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A Derivation Formula for Convex Integral Functionals Defined on $BV(\Omega)$

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Dedicated to R. T. Rockafellar on his 60th Birthday

We show that convex lower semicontinuous functionals defined on functions of bounded variation are characterized by their minima, and we prove a derivation formula which allows an integral representation of such functionals. Applications to relaxation and homogenization are given.

Keywords : integral functional, lower semicontinuity, functions of bounded variation.

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1. Introduction

In this work we study weakly lower semicontinuous integral functionals of the form

$$G(u) = \int_\Omega g(x, \nabla u)\, dx + \int_\Omega g^\infty\Big(x, \frac{dD_s u}{d|D_s u|}\Big)\, d|D_s u|,$$

defined on the space $BV(\Omega)$ of (scalar-valued) functions of bounded variation on an open set Ω of $\mathbb{R}^n$ when g is a discontinuous integrand in the space variable. For analogous integrals defined on Sobolev spaces the lower semicontinuity with respect to the weak topology is equivalent, under appropriate measurability and growth conditions, to the convexity of the integrand (with respect to the gradient). The richer structure of the functions of bounded variation yield that the hypotheses on g must be in general more complex. In particular we have to take into proper care the fact that the singular part of the derivative $D_s u$ may charge sets of zero Lebesgue measure; hence, we have to consider the values of g (or rather of its recession function g^∞) also on sets of Hausdorff dimension lower than n. The goal of this paper is to show that an integrand φ equivalent

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to the function g (that is, generating the same integrals) may be "reconstructed" from the computation of some minimum problems for the functional G (Theorem 2.1) when g is positively homogeneous of degree one in the second variable. The formula we obtain for the integrand is of "derivation-type", and it does not involve the pointwise behaviour of g. In particular it can be used to compute the lower semicontinuous envelope with respect to the $\mathrm{L}^1(\Omega)$-topology of integral functionals of linear growth also in the non-one-homogeneous case (see Theorem 2.3 and Section 4), and to obtain a homogenization formula for positively one-homogeneous integral functionals (see Section 5) defined on $\mathrm{W}^{1,1}(\Omega)$. We remark that our result utilizes only abstract properties of the functional G; hence it can be thought of as an "integral representation" theorem (see Remark 2.2).
The paper is divided as follows. In Section 2 we recall the main definitions about functions of bounded variation and sets of finite perimeter, and we state our main results. Section 3 is devoted to the proof of the derivation formula for positively one-homogeneous functionals. We remark that, by the coarea formula and by a localization argument, it suffices to prove Theorem 2.1 for characteristic functions of sets with C^1-boundary. The proof is then obtained by a direct construction. In Section 4 we apply Theorem 2.1 to give a characterization of the integrand of the relaxation of functionals defined on $\mathrm{W}^{1,1}(\Omega)$, and of functionals defined on "partitions of Ω in sets of finite perimeter". Finally, in Section 5 we prove a homogenization formula for positively one-homogeneous functionals.
A study of the same type of functionals by duality theory methods can be found in Bouchitté and Dal Maso [7]. For a comparison with a derivation formula for functionals defined on Sobolev spaces $W^{1,p}(\Omega)$ with $p > 1$ we refer to Dal Maso and Modica [13]. Some results related to the present work can be found in [2], [3], [4], [5], [15], [16], where different derivation formulas were given in the case of continuous integrands. Note that all these formulas may not give the correct result when g is discontinuous in the first variable.

2. Preliminaries and Statement of the Main Result

Let Ω be an open subset of $\mathbb{R}^n$; we will use standard notation for the Sobolev and Lebesgue spaces $\mathrm{W}^{1,p}(\Omega)$ and $\mathrm{L}^p(\Omega)$. We denote by $\mathcal{A}(\Omega)$ (respectively, $\mathcal{B}(\Omega)$) the family of the open (respectively, Borel) subsets of Ω, and if $x, y \in \mathbb{R}^n$ then $\langle x, y\rangle$ stands for their scalar product.
The Lebesgue measure and the Hausdorff $(n-1)$-dimensional measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}_n$ and $\mathcal{H}^{n-1}$, respectively. Sometimes we use the shorter notation $\{u < t\}$ for $\{x \in \mathbb{R}^n : u(x) < t\}$ (and similar) when no confusion is possible. If E is a subset of $\mathbb{R}^n$ then χ_E is its *characteristic function*, defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

If $f : \mathbb{R}^n \to [0, +\infty[$ is convex, we define the *recession function* $f^\infty : \mathbb{R}^n \to [0, +\infty]$ by

$$f^\infty(z) = \lim_{t\to+\infty} \frac{f(tz)}{t}.$$

We remark that f^∞ is a Borel function, and it is convex and positively homogeneous of degree 1. If $f : \mathbb{R}^n \to [0, +\infty]$ is a Borel function we denote by f^{**} the greatest convex and lower semicontinuous function less than or equal to f.

Given a vector-valued measure μ on Ω, we adopt the notation $|\mu|$ for its total variation (see Federer [14]), and $\mathcal{M}(\Omega)$ is the set of all signed measures on Ω with bounded total variation. If μ is a (vector-valued) measure on Ω, and f is a Carathéodory function, positively one-homogeneous in the second variable we write $\int_\Omega f(x,\mu)$ in place of $\int_\Omega f(x,\frac{d\mu}{d|\mu|})d|\mu|$.
We will use Besicovitch Covering Theorem in the following generalized form (which is a particular case of Morse [18] Theorem 5.13): let μ be a positive Radon measure on Ω, and let $\mathcal{Q}$ be a collection of closed cubes which covers finely Ω (i.e., for μ-a.e. $x \in \Omega$ we have $\inf\{|Q| : x \in Q \in \mathcal{Q}\} = 0$); then there exists a (finite or) countable family $(Q_i)_i \subset \mathcal{Q}$ such that $\mu(\Omega \setminus \bigcup_i Q_i) = 0$.
We say that $u \in L^1(\Omega)$ is a *function of bounded variation*, and we write $u \in BV(\Omega)$, if its distributional first derivatives $D_i u$ belong to $\mathcal{M}(\Omega)$. We denote by Du the $\mathbb{R}^n$-valued measure whose components are $D_1 u, \dots, D_n u$. For the general exposition of the theory of functions of bounded variation we refer to Federer [14], Giusti [17], Vol'pert [19], and Ziemer [20]. Next we just recall some results needed in the sequel.
The space $BV(\Omega)$ is a Banach space, if endowed with the BV norm

