nu - \delta)$, where $||\nu|| \leq 1$ and $\nu\{0\} = 0$. This implies $||\nu - \delta|| \geq ||\nu|| + 1 \geq 1$. From this one easily deduces that the intersection of Γ with every closed ball is closed (in the weak * topology). Thus by the theorem of Banach-Dieudonné Γ is itself closed. Now it has been shown by Goullet de Rugy ([19], prop. 2.3) that $K << 2K$ (K is 'conical' in the terminology of [19]) and that Γ is not the union of its caps. Thus if $\mathfrak{S} = \{\lambda K\}_{\lambda \geq 0}$ Γ is $\mathfrak{S}$-conuclear, although Γ is not the union of its caps. By the argument showing that Γ is closed it follows that every compact subset of Γ is contained in some set λK. Thus Γ is also *cc*-conuclear.

Proposition 3.8. *Let Γ be a closed convex cone which is $\mathfrak{S}$-conuclear. Then any closed convex cone $\Gamma_1 \subset \Gamma$ is $\mathfrak{S}_1$-conuclear where $\mathfrak{S}_1$ is the set of intersections $A \cap \Gamma_1$, with $A \in \mathfrak{S}$.*

This is obvious.

Proposition 3.9. *Let Γ be a closed convex cone which is $\mathfrak{S}$-conuclear. Then for every point $f \in \Gamma$ the order interval $\Gamma \cap (f - \Gamma)$ is a bounded subset of Γ. In particular $\Gamma \cap (-\Gamma) = \{0\}$, i.e. Γ is proper. If Γ is cc-conuclear, $\Gamma \cap (f - \Gamma)$ is compact for each $f \in \Gamma$.*

Proof. Let $f \in A \in \mathfrak{S}$. Choose $B \in \mathfrak{S}$ such that $A << B$. Then $\Gamma \cap (f - \Gamma) \subset B$. In fact, if $g \in \Gamma \cap (f - \Gamma)$ one has $f = g + (f - g)$, both term belonging to Γ, so $p_B(g) + p_B(f - g) \leq 1$. In particular $p_B(g) \leq 1$, i.e. $g \in B$. The set $\Gamma \cap (-\Gamma)$, being a bounded linear subspace, equals $\{0\}$. Finally, if B is compact (resp. compact metrizable) the set $\Gamma \cap (f - \Gamma)$, being a closed subset of B, has the same property. $\square$

Theorem 3.10. *Let F be a quasi-complete conuclear space. Let $\Gamma \subset F$ be a closed convex cone. Then the following conditions are equivalent :*

1. *$\Gamma \cap (f - \Gamma)$ is bounded for every $f \in \Gamma$.*
2. *$\Gamma \cap (f - \Gamma)$ is compact for every $f \in \Gamma$.*
3. *Γ is cc-conuclear.*
4. *There exists $\mathfrak{S}$ satisfying conditions (3.6) such that Γ is $\mathfrak{S}$-conuclear.*

We shall call a closed convex cone Γ in a quasi-complete conuclear space, which satisfies the conditions in theorem 3.10, simply a **conuclear cone**.

Corollary 3.11. *Let F be a quasi-complete conuclear space and let $\Gamma \subset F$ be a weakly complete proper convex cone. Then Γ is conuclear.*

In fact it is known that in that case $F' = \Gamma' - \Gamma'$ (cf. (2.6)), and so the order intervals are weakly bounded, hence bounded by Mackey's theorem (cf. [8 prop. 30.10]).

Proof of 3.10. 1. $\Longrightarrow$ 2. by theorem 3.1. ; 3. $\Longrightarrow$ 4. is trivial ; 4. $\Longrightarrow$ 1. is a consequence of 3.9. For the proof of the main assertion 2. $\Longrightarrow$ 3. we need the following lemma. In it and in the sequel we use the notation :

$$I_f = \Gamma \cap (f - \Gamma) \tag{3.7}$$

for the the interval between 0 and f with respect to the proper order of Γ, and

$$A^* = \Gamma \cap (A - \Gamma) \tag{3.8}$$

for the set $\bigcup_{f \in A} I_f$.

Lemma 3.12. *Let Γ be a closed convex cone in a sequentially complete locally convex space F such that the order intervals I_f are compact for all $f \in \Gamma$. Then for every compact subset $A \subset \Gamma$, the set A^* is closed and bounded.*

Proof. Since $A - \Gamma$ is closed so is A^*. To prove that A^* is bounded it is sufficient to show that every sequence $(x_n)_{n \in \mathbb{N}}$ in A^* is bounded. Let $y_n \in A$ be such that $0 \leq x_n \leq y_n$. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of non negative numbers such that $\sum_n \lambda_n < +\infty$. Then for every continuous seminorm p, we have $\sum_n \lambda_n p(y_n) < +\infty$. The space F being sequentially complete the series $\sum_{n=1}^{\infty} \lambda_n y_n$ converges. Let $f = \sum_{n=1}^{\infty} \lambda_n y_n$ and put $s_n = \sum_{k=1}^{n} \lambda_k x_k$. Then $0 \leq s_n \leq s_{n+1} \leq f$. Consequently, for every $\ell \in \Gamma'$ (cf. (2.6)) we have $\ell(s_n) \leq \ell(s_{n+1}) \leq \ell(f)$, which implies that $\lim_{n\to\infty} \ell(s_n)$ exists. Now since $\Gamma \cap (-\Gamma) = \{0\}$, Γ' separates the points of F. Therefore on the compact set I_f the uniform structure induced by F coincides with the uniform structure defined by the topology $\sigma(F, \Gamma')$. Thus $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. It converges and so the series $\sum_n \lambda_n x_n$ converges. In particular $\lim_{n\to\infty} \lambda_n p(x_n) = 0$ for every continuous seminorm p. This being the case for every such sequence $(\lambda_n)_{n\in\mathbb{N}}$ we have $\sup_n p(x_n) < +\infty$, i.e. $(x_n)_{n\in\mathbb{N}}$ is bounded. $\square$

