Journal of Convex Analysis
Volume 2 (1995), No.1/2, 103–115

On a Problem of Potential Wells

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Received 30 August 1994
Revised manuscript received 16 January 1995

Dedicated to R. T. Rockafellar on his 60th Birthday

1991 Mathematics Subject Classification: 49A22, 34A60

1. Introduction

Problems in elasticity, crystallography and phase transitions lead to the consideration of energy functionals of the kind

$$\int_\Omega g(\nabla u(x))dx$$

where g is non negative and is zero only on potential wells described by rotations of finitely many matrices $A_1, \ldots, A_r$, i.e.

$$g(F) = 0 \quad \text{for} \quad F \in \bigcup_{i=1}^{r} SO(3)A_i.$$

In general the matrices A_i describe symmetries of the material and are connected by a symmetry group. See, for instance [1], [2], [3], [4], [5] and [6].
Finding a minimizer of the energy satisfying the homogeneous condition at the boundary of $\Omega : u|_{\partial\Omega} = 0$, is then equivalent to solving the differential inclusion

$$\nabla u(x) \in \bigcup_{i=1}^{r} SO(3)A_i,$$

with the boundary condition: $u|_{\partial\Omega} = 0$. This needs not always be possible : from a result of Reshetnyak, see [7] and [5], it follows that the problem

$$\begin{cases} \nabla u(x) \in SO(3)I, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits no solution on any open and bounded $\Omega \subset \mathbb{R}^3$.

ISSN 0944-6532 / $ 2.50

The purpose of the present paper is to show that for any open and bounded $\Omega \subset \mathbb{R}^3$, the problem

$$\begin{cases} \nabla u(x) \in SO(3)I \cup SO(3)I^-, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where

$$I^- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(or any other matrix giving a change of orientation in $\mathbb{R}^3$) does indeed admit a solution, a Lipschitz continuous map $u : \overline{\Omega} \to \mathbb{R}^3$. More precisely, the matrix $\nabla u(x)$ will belong, for a.e x in Ω, to a subset of $O(3) = SO(3)I \cup SO(3)I^-$, the set $\mathcal{R}$ of those orthogonal matrices having rows $\pm e_j$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. Notice that our result is contrary to the intuition: when $\partial\Omega$ is smooth, in case u was smooth as well, the three components of u would have $\partial\Omega$ as a level set, hence their gradients would all be orthogonal to $\partial\Omega$, i.e. parallel to each other. In particular our result shows that the minimum of the functional

$$\int_\Omega g(\nabla u(x))dx$$

with homogeneous boundary condition is zero. Hence the functional is not quasiconvex since the (affine) boundary datum is not a solution to the minimum problem. The boundary datum zero need not be the only case yielding a zero infimum for the minimum problem. Characterizations of such boundary data under different assumptions are presented in [2] and [8].

An unknown referee, whom we thank for the careful reading of the proof, has pointed out that related results on the existence problem, have been announced by Muller and Sverak for the case $\Omega \subset \mathbb{R}^2$.

2. Notation and preliminary results

For x in $\mathbb{R}^3$, define the three maps $x \mapsto |X_s|(x)$, $x \mapsto |X_m|(x)$, $x \mapsto |X_i|(x)$, as follows:

$$|X_s|(x_1, x_2, x_3) = \sup\{|x_j| \,:\, j = 1, 2, 3\}.$$

Let $k \in \{1, 2, 3\}$ be such that $|X_s|(x) = |x_k|$ and set

$$|X_m|(x_1, x_2, x_3) = \sup\{|x_j| \,:\, j = 1, 2, 3;\ j \neq k\}.$$

Remark that $|X_m|$ is unambigously defined: in case k_1 and k_2 are such that $|x_{k_1}| = |X_s|(x) = |x_{k_2}|$, then $|X_m|(x) = |X_s|(x)$ independently of the choice of k. Set also

$$|X_i|(x_1, x_2, x_3) = \inf\{|x_j| \,:\, j = 1, 2, 3\}.$$

Proposition 2.1.
a) *The maps $|X_s|$, $|X_m|$, $|X_i|$ are continuous.*
b) $|X_s|(x_1, x_2, x_3) = |X_s|(|x_1|, |x_2|, |x_3|)$*, and the same for $|X_m|$ and $|X_i|$.*

c) $|X_s|(x_{j_1}, x_{j_2}, x_{j_3}) = |X_s|(x_1, x_2, x_3)$ *for any permutation* $(x_{j_1}, x_{j_2}, x_{j_3})$ *of* (x_1, x_2, x_3), *and the same is true for* $|X_m|$ *and* $|X_i|$.

Remark 2.2. The composition of a continuous function on $\mathbb{R}^3$ with $(|X_s|, |X_m|, |X_i|)$, is a continuous function of x, and is invariant under a permutation of (x_1, x_2, x_3).
For x in $\mathbb{R}^3$ and such that $|x_i| \neq |x_j|$, for $i, j = 1, 2, 3$ and $i \neq j$, set $s(x)$, $m(x)$, $i(x)$ to be such that

$$|x_{s(x)}| = |X_s|(x), \quad |x_{m(x)}| = |X_m|(x), \quad |x_{i(x)}| = |X_i|(x).$$

The maps $x \mapsto s(x)$, $x \mapsto m(x)$, $x \mapsto i(x)$, are locally constant on their (open) domains. We have the following technical proposition.

