Quasi-convex Functions
and Quasi-monotone Operators

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Dedicated to R. T. Rockafellar on his 60th Birthday

The notions of a quasi-monotone operator and of a cyclically quasi-monotone operator are introduced, and relations between such operators and quasi-convex functions are established.

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1. Introduction

Let $E$ be a Hausdorff locally convex space, $X$ a convex set in it, and $E^*$ the dual space. We say an operator $f : X \to E^*$ is quasi-monotone if

$$\min \{ \langle y - x, f(x) \rangle, \langle x - y, f(y) \rangle \} \leq 0$$

(1.1)

for all $x, y \in X$, and cyclically quasi-monotone if

$$\min \{ \langle x_{i+1} - x_i, f(x_i) \rangle : i = 0, \ldots, k \} \leq 0$$

(1.2)

for all integer $k$ and all cycles $x_0, x_1, \ldots, x_k, x_{k+1} = x_0$ in $X$.

Such operators are closely related to the so-called demand functions in mathematical economics [4]. Denote $X = \text{int} \mathbb{R}_+^n$ and suppose that there exists the single solution $d(p)$ for the extremal problem

$$u(q) \to \max, \quad pq \leq 1,$$

where an increasing utility function $u$ on $\mathbb{R}_+^n$ is assumed to be given, and a price vector $p \in X$ is considered as a parameter. The demand function $d$, showing demand for $n$ products

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as a vector function of their prices, satisfies the strong axiom of revealed preference due to Houthakker: no cycle $p^0, p^1, \ldots, p^k, p^{k+1} = p^0$ can exist in $X$ such that

$$(p^{i+1} - p^i)d(p^i) \leq 0, \quad i = 0, 1, \ldots, k,$$

and $d(p^i) \neq d(p^j)$ for at least one pair of indices $i, j = 0, \ldots, k$ with $i \neq j$. The same condition for $k = 1$ only is known as the weak axiom of revealed preference due to Samuelson. It is easily seen that each axiom implies the corresponding quasi-monotonicity property for the operator $f = -d : X \to \mathbb{R}^n$, the strong axiom implies (1.2), and the weak axiom implies (1.1).

The simplest examples of quasi-monotone (resp. cyclically quasi-monotone) operators are monotone (resp. cyclically monotone) ones. Their definitions are obtained by replacing minima with sums in (1.1) and (1.2) respectively.

The notion of a (not necessarily single-valued) monotone operator was first proposed by Kachurovskii [1] (see also [2, 3]), and properties and applications of such operators were studied later by many authors. Cyclically monotone operators were introduced and investigated by Rockafellar [6–9]. Given a smooth function $u$ on a convex domain, characterizations of its convexity in terms of monotonicity or cyclical monotonicity of $\operatorname{grad}u$ are known as follows:

$$u \text{ is convex} \iff \operatorname{grad}u \text{ is monotone} \iff \operatorname{grad}u \text{ is cyclically monotone}.$$ 

The goal of the present paper is to get similar characterization theorems connecting quasi-convex functions with quasi-monotone and cyclically quasi-monotone operators.

2. Characterization theorems

Let $X$ be a convex set in a real vector space. Recall that a function $u : X \to \mathbb{R}^1$ is said to be quasi-convex if its sublevel sets

$$\{ x : u(x) \leq \alpha \}, \quad \alpha \in \mathbb{R}^1,$$

are convex or, which is the same thing, if

$$u((1 - t)x + ty) \leq \max\{u(x), u(y)\}$$

whenever $x, y \in X$, $0 < t < 1$.

It follows from this definition that $u$ is quasi-convex if and only if all the functions $\varphi(u, x, y; \cdot)$, $x, y \in X$, on the segment $[0, 1]$ are so, where

$$\varphi(u, x, y; t) := u((1 - t)x + ty). \quad (2.1)$$

**Theorem 2.1.** Let $X$ be a convex open set in a Hausdorff locally convex space $E$, and suppose that a function $u : X \to \mathbb{R}^1$ has two properties as follows:

(i) $u$ is Gâteaux differentiable on $X$, i.e. for every $x \in X$ there exists an element $\operatorname{grad}u(x) \in E^*$ such that

$$\lim_{t \to 0} \frac{u(x + th) - u(x)}{t} = \langle h, \operatorname{grad}u(x) \rangle \quad \text{for all } h \in E; \quad (2.2)$$
for every $x, y \in X$ the function $\varphi(u, x, y; \cdot)$ given by (2.1) is absolutely continuous on $[0, 1]$.