One can now prove the implication 2. $\Longrightarrow$ 3. in 3.10 : Let A be a convex compact subset of Γ. Then A^* is convex and compact by the lemma and by proposition 3.1. Thus the set $A_s = A^* - A^*$ is convex compact and symmetric. The space F being conuclear, there exists a convex compact symmetric set B_s containing A_s such that the inclusion map $F_{A_s} \hookrightarrow F_{B_s}$ is absolutely summing. Replacing B_s by MB_s if necessary, one can assume that the constant M occurring in the analogue of (3.3) equals 1. Then if $B = \Gamma \cap B_s$ we have $A << B$. In fact, if $x_i \in \Gamma$ and $\sum_{i\in I} x_i \in A$, we have $\sum_{i\in J} x_i \in A^* \subset A_s$ for all subsets $J \subset I$, and so, since $p_B(x) = p_{B_s}(x)$ for all $x \in \Gamma$, we have $\sum_{i\in I} p_B(x_i) \leq 1$ $\square$

4. Approximation lemmas.

Let F be any locally convex Hausdorff space over $\mathbb{R}$. Let $s(F)$ be the set of functions of the form

$$\varphi = \sup_i \ell_i \tag{4.1}$$

i.e. supremum of a finite family of continuous linear forms. Then $s(F)$ is a convex cone in $h(F)$, and $h(F) = s(F) - s(F)$ (cf. (1.1)).

Given conical measures μ and ν one defines, following Choquet [8], an order relation as follows :

$$\mu \prec \nu \Longleftrightarrow \mu(\varphi) \leq \nu(\varphi) \qquad \forall\, \varphi \in s(F) \tag{4.2}$$

Since $s(F)$ generates $h(F)$ this relation is in fact anti-symmetric.

Also note that for $\ell \in F'$, both ℓ and $-\ell$ belong to $s(F)$. Thus

$$\mu \prec \nu \Longrightarrow r(\mu) = r(\nu) \tag{4.3}$$

For any point x belonging to F, or its weak completion, one defines the conical measure ε_x, for $\varphi \in h(F)$, by :

$$\varepsilon_x(\varphi) = \varphi(x) \tag{4.4}$$

Note that by (1.2)

$$\lambda\varepsilon_x = \varepsilon_{\lambda x} \quad \forall\, \lambda \geq 0 \tag{4.5}$$

expressing the fact that we are dealing with rays in cones rather than with points in convex sets. The conical measure ε_x is localizable if and only if x belongs to F. If $f = r(\mu)$ one obviously has :

$$\varepsilon_f \prec \mu \tag{4.6}$$

Let D be the set of conical integrals which are finite sums $\sum\limits_i \varepsilon_{x_i}$, with $x_i \in F$.

The following definitions will be useful when dealing with cones which are not necessarily weakly complete :

Definition 4.1. A conical measure $\mu \in M^+(F)$ is **approximable** if one has,

$$\mu(\varphi) = \sup\{\nu(\varphi) : \nu \prec \mu,\ \nu \in D\} \qquad \forall\, \varphi \in s(F) \tag{4.7}$$

A conical measure $\mu \in M^+(F)$ is strictly approximable if μ belongs to the closure, for the topology $\sigma(M^+(F), h(F))$, of the set of conical integrals $\nu \in D$ such that $\nu \prec \mu$.

Remark. Every strictly approximable conical measure is obviously approximable. No example seems to be known of an approximable conical measure which is not strictly approximable. By the Hahn-Banach theorem D is dense in $M^+(F)$ for the topology $\sigma(M^+(F), h(F))$.

If Γ is a weakly closed cone, in particular if Γ is a closed convex cone, a conical measure μ is said to be carried by Γ if $\mu(\varphi) = 0$ for all $\varphi \in h(F)$ such that $\varphi(x) = 0$ for all $x \in \Gamma$. Equivalently $\mu(\varphi) \geq 0$ for all $\varphi \in h(F)$ such that $\varphi(x) \geq 0$ for all $x \in \Gamma$ (cf. [8, 30.5, 30.6]).

Proposition 4.2. *Let μ be an approximable conical measure carried by a closed convex cone Γ. Then, if $\nu = \sum\limits_i \varepsilon_{x_i} \prec \mu$, we have $x_i \in \Gamma$ for all i. Also $r(\mu) \in \Gamma$.*

Proof. Let $\ell \in \Gamma'$ and let $\ell^- = sup(-\ell, 0)$ be its negative part. Then $\sum\limits_i \ell^-(x_i) = \nu(\ell^-) \leq \mu(\ell^-) = 0$. Thus $\ell^-(x_i) = 0$ for all i. Consequently $\ell(x_i) \geq 0$ for all $\ell \in \Gamma'$, and so by the Hahn-Banach separation theorem x_i belongs to Γ. Finally $r(\mu) = r(\nu) = \sum\limits_i x_i$ belongs to Γ. □

Applying this with $\Gamma = F$ we get :

Corollary 4.3. *If μ is approximable $r(\mu)$ belongs to F.*

In the remainder of this section it is assumed that Γ is a closed convex cone such that the closed convex hull of any compact subset $K \subset \Gamma$ is compact, i.e. Γ has the convex envelope property : $\Gamma \in (CE)$.