$$\|u\|_{BV} = \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

We say that $u_h \rightharpoonup u$ in BV-w^* (weakly* in $BV(\Omega)$) if $\sup_h |Du_h|(\Omega) < +\infty$ and $u_h \to u$ in $L^1(\Omega)$. Recall that if Ω is bounded and $\partial\Omega$ is Lipschitz, then every bounded sequence in $BV(\Omega)$ admits a subsequence converging in BV-w^*.
We denote by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure, and $D_s u$ stands for the singular part of Du with respect to the Lebesgue measure, so that

$$Du = \nabla u \cdot \mathcal{L}_n + D_s u.$$

We denote by S_u the complement of the Lebesgue set of u. If $u \in BV(\Omega)$, then the Hausdorff dimension of S_u is at most $(n-1)$.
We will say that a set E is of *finite perimeter in* Ω, or a Caccioppoli set, if $\chi_E \in BV(\Omega)$. We will set $\partial^* E \cap \Omega = S_{\chi_E} \cap \Omega$ the *reduced boundary* of E in Ω. For $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$ it is possible to define a measure theoretical *interior normal* to E $\nu_E(x) \in S^{n-1}$ such that

$$D\chi_E(B) = \int_{B\cap\partial^* E} \nu_E(x)\, d\mathcal{H}^{n-1}(x)$$

for every $B \in \mathcal{B}(\Omega)$. Remark that $|D\chi_E|(\Omega) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$ for every E of finite perimeter in Ω. Moreover if E is a set of finite perimeter, there is a countable sequence of C^1 hypersurfaces Γ_i which covers $\mathcal{H}^{n-1}$-almost all of $\partial^* E$; i.e., $\mathcal{H}^{n-1}(\partial^* E \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$.
The total variation of a function $u \in BV(\Omega)$ and its level sets are linked by the coarea formula

$$|Du|(A) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap A)\, dt = \int_{\mathbb{R}} |D\chi_{\{u>t\}}|(A)\, dt$$

for all $A \in \mathcal{A}(\Omega)$.
Finally, we recall the notion of *relaxed functional*. Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a functional on a topological space (X,τ). The relaxed functional $\overline{F}$ of F, or *relaxation* of F (in the τ topology), is the greatest τ lower semicontinuous functional less than or equal to F. For

a general treatment of this subject we refer to the books by Buttazzo [10] and by Dal Maso [12].
The letter c will denote throughout the paper a strictly positive constant, whose value may vary from line to line, which is independent from the parameters of the problems each time considered.

In order to state our main result we have to introduce some definitions. We say that a function $h : \Omega \times \mathbb{R}^n \to [0, +\infty[$ satisfies a growth condition of order 1 if there exist c_1, C_1 strictly positive constants such that

$$c_1|\xi| - C_1 \ \leq\ h(x,\xi) \ \leq\ C_1(1+|\xi|) \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{2.1}$$

Let $x \in \mathbb{R}^n$, $\rho > 0$, $\nu \in S^{n-1}$. We denote by $Q^\nu_\rho(x)$ an open cube centered in x, of side length ρ and one face orthogonal to ν. We suppose that fixed x and ν for each ρ and $\sigma > 0$ the cube $Q^\nu_\sigma(x)$ is obtained from $Q^\nu_\rho(x)$ by an homothety of center x. We also define the function $u^{\nu,x}$ by

$$u^{\nu,x}(y) = \begin{cases} 1 & \text{if } \langle y - x, \nu\rangle > 0 \\ 0 & \text{if } \langle y - x, \nu\rangle \leq 0; \end{cases} \tag{2.2}$$

i.e., the characteristic function of the half space $\{y \in \mathbb{R}^n : \langle y - x, \nu\rangle > 0\}$.

Theorem 2.1. *Let $g : \Omega \times \mathbb{R}^n \to [0, +\infty[$ be a Borel function, positively 1-homogeneous in the second variable, satisfying a growth condition of order 1, and let us suppose that the integral functional defined by*

$$\mathcal{G}(u) = \int_\Omega g(x, Du) \qquad u \in BV(\Omega) \tag{2.3}$$

be lower semicontinuous with respect to the $\mathrm{L}^1(\Omega)$ topology. Let us define the positively 1-homogeneous function $\varphi : \Omega \times \mathbb{R}^n \to [0, +\infty[$ by setting for all $x \in \Omega$ and $\nu \in S^{n-1}$

$$\varphi(x,\nu) = \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \Phi(x,\nu,\rho), \tag{2.4}$$

where

$$\Phi(x,\nu,\rho) = \inf\Big\{\mathcal{G}(w, Q^\nu_\rho(x)) :\ w \in BV(Q^\nu_\rho(x)), w = u^{\nu,x} \text{ on } \partial Q^\nu_\rho(x)\Big\}, \tag{2.5}$$

and $\mathcal{G}(u, A) = \int_A g(y, Dw)$. Then we have

$$\mathcal{G}(u) = \int_\Omega \varphi(x, Du)$$

for every $u \in BV(\Omega)$.

Remark 2.2. We can restate Theorem 2.1 as an integral representation result. Let us consider a functional $\mathcal{G} : BV(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty[$ satisfying the following properties:
(H1) for every $u \in BV(\Omega)$ the set function $B \mapsto \mathcal{G}(u, B)$ is a Borel measure;

(H2) for every $A \in \mathcal{A}(\Omega)$ the function $\mathcal{G}(\cdot, A)$ is convex, positively 1-homogeneous and $\mathrm{L}^1(\Omega)$-lower semicontinuous on $BV(\Omega)$;

(H3) there exist positive constants c_1, C_1 such that

$$c_1|Du|(B) \ \leq \ \mathcal{G}(u,B) \ \leq \ C_1|Du|(B)$$

for every $u \in BV(\Omega)$ and $B \in \mathcal{B}(\Omega)$.