The following assertions are then equivalent:

(a) $u$ is quasi-convex,
(b) the operator $\text{grad} u$ is cyclically quasi-monotone,
(c) the operator $\text{grad} u$ is quasi-monotone.

Remark 2.2. Assumptions (i) and (ii) are clearly satisfied if $u$ is $C^1$. For the finite-dimensional case of the theorem, see also [5, Proposition 3.1].

Before to pass on to a more general characterization theorem, let us formulate two assumptions, (D1) and (D2), on a function $u : X \to \mathbb{R}$, where $X$ is a convex set in a vector space. The assumptions are expressed in terms of the functions $\varphi(u, x, y; \cdot)$ (see (2.1)) as follows:

(D1) for every $x, y \in X$ and every $t, 0 < t < 1$, there exists the right derivative

$$D\varphi(u, x, y; t) := \lim_{\Delta t \downarrow 0} \frac{\varphi(u, x, y; t + \Delta t) - \varphi(u, x, y; t)}{\Delta t};$$

(D2) for every $x, y \in X$, $\varphi(u, x, y; \cdot)$ is absolutely continuous on $[0, 1]$.

It follows from (D1) and (D2) that

$$\int_0^1 D\varphi(u, x, y; t) \, dt = u(y) - u(x) \quad \text{whenever } x, y \in X. \quad (2.3)$$

Note that if (D1) holds, then for every $x, y \in X$ the directional derivative is defined as follows:

$$u'(x, y - x) := D\varphi(u, x, y; 0) = \lim_{t \downarrow 0} \frac{u(x + t(y - x)) - u(x)}{t}. \quad (2.4)$$

Theorem 2.3. Let $X$ be a convex subset in a vector space and $u : X \to \mathbb{R}$ satisfy (D1) and (D2). The following assertions are then equivalent:

(a) $u$ is quasi-convex;
(b) for every integer $k$ and every cycle $x_0, x_1, \ldots, x_k, x_{k+1} = x_0$ in $X$, the inequality

$$\min\{u'(x_i, x_{i+1} - x_i) : i = 0, \ldots, k\} \leq 0$$

holds;
(c) for every $x, y \in X$, the inequality

$$\min\{u'(x, y - x), u'(y, x - y)\} \leq 0$$

holds.

Remark 2.4. Similarly to the case of cyclically monotone operators, the notion of cyclical quasi-monotonicity can be generalized in the natural way on multivalued operators. A well-known result of Rockafellar [6,9] asserts that subdifferentials of proper
semi-continuous convex functions are characterized as maximal cyclically monotone operators. Here a multivalued cyclically monotone (resp. cyclically quasi-monotone) operator \( f \) is called maximal if no cyclically monotone (resp. cyclically quasi-monotone) operator \( g \) exists with \( \text{gr} f \subseteq \text{gr} g \). This Rockafellar’s characterization theorem cannot be generalized on quasi-convex functions, because for any (single-valued) cyclically quasi-monotone operator \( f \), the operator \( g(x) := \{ \alpha f(x) : \alpha \geq 0 \} \) is cyclically quasi-monotone as well, and \( \text{gr} f \subseteq \text{gr} g \) provided \( f \neq 0 \). It follows that for every non-constant smooth quasi-convex function \( u \) the single-valued cyclically quasi-monotone operator \( f = \text{grad} u \) is not maximal.

3. Proofs

Proof of Theorem 2.1 Observe that if \( u \) is Gâteaux differentiable on \( X \), then (D1) holds and \( u'(x, y - x) = \langle y - x, \text{grad} u(x) \rangle \) for all \( x, y \in X \). Theorem 2.1 is then a direct consequence of Theorem 2.3.