If F is quasi-complete this is the case for every closed convex cone.

Proposition 4.4. *Let Γ be a closed convex cone satisfying condition (CE). Then every conical integral concentrated on Γ, with $r(\mu) \in F$, is strictly approximable and carried by Γ.*

Proof. Let μ be localized in m on Γ. Let K be a compact subset of Γ and let μ_K be the conical integral defined by $\mu_K(\varphi) = \int_K \varphi(x) dm(x)$. By our hypothesis on $\Gamma, r(\mu_K) = \int_K x dm(x)$ belongs to Γ. Consequently $r(\mu - \mu_K) = r(\mu) - r(\mu_K)$ belongs to F, hence to Γ, $\mu - \mu_K$ being concentrated on Γ (cf. 2.1).

Now fix $\varphi \in h(F)$ and let $\epsilon > 0$ be given. Choose K convex and compact in Γ such that $|\mu(\varphi) - \mu_K(\varphi)| \leq \epsilon$ and such that, if $x_o = r(\mu - \mu_K), |\varphi(x_o)| \leq \epsilon$. The restriction of φ to K being uniformly continuous for the weak topology, there exists a partition of K into convex Borel sets $A_i,\ i = 1, \ldots, n$, intersections of polyhedral sets with K, such that for any choice of points a_i belonging to the closure $\overline{A_i}$ one has the following approximation of the integral by Riemann-Lebesgue sums :

$$| \int_K \varphi dm - \sum_{i=1}^{n} \varphi(a_i) m(A_i)| \leq \epsilon \tag{4.8}$$

in particular for $a_i = \frac{1}{m(A_i)} \int_{A_i} x dm(x)$ if $m(A_i) > 0$. Thus, if $m_i = 1_{A_i} m$ and $x_i = r(m_i)$ we have :

$$|\mu_K(\varphi) - \sum_{i=1}^{n} \varphi(x_i)| \leq \epsilon \tag{4.9}$$

Also, if $\psi \in s(F)$, we have :

$$\sum_{i=1}^{n} \psi(x_i) = \sum_{i=1}^{n}{}' m(A_i) \psi(a_i) \leq \sum_{i=1}^{n} \int_{A_i} \psi dm = \int_K \psi dm \tag{4.10}$$

the primed sum being over all the indices i such that $m(A_i) > 0$. Thus we have $\sum\limits_{i=1}^{n} \varepsilon_{x_i} \prec \mu_K$. Since $\varepsilon_{x_o} \prec \mu - \mu_K$ we have $\sum\limits_{i=0}^{n} \varepsilon_{x_i} \prec \mu$. By our construction it follows that $|\mu(\varphi) - \sum\limits_{i=0}^{n} \varepsilon_{x_i}(\varphi)| \leq 3\epsilon$. It is clear that if $\{\varphi\}$ is any finite subset of $h(F)$, one could have chosen K and the partition (A_i) in such a way that the approximation is valid simultaneously for all the functions φ under consideration. Thus μ belongs to the closure of the set of $\nu \in D$ such that $\nu \prec \mu$. Since any conical integral concentrated on Γ is obviously carried by Γ this ends the proof. □

Proposition 4.5. *Let Γ be a closed convex cone satisfying the condition (CE). Let Γ be $\mathfrak{S}$-conuclear, $\mathfrak{S}$ being a set of compact subsets of Γ satisfying the conditions* (3.6). *Then the following conditions on the conical measure $\mu \in M^+(F)$ are equivalent :*

1. *μ is localizable on Γ and $r(\mu) \in F$, briefly : $\mu \in \mathcal{M}^+(\Gamma)$.*

2. *μ is strictly approximable and carried by Γ.*

3. *μ is localizable on a set $B \in \mathfrak{S}$, in a measure m with $\int_B dm \leq 1$.*

Proof. 1. $\Longrightarrow$ 2. This is a particular case of the previous result 4.4. ; 2.$\Longrightarrow$ 3. By 4.2 we know that $r(\mu)$ belongs to Γ. Let $f = r(\mu)$ belong to $A \in \mathfrak{S}$.

Choose B such that $A << B$. If $\nu = \sum_{i \in I} \varepsilon_{x_i} \prec \mu$, we have $x_i \in \Gamma$ and $\sum_{i \in I} x_i = f$. Consequently $\sum_{i \in I} p_B(x_i) \leq 1$. Thus, δ_y being the Dirac measure at y, ν is localizable on B in the measure $\sum_{i \in I} p_B(x_i)\delta_{y_i}$, where $y_i = x_i/p_B(x_i)$. This measure has a total mass ≤ 1. (Since $\varepsilon_0 = 0$, we may assume without loss of generality that $x_i \neq 0$ for all $i \in I$). Now let (ν_α) be a net in D, with $\nu_\alpha \prec \mu$, and $\mu(\varphi) = \lim_\alpha \nu_\alpha(\varphi)$ for all $\varphi \in h(F)$. Let m_α be a localization of ν_α on B such that $\int dm_\alpha \leq 1$. The set of Radon measures on B, with total mass ≤ 1, is compact in the vague topology. If m is a limit point of the net (m_α) we have :

$$\mu(\varphi) = \lim_\alpha \nu_\alpha(\varphi) = \lim_\alpha \int_B \varphi dm_\alpha = \int_B \varphi dm \tag{4.11}$$

Thus the restriction of m to $B \backslash \{0\}$ is a localization of μ on B with total mass at most 1.