Then if we define the positively 1-homogeneous function $\varphi : \Omega \times \mathbb{R}^n \to [0,+\infty[$ as in (2,4), (2.5), then we have for every $u \in BV(\Omega)$

$$\mathcal{G}(u,B) = \int_B \varphi(x, Du)$$

for every $B \in \mathcal{B}(\Omega)$. In fact, it suffices to recall that by [7] Corollary 5.5 conditions (H1)–(H3) assure that the functional $\mathcal{G}$ can be represented in an integral form, and then apply Theorem 2.1.

As a Corollary to Theorem 2.1 we will obtain in Section 4 the following relaxation result for integral functionals defined on $\mathrm{W}^{1,1}(\Omega)$.

Theorem 2.3. *Let $f : \Omega \times \mathbb{R}^n \to [0,+\infty[$ be a Borel function, satisfying a growth condition of order* 1*, and such that $f(x,0) = 0$ for all $x \in \Omega$, and $f(x,\xi) \geq f^\infty(x,\xi) - a(x)$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, with $a \in \mathrm{L}^1(\Omega)$. Then the relaxed functional $\overline{\mathcal{F}}$ with respect to the $\mathrm{L}^1(\Omega)$-topology of the functional*

$$\mathcal{F}(u) = \begin{cases} \int_\Omega f(x, \nabla u(x))\, dx & \text{if } u \in \mathrm{W}^{1,1}(\Omega) \\ +\infty & \text{if } u \in BV(\Omega) \setminus \mathrm{W}^{1,1}(\Omega) \end{cases} \tag{2.6}$$

is given by

$$\overline{\mathcal{F}}(u) = \int_\Omega h(x, \nabla u(x))\, dx + \int_\Omega \varphi_f(x, D_s u), \qquad u \in BV(\Omega) \tag{2.7}$$

where φ_f is the positively 1*-homogeneous function defined by setting for all $x \in \Omega$, $\nu \in S^{n-1}$*

$$\begin{aligned} \varphi_f(x,\nu) = \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \lim_{t\to+\infty} \frac{1}{t} \inf\Big\{ \int_{Q^\nu_\rho(x)} f(y, \nabla w(y)) dy :& \\ : w \in \mathrm{W}^{1,1}(Q^\nu_\rho(x)), w = tu^{\nu,x} \text{ on } \partial Q^\nu_\rho(x) \Big\}&, \end{aligned} \tag{2.8}$$

and $h(x,\cdot) = (f(x,\cdot) \wedge \varphi_f(x,\cdot))^{**}$. If f is positively 1*-homogeneous then we can take $h = h^\infty = \varphi_f$ equal to*

$$\begin{aligned} \varphi_f(x,\nu) = \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \inf\Big\{ \int_{Q^\nu_\rho(x)} f(y, \nabla w(y)) dy :& \\ : w \in \mathrm{W}^{1,1}(Q^\nu_\rho(x)), w = u^{\nu,x} \text{ on } \partial Q^\nu_\rho(x) \Big\}&, \end{aligned}$$

for all $x \in \Omega$, $\nu \in S^{n-1}$.

Other relaxation results and an application to the asymptotic behaviour as ε tends to 0 of integrals of the form $\int_\Omega f(\frac{x}{\varepsilon}, \nabla u(x))\,dx$ with f positively 1-homogeneous will be obtained in Sections 4 and 5.

3. Proof of Theorem 2.1

Let $\mathcal{G}$ be as in Theorem 2.1. Note first that we can write by the Fleming and Rishel coarea formula

$$\mathcal{G}(u) = \int_{\mathbb{R}} \mathcal{G}(\chi_{E_t})\,dt, \tag{3.1}$$

where $E_t = \{u > t\}$. Hence, Theorem 2.1 is equivalent to the following one.

Theorem 3.1. *For every set of finite perimeter in Ω we have*

$$\int_{\Omega\cap\partial^* E} g(x, \nu_E(x))\,d\mathcal{H}^{n-1} = \int_{\Omega\cap\partial^* E} \varphi(x, \nu_E(x))\,d\mathcal{H}^{n-1}, \tag{3.2}$$

where $\nu_E(x)$ denotes the interior normal to E.

Remark 3.2. In the introduction we mentioned the fact that integrands of functionals defined on BV functions are not merely defined almost everywhere. The statement of Theorem 3.1 specifies that for every fixed E of finite perimeter the function g must be defined $\mathcal{H}^{n-1}$-a.e. on $\partial^* E$, but only "in the direction" of the normal to E. For example, the function $g : \mathbb{R}^2 \times S^1 \to [0, +\infty[$ defined by

$$g(x, \nu) = \begin{cases} 1 & \text{if } x_1 \neq 0 \text{ or } \nu = \pm e_1 \\ \text{any function} & \text{if } x_1 = 0 \text{ and } \nu \neq \pm e_1 \end{cases} \tag{3.3}$$

gives

$$\int_{\Omega\cap\partial^* E} g(x, \nu_E(x))\,d\mathcal{H}^1 = \mathcal{H}^1(\Omega \cap \partial^* E). \tag{3.4}$$

Note in fact that, by standard properties of the approximate tangent spaces, we have $\nu_E = \pm e_1$ $\mathcal{H}^1$-a.e. on $\partial^* E \cap \{x_1 = 0\}$ for every set of finite perimeter E.

Proof of Theorem 3.1
We localize the functional $\mathcal{G}$ by setting

$$\mathcal{G}(u, A) = \int_A g(x, Du), \tag{3.5}$$

for every $A \in \mathcal{A}(\Omega)$ and $u \in BV(\Omega)$, in particular

$$\mathcal{G}(\chi_E, A) = \int_{A\cap\partial^* E} g(x, \nu_E(x))\,d\mathcal{H}^{n-1} \tag{3.6}$$

for every $A \in \mathcal{A}(\Omega)$ and E set of finite perimeter. Since the set function $A \mapsto \mathcal{G}(\chi_E, A)$ is a measure for every E, and $\partial^* E$ can be expressed, up to a $\mathcal{H}^{n-1}$-negligible set, as the union of C^1 hypersurfaces, it will suffice to prove the equality

$$\mathcal{G}(\chi_E, A) = \int_{A\cap\partial^* E} \varphi(x, \nu_E(x))\,d\mathcal{H}^{n-1} \tag{3.7}$$

when $\partial^* E \cap A$ is a C^1 hypersurface (see [1] Lemma 4.2).