Proposition 2.3. *Let* $E \subset \mathbb{R}^2$ *be defined by*

$$E = \{x : \| x \|_\infty \leq 1,\ |x_1| + |x_2| \leq 1,\ |x_2| \leq |x_1|\} \\ \cup \{x : \| x \|_\infty \leq 1,\ |x_1| + |x_2| \geq 1,\ |x_1| \leq |x_2|\}.$$

Then, (x_1, x_2) *belongs to* E *if and only if* $((x_1)_{mod\,1}, (x_2)_{mod\,1})$ *belongs to* E.

Proof. Set $y_1 = (x_1)_{mod\,1}$ and $y_2 = (x_2)_{mod\,1}$. Four cases are possible: $(x_1, x_2) = (y_1, y_2)$, $(x_1, x_2) = (y_1 - 1, y_2)$, $(x_1, x_2) = (y_1, y_2 - 1)$, $(x_1, x_2) = (y_1 - 1, y_2 - 1)$. One verifies easily the claim, separately for each case. □

We wish to have indices i in $\{1, 2, 3\}$. It is convenient to set $(r)_3 = (r - 1)_{mod\,3} + 1$, for any integer r.

We shall need three functions f^1, f^2, f^3, from $\mathbb{R}$ to $\mathbb{R}$. On $[0, 1]$ set

$$f^1(y) = \inf\{y, 1 - y\},$$

and consider f^1 on $\mathbb{R}$ to be its extension by periodicity. We have that f^1 is continuous and that $f^1(y) = f^1(|y|)$. Set also

$$f^2(y) = \frac{1}{2} f^1(2y); \quad f^3(y) = \frac{1}{4} f^1(4y).$$

3. Main result

It is our purpose to define a function $u : \overline{\Omega} \to \mathbb{R}^3$, Lipschitz continuous on $\overline{\Omega}$, such that $u|_{\partial\Omega} = 0$ and $\nabla u(x)$ is in $\mathcal{R} \subset SO(3)I \cup SO(3)I^-$ for a.e. x in Ω.

Theorem 3.1. *Let* Ω *be a bounded open subset of* $\mathbb{R}^3$. *Then there exists* $\tilde{u} : \Omega \to \mathbb{R}^3$, *Lipschitz continuous with Lipschitz constant one, such that*

i) $\tilde{u}|_{\partial\Omega} = 0$;
ii) $\nabla \tilde{u}(x) \in \mathcal{R}$, *for a.e.* x *in* Ω.

Proof. The proof consists of the following steps:

a) We define first a map u^1 on the sphere $\| x \|_\infty \leq 1$, satisfying the differential inclusion ii) on $\| x \|_\infty < 1$ but not the boundary condition i) at $\| x \|_\infty = 1$.

b) We recursively extend this map, by defining a function u^n on the set of x such that $\sum_{i=0}^{n-2} \frac{1}{2^i} \leq \| x \|_\infty \leq \sum_{i=0}^{n-1} \frac{1}{2^i}$, a Lipschitz continuous map satisfying condition ii) and such that, for all $j \in \{1,2,3\}$, $\sup |u_j^n(x)| \leq \frac{1}{2^{n-1}}$.

c) We define a function u satisfying properties i) and ii) for $\overline{\Omega} = B_2$, the sphere $\| \ \|_\infty$ $x \leq\ 2$.

d) Exploiting Vitali's covering theorem, we define $\tilde{u}$ on $\overline{\Omega}$, with the properties i) and ii) of the theorem.

a) On B_1, the unit ball $\| x \|_\infty \leq 1$, set:

$$
\begin{aligned}
u_1^1(x) &= 1 - \| x \|_\infty = 1 - |X_s|(x); \\
u_2^1(x) &= \inf\{f^1(|X_i|(x)), f^1(|X_m|(x))\}; \\
u_3^1(x) &= \begin{cases} f^2(|X_m|(x)) & \text{on } |X_i|(x) + |X_m|(x) \leq 1 \\ f^2(|X_i|(x)) & \text{on } |X_i|(x) + |X_m|(x) \geq 1. \end{cases}
\end{aligned}
$$

Notice that on the set $\{x : |X_i|(x) + |X_m|(x) = 1\}$, one has $|X_i|(x) = 1 - |X_m|(x)$, hence $f^2(|X_i|(x)) = f^2(1 - |X_m|(x)) = f^2(|X_m|(x) - 1) = f^2(|X_m|(x))$ by the periodicity of f^2. Recalling Proposition 2.1, the map u^1 is continuous, actually piecewise affine. In particular consider $u^1(x_1, x_2, 1)$.

Claim 1. $u^1(x_1, x_2, 1) = u^1((x_1)_{mod\,1}, (x_2)_{mod\,1}, 1)$.

