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# Dualities Associated To Binary Operations On $\overline{R}$

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**Dedicated to R. T. Rockafellar on his 60th Birthday**

Continuing our papers [14]–[16] and [6]–[9], where we have given an axiomatic approach to generalized conjugation theory, we introduce and study dualities $\Delta\colon \overline{R}^X \to \overline{R}^W$ associated to a binary operation $*$ on $\overline{R}$, where $X$ and $W$ are two arbitrary sets and $\overline{R} = [-\infty, +\infty]$, which encompass, as particular cases, conjugations, $\vee$-dualities and $\perp$-dualities in the sense of [14] and [7]. We show that this class of dualities can be extended so as to encompass also the $*$-dualities $\Delta\colon \overline{A}^X \to \overline{A}^W$ in the sense of [8], where $\overline{A}$ is the canonical enlargement of a complete totally ordered group.

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## 1. Introduction

Since various concepts of conjugation have important applications to duality in optimization theory, an axiomatic approach to generalized conjugation theory was started in [14] and continued in [15], [16] and [6]–[9]. Let us recall that if $X$ and $W$ are two sets (which we shall assume non-empty throughout the sequel), a mapping $\Delta\colon \overline{R}^X \to \overline{R}^W$ (where $\overline{R}^X$ denotes the family of all functions $f\colon X \to \overline{R} = [-\infty, +\infty]$) is called
a) a *duality* ([16],[6]), if for any index set $I$ we have

$$(\inf_{i\in I} f_i)^{\Delta} = \sup_{i\in I} f_i^{\Delta} \qquad (\{f_i\}_{i\in I} \subseteq \overline{R}^X), \tag{1.1}$$

where $\inf_{i\in I} f_i \in \overline{R}^X$ and $\sup_{i\in I} f_i^{\Delta} \in \overline{R}^W$ are defined pointwise on $X$ and $W$ respectively (i.e., $(\inf_{i\in I} f_i)(x) = \inf_{i\in I} f_i(x)$ for all $x \in X$), with the usual conventions

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$$\inf \emptyset = +\infty, \quad \sup \emptyset = -\infty; \tag{1.2}$$

b) a *conjugation* [14], if we have (1.1) and

$$(f \dot{+} d)^{\Delta} = f^{\Delta} \underset{\cdot}{+} -d \qquad\qquad (f \in \overline{R}^X, d \in \overline{R}), \tag{1.3}$$

where we identify each $d \in \overline{R}$ with the constant function taking everywhere the value $d$, the operations $\dot{+}$ and $\underset{\cdot}{+}$ on $\overline{R}^X$ (respectively, $\overline{R}^W$) are defined pointwise, and the binary operations $\dot{+}$ and $\underset{\cdot}{+}$ on $\overline{R}$ are the "upper addition" and "lower addition" defined ([10], [11]) by

$$a \dot{+} b = a \underset{\cdot}{+} b = a + b \quad \text{if} \quad R \cap \{a, b\} \neq \emptyset \quad \text{or} \quad a = b = \pm\infty, \tag{1.4}$$

$$a \dot{+} b = +\infty, a \underset{\cdot}{+} b = -\infty \quad \text{if} \quad a = -b = \pm\infty; \tag{1.5}$$

c) a $\vee$-*duality* [7], if we have (1.1) and

$$(f \vee d)^{\Delta} = f^{\Delta} \wedge -d \qquad\qquad (f \in \overline{R}^X, d \in \overline{R}), \tag{1.6}$$

where $\vee$ and $\wedge$ stand for (pointwise) sup and inf, in $\overline{R}^X$ and $\overline{R}^W$ respectively;
d) a $\perp$-*duality* [7], if we have (1.1) and

$$(f \perp d)^{\Delta} = f^{\Delta} \top -d \qquad\qquad (f \in \overline{R}^X, d \in \overline{R}), \tag{1.7}$$

where $\perp$ and $\top$ are the binary operations defined [6] on $\overline{R}$ by

$$a \perp b = \begin{cases} a & \text{if } a < b \\ +\infty & \text{if } a \geq b, \end{cases} \tag{1.8}$$