3. $\Longrightarrow$ 1. If μ is localized on B in m, with $\int_B dm \leq 1$, we have $r(\mu) = r(m) \in B$. Thus $r(\mu)$ belongs to F, and μ belongs to $\mathcal{M}^+(\Gamma)$. □

Remark 4.6. It has been shown in the course of the proof that if A and B are compact subsets of Γ such that $A << B$, every $\mu \in \mathcal{M}^+(\Gamma)$, such that $r(\mu) \in A$, can be localized on B in a measure with total mass ≤ 1.

5. The main integral representation theorem.

In this section it is again assumed that F is an arbitrary locally convex Hausdorff space and that Γ is a closed convex cone in F satisfying the condition (CE) i.e. the closed convex envelope of every compact subset of Γ is compact. If F is quasi-complete every closed convex cone Γ in F has this property.

For $f \in \Gamma$ let $I_f = \Gamma \cap (f - \Gamma)$ be the order interval between 0 and f, and let $\Gamma(f) = \bigcup_{\lambda \geq 0} I_{\lambda f}$ be the face generated by f in Γ. Then $\Gamma(f)$ is a convex subcone of Γ whose proper order is identical to the order induced by Γ. Clearly Γ is a lattice if and only if $\Gamma(f)$ is a lattice for all $f \in \Gamma$.

Theorem 5.1. *Let Γ be a closed convex cone in F satisfying condition (CE). Assume that Γ is $\mathfrak{S}$-conuclear, $\mathfrak{S}$ being a set of compact metrizable subsets of Γ whose union is Γ. Then we have :*

A) *The map $r : \mathcal{M}^+(\mathrm{ext}(\Gamma)) \longrightarrow \Gamma$ is surjective, i.e. every point $f \in \Gamma$ has an integral representation.*

B) *The point* $f \in \Gamma$ *has a unique integral representation, i.e. the set* $\{\mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma)) : r(\mu) = f\}$ *is a singleton, if and only if* $\Gamma(f)$ *is a lattice. In particular the map* $r : \mathcal{M}^+(\mathrm{ext}(\Gamma)) \to \Gamma$ *is bijective if and only if* Γ *is a lattice.*

Before proving this theorem we note two consequences : the theorem mentioned in the introduction (in terms of conical integrals), and the theorem on well-capped cones :

Theorem 5.2. *Let* F *be a quasi-complete conuclear space. Let* $\Gamma \subset F$ *be a closed convex cone such that the order intervals* $\Gamma \cap (f - \Gamma)$, $f \in \Gamma$, *are bounded subsets of the topological vector space* F. *Then* Γ *has the properties* A) *and* B) *of* theorem 5.1. *In particular this is the case for every weakly complete proper convex cone* $\Gamma \subset F$.

The first statement is a consequence of theorem 5.1 because Γ is *cc*-conuclear (theorem 3.10) and because the compact subsets of Γ are metrizable (proposition 3.1). By corollary 3.11 weakly complete cones have bounded order intervals.

Theorem 5.3. *Let* F *be an arbitrary locally convex Hausdorff space. Any closed convex cone* $\Gamma \subset F$, *satisfying the condition* (CE), *which is the union of metrizable caps, has the properties A) and B).*

In fact, if $\mathcal{K}_o$ is the set of metrizable caps in Γ, Γ is $\mathcal{K}_o$-conuclear. Note that in this case the surjectivity of the map $r : \mathcal{M}^+(\mathrm{ext}(\Gamma)) \to \Gamma$ is an immediate consequence of Choquet's theorem applied to the caps.

Corollary 5.4. *Under the assumption of* theorem 5.1 Γ *equals the closed convex hull of its extreme generators :* $\Gamma = \overline{\mathrm{co}}(\mathrm{ext}(\Gamma))$.

Proof. By proposition 2.1 $\Gamma = r(\mathcal{M}^+(\mathrm{ext}(\Gamma)))$ is contained in $\overline{\mathrm{co}}(\mathrm{ext}(\Gamma)) \subset \Gamma$. □

Corollary 5.5. *Under the hypotheses of* theorem 5.1 Γ *has the integral representation property (cf.* section 2*).*

In fact by proposition 3.8. every closed convex subcone of Γ satisfies the conditions of the theorem.

Corollary 5.6. (Choquet). *Any closed convex proper cone in* $\mathbb{R}^{\mathbb{N}}$ *has the integral representation property.*

In fact $\mathbb{R}^{\mathbb{N}}$ being both weakly complete and conuclear, this immediately results from theorem 5.2. In this case Choquet has shown that Γ is actually well-capped [8].

Remark 5.7. a) It can happen that some but not all elements of Γ have a unique integral representation. For instance, any element $e \in \mathrm{ext}(\Gamma) \backslash \{0\}$ has a unique integral representation : $\Gamma(e)$ is a ray (cf. [29, 1.4]).

If Γ is a cone with four extreme rays in $\mathbb{R}^3$ it is clear that the elements on the topological boundary have a unique "integral representation" while those in the interior do not.