Proof of claim 1. We have $u_1^1(x_1, x_2, 1) = 0$. Moreover,

$$
\begin{aligned}
u_2^1(x_1, x_2, 1) &= \inf\{f^1(|x_1|), f^1(|x_2|)\} = \inf\{f^1(x_1), f^1(x_2)\} \\
&= \inf\{f^1((x_1)_{mod\,1}), f^1((x_2)_{mod\,1})\}.
\end{aligned}
$$

Finally consider $u_3^1(x_1, x_2, 1)$. Recalling Proposition 2.3, we have

$$(x_1, x_2) \in E \Leftrightarrow ((x_1)_{mod\,1}, (x_2)_{mod\,1}) \in E.$$

Then if $(x_1, x_2) \in E$ we have

$$u_3^1(x_1, x_2, 1) = f^2(|x_1|) = f^2(x_1)$$

and

$$u_3^1((x_1)_{mod\,1}, (x_2)_{mod\,1}, 1) = f^2(|(x_1)_{mod\,1}|) = f^2((x_1)_{mod\,1}).$$

Since $f^2(x_1) = f^2((x_1)_{mod\,1})$ the claim follows in this case. Analogously when (x_1, x_2) belongs to the complement of E. This proves claim 1. $\square$

Moreover, we have: $\sup\{|u_j^1(x)| \,:\, x \in B_1,\, j = 1,2,3\} = 1$.
Whenever the gradients exist, we have:

$$\begin{aligned}
\nabla u_1^1(x) &= -\operatorname{sign}(x_{s(x)})e_{s(x)} \\
\nabla u_2^1(x) &= \begin{cases} \operatorname{sign}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < 1 \\ -\operatorname{sign}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > 1 \end{cases} \\
\nabla u_3^1(x) &= \begin{cases} f^{2\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < 1 \\ f^{2\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > 1. \end{cases}
\end{aligned}$$

Since $|f^{2\prime}(t)| = 1$, for $t \notin \{(1/4)z \,:\, z \,\text{integer}\}$, we have that, a.e. on B_1, $\nabla u^1(x) \in \mathcal{R}$.

b) We begin by defining two auxiliary functions v and ℓ^1. The function ℓ^1 will, in turn, extend u^1 as u^2 on the layer $1 \leq \| x \|_\infty \leq 1 + \frac{1}{2}$. To do so, we have to carefully consider the continuity of u^2 at $\{x : \| x \|_\infty = 1\}$. An induction argument carries this construction to u^n.
We begin by defining the subsets Q_s^3, Q_m^3, Q_i^3 of $B_{\frac{1}{2}} = \{x : \| x \|_\infty \leq \frac{1}{2}\}$ as

$$\begin{aligned}
Q_s^3 &= \{x \in B_{\frac{1}{2}} \,:\, |x_3| = |X_s|(x)\}; \\
Q_m^3 &= \{x \in B_{\frac{1}{2}} \,:\, |x_3| = |X_m|(x)\}; \\
Q_i^3 &= \{x \in B_{\frac{1}{2}} \,:\, |x_3| = |X_i|(x)\}.
\end{aligned}$$

Set $v : B_{\frac{1}{2}} \to \mathbb{R}^3$ to be:

$$v_2(x) = \frac{1}{2} - \| x \|_\infty = \frac{1}{2} - |X_s|(x);$$

and, for x in $Q_s^3 \cup Q_m^3$,

$$\begin{aligned}
v_1(x) &= \begin{cases} f^3(|X_m|(x)) & \text{on } |X_i|(x) + |X_m|(x) \leq \frac{1}{2} \\ f^3(|X_i|(x)) & \text{on } |X_i|(x) + |X_m|(x) \geq \frac{1}{2}; \end{cases} \\
v_3(x) &= \inf\{f^2(|X_i|(x)), f^2(|X_m|(x))\};
\end{aligned}$$

for x in Q_i^3,

$$\begin{aligned}
v_1(x) &= \begin{cases} f^3(|X_i|(x)) & \text{on } |X_i|(x) + |X_m|(x) \leq \frac{1}{2} \\ f^3(|X_m|(x)) & \text{on } |X_i|(x) + |X_m|(x) \geq \frac{1}{2}; \end{cases} \\
v_3(x) &= \sup\{f^2(|X_i|(x)), f^2(|X_m|(x))\}.
\end{aligned}$$

The same arguments as used for u^1 show that v is lipschitzean on $B_{\frac{1}{2}}$ and that

$$\sup\{|v_j(x)| \,:\, x \in B_{\frac{1}{2}},\, j = 1,2,3\} = \frac{1}{2}.$$

Notice, for future use, the following properties of v:

α) $v(x_1, x_2, x_3) = v(|x_1|, |x_2|, |x_3|)$;
β) $v(x_1, x_2, x_3) = v(x_2, x_1, x_3)$.
To prove β) above, remark that when (x_1, x_2, x_3) belongs to one of the set Q_s^3, Q_m^3, Q_i^3, so does (x_2, x_1, x_3). Then v is defined through the maps $|X_s|$, $|X_m|$, $|X_i|$ that assume the same values on (x_1, x_2, x_3) and any of its permutations.
The map v is differentiable a.e. on $B_{\frac{1}{2}}$ and, whenever ∇v exists, has the form, for x in $Q_s^3 \cup Q_m^3$:

$$\nabla v_1(x) = \begin{cases} f^{3\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < \frac{1}{2} \\ f^{3\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > \frac{1}{2} \end{cases}$$

$$\nabla v_2(x) = -\operatorname{sign}(x_{s(x)})e_{s(x)}$$

$$\nabla v_3(x) = \begin{cases} f^{2\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < \frac{1}{2} \\ f^{2\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > \frac{1}{2}. \end{cases}$$

For x in Q_i^3:

$$\nabla v_1(x) = \begin{cases} f^{3\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < \frac{1}{2} \\ f^{3\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > \frac{1}{2} \end{cases}$$

$$\nabla v_2(x) = -\operatorname{sign}(x_{s(x)})e_{s(x)}$$

$$\nabla v_3(x) = \begin{cases} f^{2\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < \frac{1}{2} \\ f^{2\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > \frac{1}{2}. \end{cases}$$

Hence, a.e. on $B_{\frac{1}{2}}$, $\nabla v(x) \in \mathcal{R}$.
The following properties will be essential to show the continuity of the extension of the map u^1.