Fix E and A such that $\partial^* E \cap A$ is a C^1 hypersurface. Note that if $x \in A \cap \partial^* E$, $\nu = \nu_E(x)$, and we set

$$w = \begin{cases} \chi_E & \text{in } Q^\nu_{(\rho-\rho^2)}(x) \\ u^{\nu,x} & \text{in } Q^\nu_\rho(x) \setminus Q^\nu_{(\rho-\rho^2)}(x), \end{cases} \tag{3.8}$$

then w satisfies the boundary condition $w = u^{\nu,x}$ in $Q^\nu_\rho(x)$, and we have

$$\int_{Q^\nu_\rho(x)} g(y, Dw) \leq \int_{Q^\nu_\rho(x)\cap\partial^* E} g(y, \nu_E(y)) d\mathcal{H}^{n-1} + \rho^{n-2} o(\rho), \tag{3.9}$$

so that by (2.4)

$$\varphi(x, \nu_E(x)) \leq \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \int_{Q^\nu_\rho(x)\cap\partial^* E} g(y, \nu_E(y)) d\mathcal{H}^{n-1}. \tag{3.10}$$

By Lebesgue derivation theorem applied to the measure

$$\mu(A) = \int_{A\cap\partial^* E} g(y, \nu_E(y)) d\mathcal{H}^{n-1}$$

(note that $\lim_{\rho\to 0+} \rho^{1-n} \mathcal{H}^{n-1}(Q^\nu_\rho(x) \cap \partial^* E) = 1$) we obtain that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E \cap A$ the right hand side of (3.10) equals $g(x, \nu_E(x))$. Hence, we deduce the inequality

$$\int_{A\cap\partial^* E} \varphi(x, \nu_E(x))\, d\mathcal{H}^{n-1} \leq \mathcal{G}(\chi_E, A). \tag{3.11}$$

The converse inequality will be proven with the aid of formula (2.4) and of the lower semicontinuity of $\mathcal{G}$. We will exhibit a sequence u_h converging to χ_E in $L^1(A)$ such that

$$\liminf_h \mathcal{G}(u_h, A) \leq \int_{A\cap\partial^* E} \varphi(x, \nu_E(x))\, d\mathcal{H}^{n-1}. \tag{3.12}$$

The construction of such u_h will be obtained via a proper combination of minimizers for (2.5).
Let us denote by

$$\Gamma = \Big\{ x \in \partial^* E : \lim_{\rho\to 0+} \frac{1}{\rho^{n-1}} \int_{Q^\nu_\rho(x)\cap\partial^* E} \varphi(y, \nu_E(y)) d\mathcal{H}^{n-1} = \varphi(x, \nu_E(x)) \Big\}.$$

By Lebesgue derivation theorem we have $\mathcal{H}^{n-1}(\partial^* E \setminus \Gamma) = 0$.
Fix $h \in \mathbb{N}$. Let $u(x, \rho, \nu) \in BV(Q^\nu_\rho(x))$, with $u(x, \rho, \nu) = u^{\nu,x}$ on $\partial Q^\nu_\rho(x)$, satisfy

$$\int_{Q^\nu_\rho(x)} g(y, Du(x, \rho, \nu)) \leq \Phi(x, \nu_E(x), \rho) + \frac{\rho^{n-1}}{2h}.$$

We extend $u(x, \rho, \nu)$ to all A setting

$$u(x, \rho, \nu) = \chi_E$$

on $A \setminus Q_\rho^\nu(x)$.

Let us choose for each h the family $\mathcal{Q}_h$ of all closed cubes $\overline{Q_\rho^\nu(x)}$ such that $\rho \leq 1/h$, $x \in \Gamma$, $\nu = \nu_E(x)$, $Q_\rho^\nu(x) \subset A$, the orthogonal projection of $Q_\rho^\nu(x) \cap \partial^* E$ covers the faces of $Q_\rho^\nu(x)$ orthogonal to ν (so that, in particular, $\rho^{n-1} \leq \mathcal{H}^{n-1}(Q_\rho^\nu(x) \cap \partial^* E)$),

$$|Du(x,\rho,\nu)|(\partial Q_\rho^\nu(x)) \leq \frac{1}{h}\rho^{n-1}, \tag{3.13}$$

$$\varphi(x,\nu_E(x)) \leq \frac{1}{\rho^{n-1}} \int_{Q_\rho^\nu(x)\cap\partial^* E} \varphi(y,\nu_E(y))d\mathcal{H}^{n-1} + \frac{1}{h}, \tag{3.14}$$

and

$$\frac{1}{\rho^{n-1}}\Phi(x,\nu_E(x),\rho) \leq \varphi(x,\nu_E(x)) + \frac{1}{2h}. \tag{3.15}$$

The family $\mathcal{Q}_h$ covers finely $\mathcal{H}^{n-1}$-almost all $\partial^* E \cap A$; hence, by the (generalized) Besicovitch covering theorem there exists a countable sub-family of disjoint cubes $\{\overline{Q_{\rho_i}^{\nu_i}(x_i)} : i \in \mathbb{N}\}$ still covering $\mathcal{H}^{n-1}$-almost all $\partial^* E \cap A$. We define then

$$u_h(y) = \begin{cases} u(x_i,\rho_i,\nu_i)(y) & \text{if } y \in Q_{\rho_i}^{\nu_i}(x_i), \\ 1 & \text{if } y \in (E \setminus \bigcup_i Q_{\rho_i}^{\nu_i}(x_i)) \cap A \\ 0 & \text{if } y \in (A \setminus E) \setminus \bigcup_i Q_{\rho_i}^{\nu_i}(x_i). \end{cases} \tag{3.16}$$