$$a \top b = \begin{cases} a & \text{if } a > b, \\ -\infty & \text{if } a \leq b, \end{cases} \tag{1.9}$$

and extended pointwise to $\overline{R}^X$ and $\overline{R}^W$.
Let us also recall (see e.g. [16], [6]) that if $\Delta: \overline{R}^X \to \overline{R}^W$ is a duality, then so is the dual mapping $\Delta': \overline{R}^W \to \overline{R}^X$ defined by

$$g^{\Delta'} = \inf_{\substack{h \in \overline{R}^X \\ h^{\Delta} \leq g}} h \qquad\qquad (g \in \overline{R}^W), \tag{1.10}$$

and for any $f \in \overline{R}^X$ and $g \in \overline{R}^W$ we have the equivalence

$$f^{\Delta} \leq g \Leftrightarrow g^{\Delta'} \leq f. \tag{1.11}$$

In the above mentioned papers, among other results, various "representation theorems" have been given for dualities (with the aid of functions $G: X \times W \times \overline{R} \to \overline{R}$), conjugations, $\vee$-dualities and $\wedge$-dualities (with the aid of "coupling functions" from $X \times W$ into $\overline{R}$)

$\Delta: \overline{R}^X \to \overline{R}^W$ and for their duals $\Delta': \overline{R}^W \to \overline{R}^X$, as well as for the "second dual" $f^{\Delta\Delta'} = (f^{\Delta})^{\Delta'} \in \overline{R}^X$ of a function $f \in \overline{R}^X$. Also, it has been shown there that the dual $\Delta': \overline{R}^W \to \overline{R}^X$ of a conjugation $\Delta: \overline{R}^X \to \overline{R}^W$ is a conjugation, but the dual of a $\vee$-duality is a $\perp$-duality (actually, this was the main motivation for introducing in [6] the operations $\perp$ and $\top$ and the concept of $\perp$-dualities) and the dual of a $\perp$-duality is a $\vee$-duality.

Furthermore, in the recent paper [8], we have generalized the theory of conjugations (1.1), (1.3) to certain mappings $\Delta: \overline{A}^X \to \overline{A}^W$, called (in [8]) $*$-*dualities*, where $\overline{A} = (\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ is the "canonical enlargement" of a complete totally ordered group $A = (A, \leq, *)$, which contain, as particular cases, the conjugations $\Delta: \overline{R}^X \to \overline{R}^W$ and various kinds of known polarities (e.g., polarities $\Delta: \overline{R}_+^X \to \overline{R}_+^W$ in the sense of Moreau [10], p. 92, formula (14.4), Rockafellar [13], p. 136, Elster and Wolf [4], with applications to fractional programming duality, etc.).

In the present paper, continuing to develop these axiomatic approaches to generalized conjugation theory, we shall introduce and study a kind of dualities $\Delta: \overline{R}^X \to \overline{R}^W$, namely, *dualities associated to a binary operation* $*$ *on* $\overline{R}$ satisfying "condition $(\alpha)$" (i.e., condition (2.1) below), called, briefly, $*$-*dualities*, which encompass, as particular cases, the conjugations, $\vee$-dualities and $\perp$-dualities mentioned above. Also, we shall show how this theory can be extended to the case when $\overline{R}$ is replaced by the canonical enlargement $\overline{A}$ of a complete totally ordered group $A = (A, \leq, *)$, so as to encompass also the "$*$-dualities" in the sense of [8], as particular cases.

In Section 2 we shall introduce the class of binary operations $*$ on $\overline{R}$ satisfying *condition* $(\alpha)$ (defined by (2.1)), which contain, as particular cases, the binary operations $\dot{+}$, $\vee$ and $\perp$ mentioned above. Also, we shall introduce and study "*the (left) epi-hypo-inverse*" $*_l$ and "*the (left) conjugate*" $\overline{*}$ of such a binary operation $*$, which will be needed in the sequel.