Claim 2. *For $(x_1, x_2, 1)$ in B_1 we have*

$$u^1(x_1, x_2, 1) = v\left((x_1)_{mod\,1} - \frac{1}{2}, (x_2)_{mod\,1} - \frac{1}{2}, 0\right).$$

Proof of claim 2. We have already proved that

$$u^1(x_1, x_2, 1) = u^1((x_1)_{mod\,1}, (x_2)_{mod\,1}, 1),$$

hence, without loss of generality, we can assume $x_1, x_2 \geq 0$. Set y in $B_{\frac{1}{2}}$ to be $y = (x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0)$. Since $y_3 = 0$, y is in Q_i^3, and $|X_i|(y) + |X_m|(y) = |X_m|(y) \leq 1/2$, so that $v_1(y) = f_3(0) = 0$. Moreover, by the very definition, $u_1^1(x_1, x_2, 1) = 0$.
In order to prove the claim for the second and third components, consider the sets

$$A = \{x_1 + x_2 \leq 1\} \cap \{x_2 \leq x_1\},$$
$$B = \{x_1 + x_2 \geq 1\} \cap \{x_2 \geq x_1\},$$
$$C = \{x_1 + x_2 \leq 1\} \cap \{x_2 \geq x_1\},$$
$$D = \{x_1 + x_2 \geq 1\} \cap \{x_2 \leq x_1\}.$$

Since $v_2(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0) = \frac{1}{2} - \sup\{|x_1 - \frac{1}{2}|, |x_2 - \frac{1}{2}|\}$, we have

$$\begin{aligned} & v_2\left(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0\right) \\ & = x_2\chi_A(x_1, x_2) + (1 - x_2)\chi_B(x_1, x_2) + x_1\chi_C(x_1, x_2) + (1 - x_1)\chi_D(x_1, x_2). \end{aligned}$$

On the other hand,

$$u_2^1(x_1, x_2, 1) = \inf\{f^1(x_1), f^1(x_2)\} = \inf\{x_1, 1 - x_1, x_2, 1 - x_2\}.$$

On A we have $x_2 \leq x_1$, $1 - x_1 \geq x_2$, $x_2 \leq 1 - x_2$, so that $u_2^1(x_1, x_2, 1) = x_2$. Analogously one verifies that $u_2^1(x_1, x_2, 1) = v_2(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0)$, for $(x_1, x_2) \in B \cup C \cup D$.
Consider now the third component. Notice that

$$|X_i|(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0) = 0$$

and that

$$|X_m|(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0) = \inf\{|x_1 - \frac{1}{2}|, |x_2 - \frac{1}{2}|\},$$

hence, by definition, $v_3(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0) = f^2(\inf\{|x_1 - \frac{1}{2}|, |x_2 - \frac{1}{2}|\})$. We have

$$\inf\left\{\left|x_1 - \frac{1}{2}\right|, \left|x_2 - \frac{1}{2}\right|\right\} = \begin{cases} \left|x_1 - \frac{1}{2}\right| & \text{on } A \cup B \\ \left|x_2 - \frac{1}{2}\right| & \text{on } C \cup D \end{cases}$$

so that

$$\begin{aligned} v_3\left(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0\right) &= f^2\left(\left|x_1 - \frac{1}{2}\right|\right)\chi_{A\cup B}(x_1, x_2) + f^2\left(\left|x_2 - \frac{1}{2}\right|\right)\chi_{C\cup D}(x_1, x_2) \\ &= f^2\left(x_1 - \frac{1}{2}\right)\chi_{A\cup B}(x_1, x_2) + f^2\left(x_2 - \frac{1}{2}\right)\chi_{C\cup D}(x_1, x_2) \\ &= f^2(x_1)\chi_{A\cup B}(x_1, x_2) + f^2(x_2)\chi_{C\cup D}(x_1, x_2). \end{aligned}$$

On the other hand, by definition,

$$\begin{aligned} u_3^1(x_1, x_2, 1) &= f^2(|X_m|(x_1, x_2, 1))\chi_{A\cup C}(x_1, x_2) + f^2(|X_i|(x_1, x_2, 1))\chi_{B\cup D}(x_1, x_2) \\ &= f^2(x_1)\chi_A + f^2(x_2)\chi_C + f^2(x_1)\chi_B + f^2(x_2)\chi_D. \end{aligned}$$

This proves claim 2. □

Claim 3. *For ξ_1, ξ_2: $-\frac{1}{2} \leq \xi_1 \leq \frac{1}{2}$, $-\frac{1}{2} \leq \xi_2 \leq \frac{1}{2}$, and for $r = 1, 2, 3$, we have:*

$$v_{(r-1)_3}(\xi_1, \xi_2, 0) = 2v_r\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\xi_2 + \frac{1}{4}, \frac{1}{2}\right).$$