From the theory of the Stieltjes moment problem it follows that there exists a closed convex proper cone $\Gamma \subset \mathbb{R}^{\mathbb{N}}$ which is not a lattice, such that the set of elements in Γ having a unique integral representation is dense in Γ. In fact the closed cone $\Gamma \subset \mathbb{R}^{\mathbb{N}}$ generated by the sequences $e_\lambda = (\lambda^n)_{n\geq 0}$, $\lambda \in \mathbb{R}_+$, which can be shown to form a section of $\mathrm{ext}(\Gamma)$, is not a lattice cone. For Stieltjes [33, p.510] has shown that there are sequences in Γ having more than one integral representation (cf.[38]). On the other hand the sequences e_λ, with $\lambda \in \mathbb{R}_+$, being linearly independent, their finite linear combinations with positive coefficients, which are dense in Γ, have a unique "integral representation".

b) Under the assumptions of theorem 5.1 it can be shown (using dilations) that a conical integral $\mu \in \mathcal{M}^+(\Gamma)$ belongs to $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ if and only if μ is maximal in $\mathcal{M}^+(\Gamma)$ with respect to the Choquet order. If Γ is weakly complete every conical measure carried by Γ is strictly approximable [8, proof of 30.9], hence belongs to $\mathcal{M}^+(\Gamma)$. Therefore if Γ is weakly complete μ belongs to $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ if and only if μ is maximal in the cone $M^+(\Gamma)$ of conical measures carried by Γ.

Problem : By theorem 5.1 the cone considered in example 3.7 has the integral representation property although it is not the union of its caps. Whether the cones in conuclear spaces considered in theorem 5.2 are union of their caps (i.e. well-capped) is unknown, even in the case of weakly complete cones. If such a cone is a lattice it is a consequence of theorem 5.2 that it is well-capped (essentially because $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ is well-capped).

For the proof of A) and B), under the assumptions of theorem 5.1, we use the following abbreviations :

$$\mathcal{M}_f = \{\mu \in \mathcal{M}^+(\Gamma) : r(\mu) = f\} \tag{5.1}$$

and

$$D_f = D \cap \mathcal{M}_f \tag{5.2}$$

i.e. the set of finite sums $\nu = \sum_i \varepsilon_{x_i}$, with $x_i \in \Gamma$ and $\sum_i x_i = f$.

Without restricting the generality we may assume that $\mathfrak{S}$ is invariant under the transformations $A \mapsto \lambda A$, $\lambda \geq 0$, i.e. $\mathfrak{S}$ satisfies conditions (3.6).

For every $\varphi \in h(F)$ and $x \in \Gamma$ let

$$\varphi'(x) = \sup \sum_i \varphi(x_i) \tag{5.3}$$

where the supremum is taken over all finite families $(x_i)_{i\in I}$ such that $x_i \in \Gamma$ for all $i \in I$, and $\sum_i x_i = x$. Equivalently :

$$\varphi'(x) = \sup\{\nu(\varphi) : \nu \in D_x) \tag{5.4}$$

Lemma 5.8. *The function φ' has the following properties :*

$$\varphi(x) \leq \varphi'(x) < +\infty \qquad \forall\, x \in \Gamma \tag{5.5}$$

$$\varphi'(x+y) \geq \varphi'(x) + \varphi'(y) \qquad \forall\, x, y \in \Gamma \tag{5.6}$$

$$\varphi'(\lambda x) = \lambda\varphi'(x) \qquad \forall\, \lambda \geq 0, \quad \forall\, x \in \Gamma \tag{5.7}$$

$$\varphi'(x) = \varphi(x) \qquad \forall\, x \in \operatorname{ext}(\Gamma) \tag{5.8}$$

These are, except for the finiteness in (5.5), immediate consequences of the definition. The inequality $\varphi'(x) < +\infty$ can be obtained as follows : Let $x \in A << B$, and let $M = \sup_{y \in B} \varphi(y)$. Then by the homogeneity of p_B we have $\varphi \leq M p_B$ on Γ. Consequently $\varphi'(x) \leq M \sup(\sum_i p_B(x_i)) \leq M$, the supremum being taken over the same families as in (5.3).

Lemma 5.9. *For every set $A \in \mathfrak{S}$ the restriction $\varphi'|_A$ is upper semi-continuous.*

Proof. Instead of (5.4) we also have

$$\varphi'(x) = \sup\{\mu(\varphi) : \mu \in \mathcal{M}_x\} \tag{5.9}$$

since by proposition 4.4 D_x is dense in $\mathcal{M}_x$. If $x \in A << B$ we know (remark 4.6) that every $\mu \in \mathcal{M}_x$ can be localized on B in a measure m with $\int_B dm \leq 1$. Let $M_x^1(B)$ be the set of Radon measures on B such that $\int_B dm \leq 1$ and $r(m) = x$. Then by (5.9) $\varphi'(x) = \sup\{\int \varphi dm : m \in M_x^1(B)\}$. Now $M_x^1(B)$ being compact in the vague topology (i.e. the weak * topology in the duality with the continuous functions on B), this supremum is attained. If (x_α) is a net in A converging to x, and such that $k \leq \varphi'(x_\alpha)$ for all α, let $m_\alpha \in M_x^1(B)$ be such that $\varphi'(x_\alpha) = \int_B \varphi dm_\alpha$, and let m be a limit point of the net (m_α) with respect to the vague topology. Then $m \in M_x^1(B)$ and we have $k \leq \int_B \varphi dm \leq \varphi'(x)$. Thus $\varphi'|_A$ is upper semi-continuous. □

Lemma 5.10. *If $\mu \in \mathcal{M}_f$, φ' is μ-summable and one has :*

$$\mu(\varphi) \leq \mu(\varphi') \leq \varphi'(f) \tag{5.10}$$

Proof. Let $f \in A << B$, and let m be a probability measure on B localizing μ (if $m \in M_f^1(B)$ and $\lambda = \int dm < 1$ one can always replace m by $1/\lambda$ times the image of m under the map $x \mapsto \lambda x$). Then by (5.5) and the fact that $\varphi|_B$ is upper semi-continuous φ' is m-integrable and we have $\int_B \varphi dm \leq \int_B \varphi' dm$. Also, since by (5.6) and (5.7) φ' is concave we have $\int_B \varphi' dm \leq \varphi'(f)$ (note that, by the Hahn-Banach theorem, $\varphi'|_B$ is the infimum of functions $-\psi, \psi \in s(F)$ (cf. [8], 21.23). Thus φ' being homogeneous of degree 1, φ' is μ-summable and we have (5.10). □