In Section 3 we shall introduce the concept of a duality $\Delta: \overline{R}^X \to \overline{R}^W$ with respect to a binary operation $*$ on $\overline{R}$ satisfying condition $(\alpha)$, called, briefly, a $*$-*duality*, with the aid of a suitable "second condition" (besides (1.1)), namely, condition (3.1), encompassing, among other particular cases, (1.3), (1.6) and (1.7). Also we shall determine the dual of a $*$-duality, from which one recovers, in particular, the above mentioned results of [14] and [7] on the duals of conjugations, $\vee$-dualities and $\perp$-dualities.

In Section 4 we shall obtain some results on the representations of $*$-dualities $\Delta: \overline{R}^X \to \overline{R}^W$ and of their duals $\Delta': \overline{R}^W \to \overline{R}^X$, with the aid of coupling functions $\psi: X \times W \to \overline{R}$, which contain, as particular cases, the results of [14] and [7] on the representation of conjugations, $\vee$-dualities and $\perp$-dualities, mentioned above.

Finally, in Section 5 (Appendix) we shall show that the concept $*$-duality of the present paper can be extended to a more general notion of "$(*, s)$-duality", which encompasses, as particular cases, also the "$*$-dualities" $M: \overline{A}^X \to \overline{A}^W$, in the sense of [8], where $\overline{A} = (\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ is the "canonical enlargement" of a complete totally ordered group $A = (A, \leq, *)$. Thus, we shall obtain a unifying framework for the results of the present paper and those of [8].

## 2. Inverses and conjugates of binary operations on $\overline{R}$

**Definition 2.1.** We shall say that a binary operation $*$ on $\overline{R}$ satisfies *condition* $(\alpha)$, if for any index set $I$ we have

$$(\inf_{i\in I} b_i) * c = \inf_{i\in I}(b_i * c) \qquad (\{b_i\}_{i\in I} \subseteq \overline{R}, c \in \overline{R}). \tag{2.1}$$

**Remark 2.2.** a) For $I = \emptyset$, condition (2.1) yields

$$+\infty * c = +\infty \qquad (c \in \overline{R}). \tag{2.2}$$

b) For each $c \in \overline{R}$, define $k_c: \overline{R} \to \overline{R}$ by

$$k_c(b) = b * c \qquad (b \in \overline{R}). \tag{2.3}$$

Then, condition $(\alpha)$ means that for any index set $I$ we have

$$k_c(\inf_{i\in I} b_i) = \inf_{i\in I} k_c(b_i) \qquad (\{b_i\}_{i\in I} \subseteq \overline{R}, c \in \overline{R}), \tag{2.4}$$

or, *equivalently* (by [6], lemma 2.1, applied to $k_c$ regarded as a mapping of $(\overline{R}, \leq)$ into $(\overline{R}, \geq)$, i.e., into $\overline{R}$ endowed with the "reverse order" $\geq$), that for each $c \in \overline{R}$ the function $k_c$ *is non-decreasing and upper semi-continuous.*

Let us give now some examples of binary operations $*$ on $\overline{R}$ satisfying condition $(\alpha)$.

**Example 2.3.** Let $* = \dot{+}$. Then, by [11], formula (4.7), $*$ satisfies condition $(\alpha)$.

**Example 2.4.** Let $* = \vee$. Then it is well-known (and immediate) that $*$ satisfies condition $(\alpha)$.

**Example 2.5.** Let $* = \perp$ (of (1.8)). Then, by [7], formula (1.21), $*$ satisfies condition $(\alpha)$.

We shall denote by min (respectively, max), an inf (respectively, sup) which is attained.