Proof of claim 3. Consider $r = 2$. By definition, since $(\xi_1, \xi_2, 0)$ is in Q_i^3, we have $v_1(\xi_1, \xi_2, 0) = f^3(0) = 0$. On the other hand, since v_2 is zero at the boundary of $B_{\frac{1}{2}}$, the claim holds for $r = 2$.
Consider $r = 3$. We have

$$v_2(\xi_1, \xi_2, 0) = \frac{1}{2} - \sup\{|\xi_1|, |\xi_2|\},$$

while, since $\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\xi_2 + \frac{1}{4}, \frac{1}{2}\right)$ is in Q_s^3,

$$\begin{aligned} v_3\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\xi_2 + \frac{1}{4}, \frac{1}{2}\right) &= \inf\left\{f^2\left(\frac{1}{2}\xi_1 + \frac{1}{4}\right), f^2\left(\frac{1}{2}\xi_2 + \frac{1}{4}\right)\right\} \\ &= \inf\left\{\frac{1}{2}f^1\left(\xi_1 + \frac{1}{2}\right), \frac{1}{2}f^1\left(\xi_2 + \frac{1}{2}\right)\right\}. \end{aligned}$$

Since f^1, for $t \in [0, 1]$, can be written as $f^1(t) = \frac{1}{2} - |t - \frac{1}{2}|$,

$$\begin{aligned} &\inf\left\{\frac{1}{2}f^1\left(\xi_1 + \frac{1}{2}\right), \frac{1}{2}f^1\left(\xi_2 + \frac{1}{2}\right)\right\} = \frac{1}{2}\inf\left\{\frac{1}{2} - |\xi_1|, \frac{1}{2} - |\xi_2|\right\} \\ &= \frac{1}{2}\left(\frac{1}{2} - \sup\{|\xi_1|, |\xi_2|\}\right), \end{aligned}$$

and the claim holds in this case as well.
Consider $r = 1$. Since $(\xi_1, \xi_2, 0)$ is in Q_i^3,

$$v_3(\xi_1, \xi_2, 0) = f^2(\inf\{|\xi_1|, |\xi_2|\}).$$

On the other hand,

$$\begin{aligned} &v_1\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\xi_2 + \frac{1}{4}, \frac{1}{2}\right) \\ &= \begin{cases} f^3\left(\sup\left\{\left|\frac{1}{2}\xi_1 + \frac{1}{4}\right|, \left|\frac{1}{2}\xi_2 + \frac{1}{4}\right|\right\}\right) & \text{on } \left|\frac{1}{2}\xi_1 + \frac{1}{4}\right| + \left|\frac{1}{2}\xi_2 + \frac{1}{4}\right| \le \frac{1}{2} \\ f^3\left(\inf\left\{\left|\frac{1}{2}\xi_1 + \frac{1}{4}\right|, \left|\frac{1}{2}\xi_2 + \frac{1}{4}\right|\right\}\right) & \text{on } \left|\frac{1}{2}\xi_1 + \frac{1}{4}\right| + \left|\frac{1}{2}\xi_2 + \frac{1}{4}\right| \ge \frac{1}{2} \end{cases} \\ &= \begin{cases} f^3\left(\frac{1}{4} + \frac{1}{2}\sup\{\xi_1, \xi_2\}\right) & \text{on } \xi_1 + \xi_2 \le 0 \\ f^3\left(\frac{1}{4} + \frac{1}{2}\inf\{\xi_1, \xi_2\}\right) & \text{on } \xi_1 + \xi_2 \ge 0 \end{cases} \\ &= \begin{cases} f^3\left(\frac{1}{2}\sup\{\xi_1, \xi_2\}\right) & \text{on } \xi_1 + \xi_2 \le 0 \\ f^3\left(\frac{1}{2}\inf\{\xi_1, \xi_2\}\right) & \text{on } \xi_1 + \xi_2 \ge 0 \end{cases} \\ &= \begin{cases} \frac{1}{2}f^2(\sup\{\xi_1, \xi_2\}) & \text{on } \xi_1 + \xi_2 \le 0 \\ \frac{1}{2}f^2(\inf\{\xi_1, \xi_2\}) & \text{on } \xi_1 + \xi_2 \ge 0. \end{cases} \end{aligned}$$

Consider the four sets:

$$\begin{aligned} A &= \{(\xi_1, \xi_2) \,:\, \xi_1 + \xi_2 \le 0 \,\text{and}\, \xi_2 \ge \xi_1\}, \\ B &= \{(\xi_1, \xi_2) \,:\, \xi_1 + \xi_2 \le 0 \,\text{and}\, \xi_2 \le \xi_1\}, \\ C &= \{(\xi_1, \xi_2) \,:\, \xi_1 + \xi_2 \ge 0 \,\text{and}\, \xi_2 \le \xi_1\}, \\ D &= \{(\xi_1, \xi_2) \,:\, \xi_1 + \xi_2 \ge 0 \,\text{and}\, \xi_2 \ge \xi_1\}, \end{aligned}$$

so that

$$B \cup D = \{(\xi_1, \xi_2) \,:\, |\xi_2| \geq |\xi_1|\}$$

and

$$A \cup C = \{(\xi_1, \xi_2) \,:\, |\xi_1| \geq |\xi_2|\}.$$

We have

$$v_1\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\xi_2 + \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2}f^2(\xi_2)\chi_A + \frac{1}{2}f^2(\xi_1)\chi_B + \frac{1}{2}f^2(\xi_2)\chi_C + \frac{1}{2}f^2(\xi_1)\chi_D$$
$$= \frac{1}{2}f^2(|\xi_1|)\chi_{B\cup D} + \frac{1}{2}f^2(|\xi_2|)\chi_{A\cup C} = \frac{1}{2}f^2(\inf\{|\xi_1|, |\xi_2|\}).$$