We can now proceed with the proof of part A) of the theorem : Let $f \in \Gamma$ be given. Choose A, B and C in $\mathfrak{S}$ such that $f \in A << B << C$. Then $B^* \subset C$ and so B^*, being obviously closed (cf.(3.8)), is compact and metrizable. There then exists a sequence $(\ell_n)_{n\in\mathbb{N}}$ of continuous linear forms on F with the property that if $x, y \in B^*$ are not proportional, there exists $n \in \mathbb{N}$ such that $\ell_n(x)$ and $\ell_n(y)$ are of opposite sign, hence

$|\ell_n(x+y)| < |\ell_n(x)| + |\ell_n(y)|$. Scaling the ℓ_n if necessary, we may assume $|\ell_n(x)| \leq 1/n^2$ for all $x \in B$. Put $\Phi(x) = \sum_{n\in\mathbb{N}} |\ell_n(x)|$ for all $x \in \Gamma$. Then $\Phi : \Gamma \longrightarrow [0, +\infty]$ is positive homogeneous of degree 1, subadditive, and the restriction $\Phi|_B$ is finite and continuous. Also we have $\Phi(x_1 + x_2) < \Phi(x_1) + \Phi(x_2)$ if x_1 and x_2 belong to B^* and are non-proportional.

Let $\Phi'(x) = \sup \sum_i \Phi(x_i)$, the supremum being taken over all finite families $(x_i)_{i\in I}$ with $x_i \in \Gamma$, and $\sum_i x_i = x$. Then, except for the finiteness, Φ' has the properties of φ' described in Lemma 5.8. In particular $\Phi(x) \leq \Phi'(x)$ for all $x \in \Gamma$, with equality for $x \in \operatorname{ext}(\Gamma)$.

Lemma 5.11. *If $\mu \in \mathcal{M}_f$, Φ' is μ-summable. Moreover*

$$\mu(\Phi) \leq \mu(\Phi') \leq \Phi'(f) < +\infty \tag{5.11}$$

and

$$\Phi'(f) = \sup\{\mu(\Phi) : \mu \in \mathcal{M}_f\} \tag{5.12}$$

Proof. If $\Phi(x) \leq M$ for all $x \in B, \Phi(x) \leq M p_B(x)$ for all $x \in \Gamma$, hence $\Phi'(f) \leq M$. Let $\Phi_n(x) = \sum_{i=1}^{n} |\ell_i(x)|$. Then $\Phi_n \leq \Phi_{n+1} \leq \Phi$, $\Phi'_n \leq \Phi'_{n+1} \leq \Phi'$, $\Phi = \sup_n \Phi_n$ and so exchanging two suprema : $\Phi' = \sup_n \Phi'_n$. By lemma 5.10 we have $\mu(\Phi_n) \leq \mu(\Phi'_n) \leq \Phi'_n(f)$, and so, applying the monotone convergence theorem to some localization of μ, we obtain the inequalities (5.11). Now (5.12) is obvious since by definition $\Phi'(f) = \sup\{\nu(\Phi) : \nu \in D_f\}$.

Note that $\Phi'_n|_B$ being upper semi-continuous, $\Phi'|_B$ is a Borel function.

The proof of A) now results from :

Lemma 5.12. *The supremum in (5.12) is attained. If $\mu \in \mathcal{M}_f$ is such that $\mu(\Phi) = \Phi'(f)$, μ is concentrated on* $\operatorname{ext}(\Gamma)$. *(In particular* $\operatorname{ext}(\Gamma) \neq \{0\}$*).*

Proof. Every $\mu \in \mathcal{M}_f$ being localized in some $m \in M^1_f(B)$, the supremum in (5.12) can be written $\sup\{\int \Phi dm : m \in M^1_f(B)\}$, and this is attained, $\Phi|_B$ being continuous. This proves the first assertion.

Let $\mu \in \mathcal{M}_f$ be such that $\mu(\Phi) = \Phi'(f)$ then by (5.11) we have $\mu(\Phi) = \mu(\Phi')$. If m is a localization of μ on B : $\int \Phi dm = \int \Phi' dm$. Thus $\Phi(x) = \Phi'(x)$ for m-almost all $x \in B$. Now if $x \in B$ and $x = x_1 + x_2$, x_1 and x_2 being non proportional elements of Γ, we have $\Phi(x) < \Phi(x_1) + \Phi(x_2) \leq \Phi'(x)$. Thus m is concentrated on the set

$$B \cap \operatorname{ext}(\Gamma) = \{x \in B : \Phi(x) = \Phi'(x)\} \tag{5.13}$$

and so μ is concentrated on $\operatorname{ext}(\Gamma)$. Since $r(\mu) = f$ this ends the proof of A).

It is convenient at this point to first prove :

Proposition 5.13. *Under the assumptions of theorem 5.1,* $\operatorname{ext}(\Gamma)$ *is universally measurable. For all* $B \in \mathfrak{S}, B \cap \operatorname{ext}(\Gamma)$ *is a* G_δ *in* B*. In particular, if* Γ *is cc-conuclear* $K \cap \operatorname{ext}(\Gamma)$ *is a* G_δ *in* K *for every compact set* $K \subset \Gamma$.