We have by (3.13)–(3.15) and the definition of $u(x,\rho,\nu)$

$$\begin{aligned} \mathcal{G}(u_h,A) &= \sum_i \int_{\overline{Q_{\rho_i}^{\nu_i}(x_i)}} g(y,Du(x_i,\rho_i,\nu_i)) \\ &= \sum_i \int_{Q_{\rho_i}^{\nu_i}(x_i)} g(y,Du(x_i,\rho_i,\nu_i)) + \sum_i \int_{\partial Q_{\rho_i}^{\nu_i}(x_i)} g(y,Du(x_i,\rho_i,\nu_i)) \\ &\leq \sum_i {\rho_i}^{n-1}\Big(\varphi(x_i,\nu_E(x_i)) + \frac{1}{h}\Big) + C_1 \sum_i |Du(x_i,\rho_i,\nu_i)|(\partial Q_{\rho_i}^{\nu_i}(x_i)) \\ &\leq \sum_i {\rho_i}^{n-1}\varphi(x_i,\nu_E(x_i)) + \frac{1}{h}(1+C_1)\sum_i \rho_i^{n-1} \\ &\leq \sum_i \int_{Q_{\rho_i}^{\nu_i}(x_i)\cap\partial^* E} \varphi(y,\nu_E(y))d\mathcal{H}^{n-1} + \frac{1}{h}(2+C_1)\sum_i \rho_i^{n-1} \\ &\leq \int_{A\cap\partial^* E} \varphi(y,\nu_E(y))d\mathcal{H}^{n-1} + \frac{1}{h}(2+C_1)\mathcal{H}^{n-1}(A\cap\partial^* E). \end{aligned} \tag{3.17}$$

Letting $h \to +\infty$ we obtain then

$$\liminf_h \mathcal{G}(u_h,A) \leq \int_{A\cap\partial^* E} \varphi(y,\nu_E(y))d\mathcal{H}^{n-1}.$$

Since it is clear that $u_h \to \chi_E$ in $\mathrm{L}^1(A)$, we have proven (3.12). Due to the lower semicontinuity of $\mathcal{G}$ we have

$$\mathcal{G}(\chi_E, A) \leq \int_{A\cap\partial^* E} \varphi(y, \nu_E(y))d\mathcal{H}^{n-1}, \tag{3.18}$$

and we conclude the converse inequality of (3.11). □

Remark 3.3. We recall that if g satisfies a growth condition of order 1, and the functional $G(u,\Omega) = \int_\Omega g(x, Du)$ is $\mathrm{L}^1(\Omega)$-lower semicontinuous, then the direct methods of the calculus of variations yield the existence for minimum problems of the form

$$\min\Big\{\int_A g(x, Du)\ :\ u = \Phi \text{ on } \Omega\setminus A\Big\}$$

if A is an open set with $A \subset\subset \Omega$ and $\Phi \in BV_{\mathrm{loc}}(\Omega)$. Using this fact, it is easy to see, proceeding exactly as in Theorem 3.1, that the function φ can also be given by

$$\begin{aligned}\varphi(x,\nu) = \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \min\Big\{\int_{Q^\nu_\rho(x)} g(y, Dw) :\\ : w \in BV(\Omega), w = u^{\nu,x} \text{ on } \Omega\setminus Q^\nu_\rho(x)\Big\}.\end{aligned} \tag{3.19}$$

Note however that for some x, ρ, ν, the minimum problem in (3.19) may be different from $\Phi(x, \nu, \rho)$: take for example $n = 2$,

$$g(x,\xi) = \begin{cases} |\xi| & \text{if } |\xi_1| \vee |\xi_2| = 1 \\ 4|\xi| & \text{otherwise.}\end{cases}$$

$x = 0$, $\nu = e_1$, and $\rho = 1$.

4. Relaxation

In this section we prove Theorem 2.3, and we give a relaxation result for functionals defined on “partitions of Ω into sets of finite perimeter”.

We recall that if f, g satisfy a growth condition of order 1, for every open subset A of Ω the functional $G(u,A) = \int_A g(x, Du)$ ($u \in BV(\Omega)$) is the relaxation (in the L^1-topology) of the functional

$$F(u,A) = \begin{cases} \int_A f(x, \nabla u)\, dx & \text{if } u \in \mathrm{W}^{1,1}(A) \\ +\infty & \text{otherwise,}\end{cases}$$

and $\phi \in \mathrm{L}^1(\partial A)$, then by a standard cut-off argument near ∂A we have

$$\inf\Big\{G(u,\Omega)\ :\ u^- = \phi \text{ on } \partial\Omega\Big\} = \inf\Big\{F(u,\Omega)\ :\ u = \phi \text{ on } \partial\Omega\Big\}, \tag{4.1}$$

where u^- denotes the inner trace of u on ∂A.

Proof of Theorem 2.3
Let us localize the functional $\mathcal{F}$ setting

$$\mathcal{F}(u,A)=\begin{cases}\int_A f(x,\nabla u(x))\,dx & \text{if } u\in \mathrm{W}^{1,1}(A)\\ +\infty & \text{otherwise,}\end{cases}$$

if A is a open subset of Ω. By Theorems 4.1 and 5.1 in [7] we obtain the representation

$$\overline{\mathcal{F}}(u,A)=\int_A h(x,\nabla u(x))\,dx+\int_A \varphi_f(x,D_s u),\qquad u\in BV(A)$$

for every open subset A of Ω, for suitable functions h and φ_f with $h(x,\cdot)$ convex, and $h^\infty(x,\cdot)=\varphi_f(x,\cdot)$ for almost every $x\in\Omega$. We have to prove that h and φ_f can be represented as in (2.8) and (2.9).
We can suppose, up to changing h on a Lebesgue negligible set, that $\varphi_f(x,\xi)=h^\infty(x,\xi)$ and $h(x,\xi)\le f(x,\xi)$ for every $x\in\Omega$, $\xi\in\mathbb{R}^n$. Note that from $f(x,0)=0$ we obtain also $h(x,0)=0$, and that the sequence of functions

$$h^t(x,\xi)=\frac{1}{t}h(x,t\xi)$$

converges increasingly to h^∞ as $t\to+\infty$. Notice also that for all A open subset of Ω

$$H^t(u,A)=\int_A h^t(x,\nabla u)dx+\int_A h^\infty(x,D_s u)$$

is the relaxed functional of

$$F^t(u,A)=\int_A \frac{1}{t}f(x,t\nabla u)dx.$$

Moreover, by the Lebesgue monotone convergence theorem

$$\sup_t H^t(u)=\int_\Omega h^\infty(x,\nabla u)\,dx+\int_\Omega h^\infty(x,D_s u),$$