**Proposition 2.6.** *Let $*$ be a binary operation on $\overline{R}$, satisfying condition $(\alpha)$. Then there exists a unique binary operation $*_l$ on $\overline{R}$ such that for any $a, b, c \in \overline{R}$ we have the equivalence*

$$a \leq b * c \Leftrightarrow a *_l c \leq b, \tag{2.5}$$

*namely,*

$$a *_l c = \min\{b' \in \overline{R} \mid a \leq b' * c\} \qquad (a, c \in \overline{R}). \tag{2.6}$$

**Proof.** Since $*$ satisfies condition $(\alpha)$, for any $a, c \in \overline{R}$ we have

$$(\inf\{b' \in \overline{R} \mid a \leq b' * c\}) * c = \inf\{b' * c \mid b' \in \overline{R}, a \leq b' * c\} \geq a,$$

so $\inf\{b' \in \overline{R} \mid a \leq b' * c\} \in \{b' \in \overline{R} \mid a \leq b' * c\}$. Thus, the min in (2.6) exists.

Let us show now that the binary operation $*_l$ on $\overline{R}$ defined by (2.6) satisfies (2.5).

If $a \leq b * c$, then $b \in \{b' \in \overline{R} \mid a \leq b' * c\}$, whence, by (2.6), $a *_l c \leq b$. Conversely, assume now that $a *_l c = \min\{b' \in \overline{R} \mid a \leq b' * c\} \leq b$, so there exists $b' \in \overline{R}$ such that

$a \leq b' * c = k_c(b')$, $b' \leq b$. Then, since $k_c$ is non-decreasing (by remark 2.2 b)), we obtain $a \leq k_c(b') \leq k_c(b) = b * c$.
Finally, if we have (2.5), then for any $a, c \in \overline{R}$ we have

$$a *_l c = \min\{b' \in \overline{R} \mid a *_l c \leq b'\} = \min\{b' \in \overline{R} \mid a \leq b' * c\}.$$

□

**Definition 2.7.** Let $*$ be a binary operation on $\overline{R}$, satisfying condition $(\alpha)$. Then the unique binary operation $*_l$ on $\overline{R}$, of proposition 2.6, will be called *the (left) epi-hypo-inverse of* $*$.

**Remark 2.8.** a) The theory of inversion of functions $k: R \to \overline{R}$ (see e.g. [2], pp. 208–211, and [12]) can be easily extended to functions $k: \overline{R} \to \overline{R}$, which is, in fact, its natural framework. Then, for any $c \in \overline{R}$, since $k_c$ of (2.3) is non-decreasing and upper semi-continuous, it admits (see e.g. [12], proposition 2.6, extended to this framework), a unique "epi-hypo-inverse", i.e., a unique function $j_c: \overline{R} \to \overline{R}$ such that for any $a, b \in \overline{R}$ we have the equivalence

$$a \leq k_c(b) \Leftrightarrow j_c(a) \leq b, \tag{2.7}$$

namely,

$$j_c(a) = \min\{b' \in \overline{R} \mid a \leq k_c(b')\} \qquad (a \in \overline{R}). \tag{2.8}$$

Then, by proposition 2.6, we have

$$j_c(a) = a *_l c \qquad (a \in \overline{R}), \tag{2.9}$$

which motivates the terminology of definition 2.7.
b) If $*$ is commutative and satisfies condition $(\alpha)$, then, by (2.5),

$$a *_l c \leq b \Leftrightarrow a *_l b \leq c. \tag{2.10}$$

c) For a binary operation $*$ on $\overline{R}$, we shall also consider the binary operation $*-$ on $\overline{R}$, defined by

$$a * -c = a * (-c) \qquad (a, c \in \overline{R}). \tag{2.11}$$

Then, by (2.6) and (2.11), we have $(*-)_l = *_l-$, since

$$a(*-)_l c = \min\{b' \in \overline{R} \mid a \leq b' * -c = b' * (-c)\} = a *_l (-c) \qquad (a, c \in \overline{R}). \tag{2.12}$$

d) If $*$ satisfies condition $(\alpha)$, then so does $*-$. Since $(*-)- = *$, the converse is also true.
If a binary operation $*$ on $\overline{R}$ satisfies condition $(\alpha)$, then $*_l$ need not satisfy it, but we shall show that $*_l$ has a "dual" property.