Thus claim 3 is fully proved. □

Having proved the properties of the map v described in claims 2 and 3, we introduce the "layer" function ℓ^1, that will be used to extend the map u^1. On the set $\mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$, we define ℓ^1 as

$$\ell^1(x_1, x_2, x_3) = v\left((x_1)_{mod\,1} - \frac{1}{2}, (x_2)_{mod\,1} - \frac{1}{2}, x_3\right).$$

We shall use the following property of ℓ^1:

Claim 4.

$$\ell^1(x_1, x_2, x_3) = \ell^1(|x_1|, |x_2|, |x_3|).$$

Proof of claim 4. We have

$$\ell^1(x_1, x_2, x_3) = v\left((x_1)_{mod\,1} - \frac{1}{2}, (x_2)_{mod\,1} - \frac{1}{2}, x_3\right)$$
$$= v\left(\left|(x_1)_{mod\,1} - \frac{1}{2}\right|, \left|(x_2)_{mod\,1} - \frac{1}{2}\right|, |x_3|\right).$$

By inspection, one verifies that $\left|(|t|)_{mod\,1} - \frac{1}{2}\right| = \left|(t)_{mod\,1} - \frac{1}{2}\right|$, so that

$$\ell^1(x_1, x_2, x_3) = v\left((|x_1|)_{mod\,1} - \frac{1}{2}, (|x_2|)_{mod\,1} - \frac{1}{2}, |x_3|\right)$$
$$= \ell^1(|x_1|, |x_2|, |x_3|).$$

Claim 4 is proved. □

Having introduced ℓ^1, define , for $n \in \mathbb{N}^+$, $\ell^n : \mathbb{R}^2 \times [-\frac{1}{2^n}, \frac{1}{2^n}]$ as

$$\ell^n(x_1, x_2, x_3) = \frac{1}{2^{n-1}}\ell^1(2^{n-1}x_1, 2^{n-1}x_2, 2^{n-1}x_3).$$

Notice that, by claim 4, $\ell^n(x_1, x_2, x_3) = \ell^n(|x_1|, |x_2|, |x_3|)$.
The analogue of the property expressed by claim 3 is given by the following claim 5.

Claim 5. *For $m \in \mathbb{N}^+$ and $r = 1, 2, 3$,*

$$\ell^{m+1}_{(r-1)_3}(x_1, x_2, 0) = \ell^m_r \left(x_1, x_2, \frac{1}{2^m} \right).$$

Proof of claim 5.

$$\begin{aligned}\ell^m_r \left(x_1, x_2, \frac{1}{2^m} \right) &= \frac{1}{2^{m-1}} \ell^1_r \left(2^{m-1} x_1, 2^{m-1} x_2, \frac{1}{2} \right) \\ &= \frac{1}{2^{m-1}} v_r \left((2^{m-1} x_1)_{mod\,1} - \frac{1}{2}, (2^{m-1} x_2)_{mod\,1} - \frac{1}{2}, \frac{1}{2} \right).\end{aligned}$$

On the other hand

$$\begin{aligned}\ell^{m+1}_{(r-1)_3}(x_1, x_2, 0) &= \frac{1}{2^m} \ell^1_{(r-1)_3}(2^m x_1, 2^m x_2, 0) \\ &= \frac{1}{2^m} v_{(r-1)_3} \left((2^m x_1)_{mod\,1} - \frac{1}{2}, (2^m x_2)_{mod\,1} - \frac{1}{2}, 0 \right).\end{aligned}$$

At this point notice that, by inspection, for t in $\mathbb{R}$, $(2t)_{mod\,1} - [2(t)_{mod\,1} - 1] \in \{0, 1\}$. Hence $((2^m x_1)_{mod\,1} - \frac{1}{2}, (2^m x_2)_{mod\,1} - \frac{1}{2}, 0) - (2(2^{m-1} x_1)_{mod\,1} - 1 - \frac{1}{2}, 2(2^{m-1} x_2)_{mod\,1} - 1 - \frac{1}{2}, 0)$ has 0 or 1 at the two first components; so that, by the periodicity of v when $x_3 = 0$,

$$\begin{aligned}& v_{(r-1)_3} \left((2^m x_1)_{mod\,1} - \frac{1}{2}, (2^m x_2)_{mod\,1} - \frac{1}{2}, 0 \right) \\ &= v_{(r-1)_3} \left(2(2^{m-1} x_1)_{mod\,1} - 1 - \frac{1}{2}, 2(2^{m-1} x_2)_{mod\,1} - 1 - \frac{1}{2}, 0 \right),\end{aligned}$$