and we conclude that the functionals $u\mapsto\int_A h^\infty(x,Du)$ are lower semicontinuous since they are the supremum of a family of lower semicontinuous functionals. Hence, for every $Q^\nu_\rho(x)$ we have

$$\begin{aligned}&\inf\Big\{\int_{Q^\nu_\rho(x)} h^\infty(y,Dw): w\in BV(Q^\nu_\rho(x)), w=u^{\nu,x}\text{ on }\partial Q^\nu_\rho(x)\Big\}\\
&=\lim_{t\to+\infty}\inf\Big\{\int_{Q^\nu_\rho(x)} h^t(y,\nabla w)+\int_{Q^\nu_\rho(x)} h^\infty(y,D_s w):\\
&\qquad\qquad : w\in BV(Q^\nu_\rho(x)), w=u^{\nu,x}\text{ on }\partial Q^\nu_\rho(x)\Big\} \qquad (4.2)\\
&=\lim_{t\to+\infty}\inf\Big\{\int_{Q^\nu_\rho(x)}\frac{1}{t}f(y,t\nabla w)dy: w\in \mathrm{W}^{1,1}(Q^\nu_\rho(x)), w=u^{\nu,x}\text{ on }\partial Q^\nu_\rho(x)\Big\}\\
&=\lim_{t\to+\infty}\inf\Big\{\int_{Q^\nu_\rho(x)}\frac{1}{t}f(y,\nabla w)dy: w\in \mathrm{W}^{1,1}(Q^\nu_\rho(x)), w=tu^{\nu,x}\text{ on }\partial Q^\nu_\rho(x)\Big\}.\end{aligned}$$

We can apply Theorem 2.1 with $g = h^\infty(= \varphi_f)$, and obtain formula (2.8).
As $h(x, 0) = 0$ we have $h \le h^\infty = \varphi_f$. Since $h \le f$, the convexity of $h(x, \cdot)$ yields

$$h(x, \cdot) \le (f(x, \cdot) \wedge \varphi_f(x, \cdot))^{**} \tag{4.3}$$

for a.e. $x \in \Omega$.
From the hypothesis $f(x, \cdot) \ge f^\infty(x, \cdot) - a(x)$ we deduce

$$(h^\infty(x, \cdot))^* \le f^*(x, \cdot) \vee \varphi_f^*(x, \cdot) \le (h^\infty(x, \cdot))^* + a(x),$$

hence, in particular, $\mathrm{dom} g^* = \mathrm{dom}\varphi_f^* = \mathrm{dom}(h^\infty)^*$. Since h^∞ gives a lower semicontinuous functional on $BV(\Omega)$, this condition yields, from [7] Theorem 4.4, that the functionals

$$G(u, A) = \int_A g(x, \nabla u)dx,$$

where

$$g(x, \xi) = (f(x, \cdot) \wedge \varphi_f(x, \cdot))^{**}(\xi),$$

are $\mathrm{L}^1(A)$-lower semicontinuous on $\mathrm{W}^{1,1}(A)$ for every open subset A of Ω. As $g \le f$, we deduce also, by definition of relaxation, that

$$\int_A g(x, \nabla u)dx = G(u, A) \le \overline{\mathcal{F}}(u, A) = \int_A h(x, \nabla u)dx$$

for every open subset A of Ω, and $u \in \mathrm{W}^{1,1}(A)$. Hence, we must have

$$g(x, \xi) \le h(x, \xi), \tag{4.4}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, and (2.9) is proven. □

If T is a finite subset of $\mathbb{R}$ then we denote by $BV(\Omega; T)$ the set of $BV(\Omega)$ functions u such that $u \in T$ a.e. on Ω. Note that if $u \in BV(\Omega; T)$ then it can be written as

$$u = \sum_{t \in T} t\chi_{E_t}, \tag{4.5}$$

where (E_t) is a partition of Ω; hence, these functions can be thought of as finite partitions of Ω into Caccioppoli sets indexed by T. We denote by $P(\Omega)$ the set of all possible finite Caccioppoli partitions, namely

$$P(\Omega) = \bigcup\{BV(\Omega; T) : T \subset \mathbb{R} \text{ finite }\}. \tag{4.6}$$

For the properties of these spaces we refer the reader to Ambrosio and Braides [1], [2], Congedo and Tamanini [11].

Theorem 4.1. *Let $f : \Omega \times \mathbb{R}^n \to [0, +\infty[$ be a positively 1-homogeneous Borel function satisfying a growth condition of order 1, and let φ_f be the positively 1-homogeneous function defined by*

$$\begin{aligned}\varphi_f(x,\nu) = \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \inf\Big\{ \int_{Q^\nu_\rho(x)} & f(y, \nabla u(y))dy : \\ & : w \in \mathrm{W}^{1,1}(Q^\nu_\rho(x)), w = u^{\nu,x} \text{ on } \partial Q^\nu_\rho(x)\Big\}\end{aligned} \tag{4.7}$$

for all $x \in \Omega$, $\nu \in S^{n-1}$. Then the relaxation of the functional $\mathcal{F}$ defined in (2.6) *coincides with the relaxation of the functional $\mathcal{J}$ defined by*

$$\mathcal{J}(u) = \begin{cases} \int_\Omega \varphi_f(x, Du) & \text{if } u \in P(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.8}$$