Proof of Theorem 2.3 (a)\( \Rightarrow \) (b). If (b) fails, then for some cycle \( x_0, x_1, \ldots, x_{k+1} = x_0 \) in \( X \),
\[
u'(x_i, x_{i+1} - x_i) > 0, \quad i = 0, 1, \ldots, k.
\]
We have \( D\varphi(u, x_i, x_{i+1}; 0) > 0 \), so
\[
\varphi(u, x_i, x_{i+1}; 0) < \varphi(u, x_i, x_{i+1}; t) \quad \text{for small } t > 0,
\]
and as \( \varphi(u, x_i, x_{i+1}; \cdot) \) is quasi-convex on \([0, 1]\), it follows from (3.1) that
\[
\varphi(u, x_i, x_{i+1}; 0) < \varphi(u, x_i, x_{i+1}; 1),
\]
i.e. \( u(x_i) < u(x_{i+1}) \). We obtain a contradictory chain of inequalities
\[
u(x_0) < u(x_1) < \ldots < u(x_k) < u(x_0),
\]
and the contradiction means that (b) is true.

(b)\( \Rightarrow \) (c). Obvious.

(c)\( \Rightarrow \) (a). Suppose \( u \) is not quasi-convex. There exist then \( x, y \in X \) and \( t_0, 0 < t_0 < 1 \), such that
\[
u((1 - t_0)x + t_0y) > \max\{u(x), u(y)\}.
\]
We claim that there exist \( t_1 \) and \( t_2, 0 < t_1 < t_0 < t_2 < 1 \), such that
\[
D\varphi(u, x, y; t_1) > 0
\]
and
\[
D\varphi(u, y, x; 1 - t_2) > 0.
\]
Indeed, if \( D\varphi(u, x, y; t) \leq 0 \) for all \( t \), \( 0 < t < t_0 \), then, taking into account (2.3) and the identity
\[
\varphi(u, x, t_0) + t_0y, t) = \varphi(u, x, t_0) + t_0y, t),
\]
we obtain
\[
\begin{align*}
u((1 - t_0)x + t_0y) &= u(x) + \int_0^1 D\varphi(u, x, (1 - t_0)x + t_0y; t) \, dt \\
&= u(x) + \int_0^1 D\varphi(u, x, y; t_0t) \, dt \\
&= u(x) + \frac{1}{t_0} \int_0^{t_0} D\varphi(u, x, y; \tau) \, d\tau \leq u(x),
\end{align*}
\]
which contradicts (3.2).
Similarly, if \(D\varphi(u, y, x; t) \leq 0\) for all \(0 < t < 1 - t_0\), then
\[
\begin{align*}
u((1 - t_0)x + t_0y) &= u(y) + \int_0^1 D\varphi(u, y, (1 - t_0)x + t_0y; t) \, dt \\
&= u(y) + \int_0^1 D\varphi(u, y, x; 1 - t_0t) \, dt \\
&= u(y) + \frac{1}{1 - t_0} \int_0^{1-t_0} D\varphi(u, y, x; \tau) \, d\tau \leq u(y),
\end{align*}
\]
which again contradicts (3.2). The claim is thus proved.

Set now
\[
x_k := (1 - t_k)x + t_k y, \quad k = 1, 2,
\]
and, by using (3.3) (3.4) and the identities
\[
\begin{align*}
\varphi(u, x_1, x_2; t) &= \varphi(u, x, y; t_1 + t(t_2 - t_1)), \\
\varphi(u, x_2, x_1; t) &= \varphi(u, y, x; 1 - t_2 + t(t_2 - t_1)),
\end{align*}
\]
one obtains
\[
\begin{align*}
u'(x_1, x_2 - x_1) &= D\varphi(u, x_1, x_2; 0) = (t_2 - t_1) \, D\varphi(u, x, y; t_1) > 0, \\
u'(x_2, x_1 - x_2) &= D\varphi(u, x_2, x_1; 0) = (t_2 - t_1) \, D\varphi(u, y, x; 1 - t_2) > 0,
\end{align*}
\]
hence
\[
\min\{u'(x_1, x_2 - x_1), u'(x_2, x_1 - x_2)\} > 0,
\]
a contradiction with (c).

Remark: After submitting the present paper, in November 1994, I have seen a manuscript by Aussel, Corvellec and Lassonde [10], where some related results on connections between (non-differentiable) quasi-convex functions and multivalued quasi-monotone operators in Banach spaces are proved in a different way.
References