and applying claim 3,

$$\begin{aligned}& v_{(r-1)_3} \left((2^m x_1)_{mod\,1} - \frac{1}{2}, (2^m x_2)_{mod\,1} - \frac{1}{2}, 0 \right) \\ &= 2 v_r \left((2^{m-1} x_1)_{mod\,1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4}, (2^{m-1} x_2)_{mod\,1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4}, \frac{1}{2} \right) \\ &= 2 v_r \left((2^{m-1} x_1)_{mod\,1} - \frac{1}{2}, (2^{m-1} x_2)_{mod\,1} - \frac{1}{2}, \frac{1}{2} \right),\end{aligned}$$

proving claim 5. □

Set

$$L^1 = \{(x_1, x_2, x_3) \,:\, |x_3| \le 1 \text{ and } \sup\{|x_1|, |x_2|\} \le |x_3|\},$$

and, for $n \ge 2$,

$$L^n = \left\{ (x_1, x_2, x_3) \,:\, \sum_{i=0}^{n-2} \frac{1}{2^i} \le |x_3| \le \sum_{i=0}^{n-1} \frac{1}{2^i} \text{ and } \sup\{|x_1|, |x_2|\} \le |x_3| \right\}.$$

On L^1 the map u^1 is already defined. For $n \geq 2$ and (x_1, x_2, x_3) in L^n, set

$$u_j^n(x_1, x_2, x_3) = \ell^{n-1}_{(j-(n+1))_3}\left(x_1, x_2, |x_3| - \sum_{i=0}^{n-2} \frac{1}{2^i}\right).$$

Remark that, from property β) of the map v and claim 4, we have, for the map u^n, the analogous properties

α') $u^n(x_1, x_2, x_3) = u^n(|x_1|, |x_2|, |x_3|)$;

β') $u^n(x_1, x_2, x_3) = u^n(x_2, x_1, x_3)$.

Notice that it follows that the map $u^1\chi_{L^1} + u^2\chi_{L^2}$ is continuous. To prove this fact, we have to show that

$$u_j^1(x_1, x_2, 1) = \ell_j^1(x_1, x_2, 0),$$

and the validity of this statement is supplied by claim 2 and by the definition of ℓ^1.

Claim 6.

$$u_j^n\left(x_1, x_2, \sum_{i=0}^{n-2} \frac{1}{2^i}\right) = u_j^{n-1}\left(x_1, x_2, \sum_{i=0}^{n-2} \frac{1}{2^i}\right).$$

Proof of claim 6. We have to show that

$$\ell^{n-1}_{(j-(n+1))_3}(x_1, x_2, 0) = \ell^{n-2}_{(j-n)_3}\left(x_1, x_2, \frac{1}{2^{n-2}}\right),$$

and this follows from claim 5 setting $m = n-2$ and $r = (j-n)_3$. Thus claim 6 is proved. □

We wish to extend each map u^n to the set $\{x : \sum_{i=0}^{n-2} \frac{1}{2^i} \leq\| x \|_\infty \leq \sum_{i=0}^{n-1} \frac{1}{2^i}\}$. Set

$$u^n(x) = u^n(|X_i|(x), |X_m|(x), |X_s|(x)).$$

It is a true extension: let x be in L^n. Then $|x_3| = |X_s|(x)$, and

$$(|X_i|(x), |X_m|(x), |X_s|(x)) \in \{(|x_1|, |x_2|, |x_3|), (|x_2|, |x_1|, |x_3|)\}.$$

From α') and β') it follows then that

$$u^n(x_1, x_2, x_3) = u^n(|X_i|(x), |X_m|(x), |X_s|(x)),$$

so that the new definition coincides with the old. Moreover each u^n is a composition of continuous maps, hence continuous.

We have in addition that, for $j = 1, 2, 3$ and $n \in \mathbb{N}^+$,

$$\sup\left\{|u_j^n(x)| \,:\, \sum_{i=0}^{n-2} \frac{1}{2^i} \leq\| x \|_\infty \leq \sum_{i=0}^{n-1} \frac{1}{2^i}\right\} \leq \frac{1}{2^{n-1}}.$$

Consider now $\nabla u^n(x)$. Recalling that the maps $i(\cdot)$, $m(\cdot)$ and $s(\cdot)$ are defined on an open set of full measure and are locally constant on it, we see that, given $\overline{x}$, there exist integer values, say $\overline{\imath}$, $\overline{m}$, $\overline{s}$, which are the values of $i(\cdot)$, $m(\cdot)$, $s(\cdot)$ respectively on a neighborhood of $\overline{x}$. For x in this neighborhood

$$u^n(x) = u^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}}).$$

If we consider the gradient with respect to the variables $x_{\overline{\imath}}$, $x_{\overline{m}}$, $x_{\overline{s}}$, we have

$$\begin{aligned}
\nabla_{\overline{\imath},\overline{m},\overline{s}} u_j^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}}) &= \nabla_{\overline{\imath},\overline{m},\overline{s}} \ell^{n-1}_{(j-(n+1))_3} \left(x_{\overline{\imath}}, x_{\overline{m}}, |x_{\overline{s}}| - \sum_{i=0}^{n-2} \frac{1}{2^i} \right) \\
&= \nabla_{\overline{\imath},\overline{m},\overline{s}} \frac{1}{2^{n-2}} \ell^1_{(j-(n+1))_3} \left(2^{n-2} x_{\overline{\imath}}, 2^{n-2} x_{\overline{m}}, 2^{n-2} \left(|x_{\overline{s}}| - \sum_{i=0}^{n-2} \frac{1}{2^i} \right) \right) \\
&= \nabla_{\overline{\imath},\overline{m},\overline{s}} \frac{1}{2^{n-2}} v_{(j-(n+1))_3} \left((2^{n-2} x_{\overline{\imath}})_{mod\,1} - \frac{1}{2}, (2^{n-2} x_{\overline{m}})_{mod\,1} - \frac{1}{2}, 2^{n-2} \left(|x_{\overline{s}}| - \sum_{i=0}^{n-2} \frac{1}{2^i} \right) \right).
\end{aligned}$$