Proof. By Theorem 2.3 the relaxed functional $\overline{\mathcal{F}}$ of $\mathcal{F}$ can be represented on $BV(\Omega)$ by

$$\overline{\mathcal{F}}(u) = \int_\Omega \varphi_f(x, Du). \tag{4.9}$$

Hence, we have $\overline{\mathcal{F}} \le \mathcal{J}$, so that $\overline{\mathcal{F}} \le \overline{\mathcal{J}}$. On the other hand, since it is not difficult to prove hypotheses (H1)–(H3) of Remark 2.2 (see [1] or [9] for a proof of (H1)), we can represent $\overline{\mathcal{J}}$ in an integral form

$$\overline{\mathcal{J}}(u) = \int_\Omega \varphi_{\mathcal{J}}(x, Du), \tag{4.10}$$

where $\varphi_{\mathcal{J}}$ is given by

$$\begin{aligned}\varphi_{\mathcal{J}}&(x,\nu) = \\ &= \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \inf\Big\{ \int_{Q^\nu_\rho(x)} \varphi_{\mathcal{J}}(x, Dw) : w \in BV(\Omega), w = u^{\nu,x} \text{ on } \partial Q^\nu_\rho(x)\Big\} \\ &= \limsup_{\rho\to 0+} \frac{1}{\rho^{n-1}} \inf\Big\{ \int_{Q^\nu_\rho(x)} \varphi_f(x, Dw) : w \in P(\Omega), w = u^{\nu,x} \text{ on } \partial Q^\nu_\rho(x)\Big\},\end{aligned}$$

where the second equality follows as in (4.1) by a standard use of the coarea formula (see [1], [9]). Proceeding then as in the proof of (3.11) (see (3.8)–(3.11)) we obtain

$$\varphi_{\mathcal{J}}(x, \nu_E(x)) \le \varphi_f(x, \nu_E(x)),$$

for every E of finite perimeter and $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$, and then $\overline{\mathcal{J}} \le \overline{\mathcal{F}}$ by the coarea formula. □

We end this section with a lemma that will be needed in the sequel.

Lemma 4.2. *Let Ω be a Lipschitz open set. If $g : \Omega \times \mathbb{R}^n \to [0, +\infty[$ be a positively 1-homogeneous Borel function, $x \in \Omega$, $\nu \in S^{n-1}$, then we have*

$$\inf\Big\{\int_\Omega g(x, Dw) : w \in P(\Omega), w = u^{\nu,x} \text{ on } \partial\Omega\Big\} \tag{4.11}$$

$$= \inf\Big\{\int_{\Omega\cap\partial^* E} g(x, \nu_E) d\mathcal{H}^{n-1} : E \text{ is a Caccioppoli set, } \chi_E = u^{\nu,x} \textit{ on } \partial\Omega\Big\}.$$

Proof. It suffices to prove that for every finite set $T \supset \{0, 1\}$ we have the inequality

$$\inf\Big\{\int_\Omega g(x, Dw) : w \in BV(\Omega; T), w = u^{\nu,x} \text{ on } \partial\Omega\Big\} \tag{4.12}$$

$$\geq \inf\Big\{\int_{\Omega\cap\partial^* E} g(x, \nu_E) d\mathcal{H}^{n-1} : E \text{ is a Caccioppoli set, } \chi_E = u^{\nu,x} \text{ on } \partial\Omega\Big\}.$$

Let us take $w \in BV(\Omega; T)$ satisfying the boundary condition in (4.12); we have to show then that there exists a set E of finite perimeter, with the same trace on $\partial\Omega$, such that

$$\int_{\Omega\cap\partial^* E} g(x, \nu_E) d\mathcal{H}^{n-1} \leq \int_\Omega g(x, Dw). \tag{4.13}$$

We prove it by induction on $\#(T)$. If $\#(T) = 2$ then we must have $T = \{0, 1\}$ and there is nothing to prove. Let us suppose that $\#(T) = N + 1$. Since $|D((w \wedge 1) \vee 0)| \leq |Dw|$ and $g(x, \cdot)$ is positively 1-homogeneous, the second integral in (4.13) clearly decreases substituting w by $(w \wedge 1) \vee 0$, we can suppose that $T \subset [0, 1]$, and we can write $T = \{a_i : i = 0, 1, \ldots, N\}$, with $0 = a_0 < a_1 < \ldots < a_N = 1$. If $w = \sum_{i=1}^N a_i \chi_{E_i}$, let us define

$$w' = \sum_{i=2}^N a_i \chi_{E_i}, \qquad w'' = a_2 \chi_{E_1} + \sum_{i=2}^N a_i \chi_{E_i}$$

(w' and w'' are obtained from w by changing a_1 into $a_0 = 0$, and a_1 into a_2, respectively). We have then

$$\int_\Omega g(x, Dw') = \int_\Omega g(x, Dw)$$
$$+ a_1\Big(\sum_{i=2}^N \int_{\partial^* E_i\cap\partial^* E_1\cap\Omega} g(x, \nu_{E_i}) d\mathcal{H}^{n-1} - \int_{\partial^* E_0\cap\partial^* E_1\cap\Omega} g(x, \nu_{E_1}) d\mathcal{H}^{n-1}\Big),$$

$$\int_\Omega g(x, Dw'') = \int_\Omega g(x, Dw)$$
$$+ (a_1 - a_2)\Big(\sum_{i=2}^N \int_{\partial^* E_i\cap\partial^* E_1\cap\Omega} g(x, \nu_{E_i}) d\mathcal{H}^{n-1} - \int_{\partial^* E_0\cap\partial^* E_1\cap\Omega} g(x, \nu_{E_1}) d\mathcal{H}^{n-1}\Big).$$

Hence, either $\int_\Omega g(x, Dw') \le \int_\Omega g(x, Dw)$ or $\int_\Omega g(x, Dw'') \le \int_\Omega g(x, Dw)$. Since $w', w'' \in BV(\Omega, T \setminus \{a_1\})$ the inequality (4.13) is true by induction. $\square$

5. Homogenization

In this section we use the previous relaxation and integral representation results to obtain a homogenization formula for positively 1-homogeneous functionals.
Let $f = f(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty[$ be a periodic (in x), positively 1-homogeneous (in ξ) Borel function satisfying a growth condition of order 1. It is well-known (see for example Bouchitté [6]) that the functionals

$$\mathcal{F}_\varepsilon(u,\Omega) = \int_\Omega f(\frac{x}{\varepsilon}, \nabla u(x))dx \qquad u \in \mathrm{W}^{1,1}(\Omega) \tag{5.1}$$