**Definition 2.9.** We shall say that a binary operation $*$ on $\overline{R}$ satisfies *condition* $(\beta)$, if for any index set $I$ we have

$$(\sup_{i\in I} a_i) * c = \sup_{i\in I}(a_i * c) \qquad (\{a_i\}_{i\in I} \subseteq \overline{R}, c \in \overline{R}). \tag{2.13}$$

**Remark 2.10.** If $*$ satisfies condition $(\beta)$, then, by (2.13) for $I = \emptyset$, we have

$$-\infty * c = -\infty \qquad (c \in \overline{R}). \tag{2.14}$$

**Proposition 2.11.** *If a binary operation $*$ on $\overline{R}$ satisfies condition $(\alpha)$, then $*_l$ satisfies condition $(\beta)$.*

**Proof.** By $(\alpha)$, the minima in (2.6) are attained, and hence

$$(\sup_{i\in I} a_i) *_l c = \min\{b' \in \overline{R} \mid \sup_{i\in I} a_i \le b' * c\} = \min \bigcap_{i\in I}\{b' \in \overline{R} \mid a_i \le b' * c\} =$$
$$= \sup_{i\in I} \min\{b' \in \overline{R} \mid a_i \le b' * c\} = \sup_{i\in I}(a_i *_l c) \quad (\{a_i\}_{i\in I} \subseteq \overline{R}, c \in \overline{R}).$$
$\square$

**Remark 2.12.** a) One can also give the following alternative proof of proposition 2.11: If $a, a', c \in \overline{R}$, $a \le a'$, then $\{b' \in \overline{R} \mid a' \le k_c(b')\} \subseteq \{b' \in \overline{R} \mid a \le k_c(b')\}$, whence, by (2.8), $j_c(a) \le j_c(a')$, so $j_c$ is *non-decreasing*. Also, by (2.7), for each $b \in \overline{R}$ we have $\{a \in \overline{R} \mid j_c(a) \le b\} = \{a \in \overline{R} \mid a \le k_c(b)\}$, which is a closed set, so $j_c$ is *lower semi-continuous*. But, by [6], lemma 2.1, we have these two properties if and only if

$$j_c(\sup_{i\in I} a_i) = \sup_{i\in I} j_c(a_i) \qquad (\{a_i\}_{i\in I} \subseteq \overline{R}, c \in \overline{R}), \tag{2.15}$$

i.e. (by (2.9)), if and only if $*_l$ satisfies condition $(\beta)$.
b) By the above, $k_c$ of (2.3) is the "*hypo-epi-inverse*" of $j_c$ (of (2.9)), in the inversion theory (of [2], pp. 208–211, and [12]) extended to functions $k: \overline{R} \to \overline{R}$, and therefore one can say that the binary operation $*$ is the *(left) hypo-epi-inverse of* $*_l$, in symbols, $* = (*_l)_u = *_{lu}$. Indeed, from (2.5) we obtain

$$b * c = \max\{a' \in \overline{R} \mid a' \le b * c\} = \max\{a' \in \overline{R} \mid a' *_l c \le b\} \qquad (b, c \in \overline{R}). \tag{2.16}$$

More generally, if we start with any $*$ satisfying condition $(\beta)$, then one can define, in the obvious way, the *(left) hypo-epi-inverse* $*_u$ of $*$, and we have $* = (*_u)_l = *_{ul}$. Also, dually to proposition 2.11, we have that *if a binary operation $*$ on $\overline{R}$ satisfies condition $(\beta)$, then $*_u$ satisfies condition $(\alpha)$.*
c) Concerning the above notations, let us mention that $l$ (and $u$) stand to indicate the lower (and, respectively, the upper) semi-continuity of $*_l$ (respectively, $*_u$) in the first component. The word "left" in the above terminology is used to indicate that we are dealing with the first component of $*_l$ (respectively, $*_u$); in the sequel, we shall omit the word "left", since this will lead to no confusion. Note also that a binary operation $*$ on $\overline{R}$ satisfying both $(\alpha)$ and $(\beta)$ (with $*_l$ replaced by $*$) has both inverses $*_l$ and $*_u$, but they need not coincide.
Let us consider now some examples.