Except on a set of measure zero, this gradient equals

$$\nabla_{\overline{\imath},\overline{m},\overline{s}} v_{(j-(n+1))_3} \left(\xi_{\overline{\imath}}, \xi_{\overline{m}}, \xi_{\overline{s}} \right)$$

computed at

$$(\xi_{\overline{\imath}}, \xi_{\overline{m}}, \xi_{\overline{s}}) = \left((2^{n-2} x_{\overline{\imath}})_{mod\,1} - \frac{1}{2}, (2^{n-2} x_{\overline{m}})_{mod\,1} - \frac{1}{2}, 2^{n-2} \left(|x_{\overline{s}}| - \sum_{i=0}^{n-2} \frac{1}{2^i} \right) \right)$$

Since, a.e.,

$$\nabla_{\overline{\imath},\overline{m},\overline{s}} v \left(\xi_{\overline{\imath}}, \xi_{\overline{m}}, \xi_{\overline{s}} \right) \in \mathcal{R}$$

and $\nabla_{\overline{\imath},\overline{m},\overline{s}} u^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}})$ is obtained from it by a permutation of the rows, then it follows that $\nabla_{\overline{\imath},\overline{m},\overline{s}} u^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}})$ belongs to $\mathcal{R}$ as well. Since the columns of $\nabla u^n(x)$ are a permutation of the columns of $\nabla_{\overline{\imath},\overline{m},\overline{s}} u^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}})$,we have

$$\nabla u^n(x) \in \mathcal{R}.$$

c) For x such that $\| x \|_\infty < 2$, set:

$$u(x) = u^\nu(x), \quad \text{when}\, x \in L^\nu.$$

By claim 6, the map u is unambigously defined and continuous, actually Lipschitz continuous, a.e. $\nabla u(x)$ is in $\mathcal{R}$ and, by the estimate on $|u_j^\nu(x)|$, one has

$$\lim_{\|x\|_\infty \to 2} u(x) = (0, 0, 0)$$

i.e. u satisfies i) and ii) with $\overline{\Omega} = B_2$.

d) The collection $\{z + rB_2 \,:\, z \in \Omega,\, r \in \mathbb{R}^+,\, r < \frac{1}{2}\operatorname{dist}(z, \partial\Omega)\}$ is a Vitali covering of Ω.
Let z_j and r_j, $j \in \mathbb{N}$, be such that:
(1) $(z_j + r_j B_2)$ are mutually disjoint;
(2) $\Omega = N \cup \left(\bigcup_{j\in\mathbb{N}}(z_j + r_j B_2)\right)$, with N a subset of Ω of zero measure.
For each $j \in \mathbb{N}$, define the vector function $\tilde{u}^j$ on Ω, by setting

$$\tilde{u}^j(x) = r_j u\left(\frac{x - z_j}{r_j}\right)\chi_{z_j + r_j B_2}(x),$$

so that $\nabla\tilde{u}^j(x) \in \mathcal{R}$ for a.e. x in $z_j + r_j B_2$.
Finally set, for x in Ω,

$$\tilde{u}(x) = \sum_{j\in\mathbb{N}} \tilde{u}^j(x).$$

Then $\tilde{u}$ is the required function: $\tilde{u}$ is Lipschitz continuous and satisfies i) and ii) of the Theorem. □

References

[1] J.M. Ball, R.D. James: Fine Phase Mixtures as Minimizers of Energy, Arch. Rational Mech. Anal., 100 (1987), 13-52.

[2] J.M. Ball, R.D. James: Proposed experimental tests of a theory of fine microstructure and the two-well problem, Phil. Trans. Royl. Soc. London, A338 (1991), 389-450.

[3] K. Bhattacharya: The microstructure of martensite and its implications for the shape-memory effect, in: Microstructure and Phase Transition, edited by D. Kinderleherer, R. James, M. Luskin and J.L. Eriksen, The IMA volumes in Mathematics and Applications, Springer, New York 1993..

[4] M. Chipot: Hyperelasticity for crystals, Euro. J. Appl. Math., 1 (1990), 113-129.

[5] I. Fonseca: Phase Transitions in Elastic Solid Materials, Arch. Rational Mech. Anal., 107 (1989), 195-223.

[6] J.P. Matos: Young measures and the absence of fine microstructures in a class of phase transitions, Euro. J. Appl. Math., 3 (1992), 31-54.

[7] Y.G. Reshetnyak: On the stability of conformal mappings in multidimensional spaces, Siberian Math. J., 8 (1967), 69-85.

[8] V. Sverak: On the problem of two wells, in: Microstructure and Phase Transition, edited by D. Kinderleherer, R. James, M. Luskin and J.L. Eriksen, The IMA volumes in Mathematics and Applications, Springer, New York 1993.

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