Γ-converge, with respect to the $\mathrm{L}^1(\Omega)$-topology, as $\varepsilon \to 0+$ to a functional

$$\mathcal{F}_0(u,\Omega) = \int_\Omega \psi(Du) \qquad u \in BV(\Omega), \tag{5.2}$$

where ψ is a positively 1-homogeneous convex function. We refer to [12] for an introduction to Γ-convergence and to its applications to the theory of homogenization.
The function ψ can be characterized by a minimum value problem on a space of periodic $\mathrm{W}^{1,1}$-functions. In this section we are going to prove an alternative formula and express ψ as a minimum value problem on sets of finite perimeter.
By the convexity of ψ it is easy to see that $u^\nu = u^{\nu,0}$ is a local minimum on $BV(\Omega;T)$ for each T finite (see [2]), so that we have

$$\psi(\nu) = \min\Big\{\int_{Q_1^\nu(0)} \psi(Dw) : w \in \mathcal{P}(Q_1^\nu(0)), w = u^\nu \text{ on } \partial Q_1^\nu(0)\Big\}. \tag{5.3}$$

Hence, by Theorem 4.1

$$\psi(\nu) = \min\Big\{\int_{\overline{Q_1^\nu(0)}} \psi(Dw) : w \in BV(Q_1^\nu(0)), w = u^\nu \text{ on } Q_1^\nu(0)\Big\}. \tag{5.4}$$

Recall that the Γ-convergence of a sequence of equicoercive functionals (which is the case) implies the convergence of minima. By standard cut-off arguments it is easy to see that Γ-convergence is maintained also after addition of boundary conditions (see [12]); hence,

$$\begin{aligned}\psi(\nu) &= \lim_{\varepsilon\to 0+} \inf\Big\{\int_{Q_1^\nu(0)} f(\frac{x}{\varepsilon}, \nabla w)dx : w \in \mathrm{W}^{1,1}(Q_1^\nu(0)), w = u^\nu \text{ on } \partial Q_1^\nu(0)\Big\} \\ &= \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{\int_{Q_T^\nu(0)} f(x, \nabla w)dx : w \in \mathrm{W}^{1,1}(Q_T^\nu(0)), w = u^\nu \text{ on } \partial Q_T^\nu(0)\Big\}.\end{aligned} \tag{5.5}$$

By Theorem 4.1 we get then

$$\begin{aligned}\psi(\nu) = \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{\int_{Q_T^\nu(0)} \varphi_f(x, Dw) :& \\ : w \in P(Q_T^\nu(0)), w = u^\nu \text{ on } \partial Q_T^\nu(0)\Big\}.&\end{aligned} \tag{5.6}$$

By Lemma 4.2, given $w \in P(Q^\nu_T(0))$ such that $w = u^\nu$ on $\partial Q^\nu_T(0)$, there exists a set of finite perimeter E such that $\chi_E = u^\nu$ on $\partial Q^\nu_T(0)$ and

$$\int_{Q^\nu_T(0)} \varphi_f(x, D\chi_E) \le \int_{Q^\nu_T(0)} \varphi_f(x, Dw). \tag{5.7}$$

Hence, we have proven the following representation result.

Theorem 5.1. *Let $f = f(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty[$ be a periodic (in x), positively 1-homogeneous (in ξ) Borel function satisfying a growth condition of order 1, and let ψ be the integrand of the homogenized functional given by (5.1), (5.2). Then we have*

$$\begin{aligned}\psi(\nu) = \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{ \int_{Q^\nu_T(0)\cap\partial^* E} \varphi_f(x, \nu_E(x)) d\mathcal{H}^{n-1} : \\ : E \text{ is a set of finite perimeter}, \chi_E = u^\nu \ \textit{on}\ \partial Q^\nu_T(0)\Big\},\end{aligned} \tag{5.8}$$

where φ_f is given by (4.7), and $u^\nu = u^{\nu,0}$ is given by (2.2).

Remark 5.2. We remark that in order to characterize ψ it is sufficient to compute this formula for rational directions ν (we call $\nu \in S^{n-1}$ a rational direction if there exists $t \in \mathbb{R}$ such that $t\nu \in \mathbf{Z}^n$), since ψ is continuous and rational directions are dense in S^{n-1}.

Example 5.3. Let $n = 2$. We can apply Theorem 5.1 to obtain the characterization of the homogenized functional of

$$\mathcal{F}_\varepsilon(u,\Omega) = \int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla u| dx, \tag{5.9}$$

where a is defined by

$$a(x,y) = \begin{cases} 1 & \text{if } [x]+[y] \text{ is even} \\ 2 & \text{otherwise,} \end{cases}$$

and $[t]$ denotes the integer part of $t \in \mathbb{R}$. The relaxation of functional $\mathcal{F}_\varepsilon$ can be expressed, by Theorem 2.1 and taking into account Remark 3.2, by changing the coefficient a into

$$\overline{a}(x,y) = \begin{cases} 1 & \text{if } [x]+[y] \text{ is even} \\ 1 & \text{if } x \in \mathbf{Z} \ \text{ or } y \in \mathbf{Z} \\ 2 & \text{otherwise.} \end{cases} \tag{5.10}$$

We can apply then formula (5.8) to compute ψ. In this case it is immediate to see (by Pythagoras' Theorem) that the boundary of the minimal set E in (5.8) "avoids" the set $\{a = 2\}$ as much as possible. If $\xi = T\nu \in \mathbf{Z}^2$ then the minimum problem in (5.8) is reduced to finding the path of minimal length through $\{a = 1\}$ from the point $(0,0)$ to the point $(\xi_2, -\xi_1)$. By a symmetry argument we can suppose that $0 \le \xi_2 \le -\xi_1$, so that the minimal path is the union of the two segments $[(0,0),(\xi_2,\xi_2)]$ and $[(\xi_2,\xi_2),(\xi_2,-\xi_1)]$). We infer then that the minimum value in (5.8) is given by

$(\sqrt{2}-1)\min\{|\xi_1|,|\xi_2|\}+\max\{|\xi_1|,|\xi_2|\}$, whenever $\xi = T\nu \in \mathbf{Z}^2$. Letting $T \to +\infty$ we obtain that

$$\psi(\nu) = (\sqrt{2}-1)\min\{|\nu_1|,|\nu_2|\}+\max\{|\nu_1|,|\nu_2|\} \tag{5.11}$$

when ν is a rational direction. By Remark 5.2 this formula determines ψ.

Remark 5.4. In the case $n = 1$ formula (5.8) reduces to $\psi(\nu) = \min \varphi_f(\cdot,\nu) = \operatorname{essinf} f(\cdot,\nu)$.

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86