Proof. By (5.13) $B \cap \operatorname{ext}(\Gamma)$ is a Borel set for all $B \in \mathfrak{S}$. More precisely $B \cap \operatorname{ext}(\Gamma) = \{x \in B : \Phi'(x) \le \Phi(x)\} = \bigcap_{n \in \mathbb{N}} \{x \in B : \Phi'_n(x) < \Phi(x) + 1/n\}$ is a G_δ in B for all $B \in \mathfrak{S}$. Now let m be any Radon measure on Γ with compact support. Then by condition (CE) $r(m)$ belongs to Γ. Thus the conical integral μ which is localized in m belongs to $\mathcal{M}^+(\Gamma)$. By proposition 4.5 μ is also localized in a measure m' on a set $B \in \mathfrak{S}$. Then as we have seen, $\operatorname{ext}(\Gamma)$ is m'-measurable. But the indicator of $\operatorname{ext}(\Gamma)$ being homogeneous of degree 0, theorem 1.10 implies that $\operatorname{ext}(\Gamma)$ is m-measurable. □

A similar argument, using lemma 5.8, shows that the functions φ' are universally measurable.

Now we can give the proof of B) in theorem 5.1 : First assume that $\Gamma(f)$ is a lattice.

Lemma 5.14. *The set* D_f *is directed with respect to the Choquet order.*

Proof. Let $\nu' = \sum_{i \in I} \varepsilon_{x_i}$ and $\nu'' = \sum_{j \in J} \varepsilon_{y_j}$ be elements of D_f. In particular, $\sum_{i \in I} x_i = \sum_{j \in J} y_j = f$. Then by the Riesz decomposition property (cf. [29, 9.1]) there exists a family $(z_{ij})_{i \in I, j \in J}$ in $\Gamma(f)$ such that $x_i = \sum_{j \in J} z_{ij}$ and $y_j = \sum_{i \in I} z_{ij}$ for every i and j. If $\nu = \sum_{i,j} \varepsilon_{z_{i,j}}$ we have $\nu \in D_f$, $\nu' \prec \nu$ and $\nu'' \prec \nu$. □

As a consequence of this lemma we have, for each $\varphi \in s(F)$:

$$\varphi'(f) = \sup_{\nu \in D_f} \nu(\varphi) = \lim_{\nu \in D_f} \nu(\varphi) \tag{5.14}$$

Since $h(F) = s(F) - s(F)$ the limit on the right hand side, with respect to the directed set D_f, also exists for every $\varphi \in h(F)$. If $\varphi \in h(F)$, let :

$$\mu_f(\varphi) = \lim_{\nu \in D_f} \nu(\varphi) \tag{5.15}$$

Then μ_f is obviously a conical measure, carried by Γ, with resultant f. Moreover μ_f is strictly approximable by construction. Thus proposition 4.5 implies that μ_f is a conical integral localizable on a set $B \in \mathfrak{S}$. Also, with the notation of Lemma 5.11 and its proof, we have $\mu_f(\Phi_n) = \mu_f(\Phi'_n) = \Phi'_n(f)$ and so $\mu_f(\Phi) = \mu_f(\Phi') = \Phi'(f)$. Consequently by lemma 5.12 we have $\mu_f \in \mathcal{M}^+(\operatorname{ext}(\Gamma))$.

To prove the uniqueness in B) we will show that for any $\mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$ with $r(\mu) = f$ one has $\mu = \mu_f$. It is clear from lemma 5.10 and the identity (5.14) that for $\mu \in \mathcal{M}^+(\Gamma)$ we have

$$\mu(\varphi) \le \mu_f(\varphi) \qquad \forall\, \varphi \in s(F) \tag{5.16}$$

To get the opposite inequality if $\mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$ it will be convenient to use the following lemmas :

Lemma 5.15. *Let $\mu \in \mathcal{M}^+(\Gamma)$. Then*

$$\mu(\varphi') = \inf \nu(\varphi') \tag{5.17}$$

where ν describes the set of discrete conical integrals obtained as follows : $\nu = \sum_{i\in I} \varepsilon_{x_i}$, where $x_i = r(\mu_i)$, with $\mu_i \in \mathcal{M}^+(\Gamma)$ and $\sum_{i\in I} \mu_i = \mu$.

Proof. By lemma 5.10, $\mu_i(\varphi') \le \varphi'(x_i)$, hence $\mu(\varphi') \le \sum_{i\in I} \varphi'(x_i) = \nu(\varphi')$. Let $r(\mu) \in A << B$, and let μ be localized on B in a measure m with $\int_B dm \le 1$. Then $\varphi'|_B$ being upper semi-continuous, there exists, $\epsilon > 0$ being given, a continuous function $g : B \to \mathbb{R}$ such that $g \ge \varphi'|_B$ and $\int_B g dm \le \int_B \varphi' dm + \epsilon$, (cf. [8], 21.23). Moreover, g being uniformly continuous with respect to the weak topology, which on K coincides with the given topology, there exists a finite partition of B : $\bigcup_{i\in I} B_i = B$, in convex Borel sets, such that the oscillation of g on each set B_i is at most ϵ. Let $b_i = \frac{1}{m(B_i)} \int_{B_i} x dm(x)$ if $m(B_i) > 0$. Then $b_i \in \overline{B_i}$ and so $g(b_i) \le g(x) + \epsilon$ for all $x \in B_i$. Let $x_i = m(B_i)b_i$ if $m(B_i) > 0$ and $x_i = 0$ otherwise. Then we have $\sum_{i\in I} \varphi'(x_i) = \sum_{i\in I} m(B_i)\varphi'(b_i) \le \sum_{i\in I} m(B_i)g(b_i) \le \sum_{i\in I} \int_{B_i} (g(x) + \epsilon) dm(x) \le \int_B g dm + \epsilon \le \mu(\varphi') + 2\epsilon$. If μ_i is defined by the formula $\mu_i(\psi) = \int_{B_i} \psi dm$, we have $\mu_i \in \mathcal{M}^+(\Gamma)$, $r(\mu_i) = x_i$, and $\sum_{i\in I} \mu_i = \mu$. □

Lemma 5.16. *Let $\mu \in \mathcal{M}_f$. Then we have :*

$$\mu_f(\varphi) \le \mu(\varphi') \qquad \forall\, \varphi \in s(F) \tag{5.18}$$

Proof. By the previous lemma it is sufficient to prove this when $\mu = \sum_{i\in I} \varepsilon_{x_i}$ belongs to D_f, which we now assume. Let $\nu \in D_f$ be such that $\mu_f(\varphi) < \nu(\varphi) + \epsilon$. This can be achieved with $\mu \prec \nu$, μ_f being the limit of elements $\succ \mu$ (5.15). Let $\nu = \sum_{j\in J} \varepsilon_{y_j}$. Using the Riesz decomposition property of $\Gamma(f)$ we may even assume that ν has been so chosen that there exists a partition $J = \bigcup_{i\in I} J_i$ such that $x_i = \sum_{j\in J_i} y_j$ for all $i \in I$. Then $\sum_{j\in J_i} \varphi(y_j) \le \varphi'(x_i)$ by definition of φ'. Adding these inequalities we obtain $\nu(\varphi) \le \mu(\varphi')$, and so $\mu_f(\varphi) \le \mu(\varphi') + \epsilon$, which proves the lemma. □

With this the proof of the uniqueness is complete, for if $\mu \in \mathcal{M}_f$ is concentrated on $\mathrm{ext}(\Gamma)$ we have $\varphi'(x) = \varphi(x)$ μ almost everywhere, whence $\mu_f(\varphi) \leq \mu(\varphi') = \mu(\varphi)$ for all $\varphi \in s(F)$. Comparing with (5.16) we see that $\mu(\varphi) = \mu_f(\varphi)$ for all $\varphi \in s(F)$, hence $\mu = \mu_f$.

Thus we have shown that if $\Gamma(f)$ is a lattice f has a unique integral representation. In particular, if Γ is a lattice the map $r : \mathcal{M}^+(\mathrm{ext}(\Gamma)) \longrightarrow \Gamma$ is a bijection.

Next consider the converse. Let us first assume that the map r is a bijection, and show that Γ is a lattice. The map r being linear it is sufficient to show that $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ is a lattice. Now $h(F)$ being a lattice, it is well known that $M^+(F)$ is a lattice. But $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ is a face in $M^+(F)$ i.e. $0 \leq \nu \leq \mu, \mu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$ implies $\nu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$. In fact, by theorem 1.15 if m is a localization of μ on a set $B \in \mathfrak{S}$ with $\int_B dm \leq 1$, ν has a localization n on B with $n \leq m$. Thus $r(\nu) = \int x dn(x)$ belongs to F and n is concentrated on $\mathrm{ext}(\Gamma)$, i.e. $\nu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$. Therefore $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ is a lattice.

To obtain the full converse, assume $f \in \Gamma$ has a unique integral representation i.e. $\mathcal{M}_f \cap \mathcal{M}^+(\mathrm{ext}(\Gamma))$ is a singleton. Then every element $g \in \Gamma$ with $g \leq f$ has a unique integral representation. In fact if $g = r(\mu') = r(\mu'')$, with μ', μ'' in $\mathcal{M}^+(\mathrm{ext}(\Gamma))$, and $\nu \in \mathcal{M}^+(\mathrm{ext}(\Gamma))$ is such that $r(\nu) = f - g$, the conical integrals $\mu' + \nu$ and $\mu'' + \nu$ are elements of $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ representing f. Since they are equal $\mu' = \mu''$. More generally, every element in $\Gamma(f)$ has a unique integral representation.

If $x \in \Gamma(f)$ let μ_x denote the unique element of $\mathcal{M}^+(\mathrm{ext}(\Gamma))$ such that $r(\mu_x) = x$. Then, if $f = g+h$, $\mu_g+\mu_h$ is a conical integral concentrated on $\mathrm{ext}(\Gamma)$ representing f. Therefore $\mu_f = \mu_g + \mu_h$, in particular $\mu_g \leq \mu_f$. Let $M^+(\mu_f)$ be the face generated by μ_f in $M^+(F)$. Then again by theorem 1.15 we have $M^+(\mu_f) \subset \mathcal{M}^+(\mathrm{ext}(\Gamma))$. In particular r maps $M^+(\mu_f)$ into Γ and so into $\Gamma(f)$. Therefore the map $r : M^+(\mu_f) \to \Gamma(f)$ is bijective, and so $M^+(\mu_f)$ being a lattice, $\Gamma(f)$ is a lattice.

Remark 5.17. If $\Gamma(f)$ is a lattice one can write instead of (5.15) :

$$\mu_f(\varphi) = \lim \sum_i \varphi(f_i) \tag{5.19}$$

equivalently :

$$\mu_f = \lim \sum_i \varepsilon_{f_i} \tag{5.20}$$

the limit being taken with respect to the directed set of finite partitions of f, i.e. the finite families $(f_i)_{i\in I}$ in Γ such that $\sum_{i\in I} f_i = f$, ordered by the relation "finer than". This will be useful in applications to direct integrals.

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