**Example 2.3** (continued). If $* = \dot{+}$, then $*_l = \underset{\cdot}{+}-$ (since these $*$ and $*_l$ satisfy (2.5), by [11], formula (3.3)).

**Example 2.4** (continued). If $* = \vee$, then $*_l = \top$ (since these $*$ and $*_l$ satisfy (2.5), by [7], formula (1.5)).

**Example 2.5** (continued). If $* = \perp$, then $*_l = \wedge$ (since these $*$ and $*_l$ satisfy (2.5), by [7], formula (1.6)).

In the sequel, for simplicity, for any binary operation $*$ on $\overline{R}$ we shall write $-a * c$ instead of $(-a) * c$, which will lead to no confusion (with $-(a * c)$).

**Definition 2.13.** Let $*$ be a binary operation $\overline{R}$. Then the binary operation $\overline{*}$ on $\overline{R}$, defined by

$$a \overline{*} c = -(-a * c) \qquad (a, c \in \overline{R}), \tag{2.17}$$

will be called *the (left) conjugate of* $*$.

**Example 2.3** (continued). If $* = \dot{+}$, then $\overline{*} = \underset{\cdot}{+}-$ (since $-(-a \dot{+} c) = a \underset{\cdot}{+} - c$, by [11], formula (3.2)).

**Example 2.4** (continued). If $* = \vee$, then $\overline{*} = \wedge-$ (since $-(-a \vee c) = a \wedge -c$).

**Example 2.5** (continued). If $* = \perp$, then $\overline{*} = \top-$ (since $-(-a \perp c) = a \top - c$, by [7], formula (1.12)).

**Remark 2.14.** a) By (2.17), we have

$$-(a \overline{*} c) = -a * c \qquad (a, c \in \overline{R}), \tag{2.18}$$

$$a * c = -(-a \overline{*} c) \qquad (a, c \in \overline{R}). \tag{2.19}$$

b) By (2.17) (applied to $\overline{*}$ instead of $*$) and (2.19), for the "biconjugate" $\overline{\overline{*}} = \overline{(\overline{*})}$ of a binary operation $*$ on $\overline{R}$ we have $\overline{\overline{*}} = *$, since

$$a \overline{\overline{*}} c = -(-a \overline{*} c) = a * c \qquad (a, c \in \overline{R}). \tag{2.20}$$

c) For any binary operation $*$ on $\overline{R}$ we have

$$\overline{*-} = \overline{*} - . \tag{2.21}$$

Indeed, by (2.17) and (2.11),

$$a \overline{*-} c = -(-a * -c) = -(-a * (-c)) = a \overline{*} (-c) = a \overline{*} - c \qquad (a, c \in \overline{R}). \tag{2.22}$$

**Proposition 2.15.**

a) *A binary operation $*$ on $\overline{R}$ satisfies $(\alpha)$ (respectively, $(\beta)$) if and only if $\overline{*}$ satisfies $(\beta)$ (respectively, $(\alpha)$).*

b) *If $*$ satisfies $(\alpha)$, then for any $a, b, c \in \overline{R}$ we have the equivalence*

$$a \overline{*} c \leq b \Leftrightarrow -b *_l c \leq -a. \tag{2.23}$$

*If $*$ is also commutative, then we also have the equivalence*

$$a\overline{*}c \leq b \Leftrightarrow -b *_l -a \leq c. \tag{2.24}$$

c) *$*$ is commutative if and only if $\overline{*}$ is "anti-commutative", i.e.,*

$$a\overline{*}c = -c\overline{*} - a \qquad\qquad (a, c \in \overline{R}). \tag{2.25}$$

**Proof.** a) If $*$ satisfies condition ($\alpha$), then

$$\begin{aligned}(\sup_{i\in I} a_i)\overline{*}c &= -\left[(-\sup_{i\in I} a_i) * c\right] = -\left[(\inf_{i\in I}(-a_i)) * c\right] = \\ &= -\inf_{i\in I}(-a_i * c) = \sup_{i\in I}[-(-a_i * c)] = \sup_{i\in I}(a_i\overline{*}c) \quad (\{a_i\}_{i\in I} \subseteq \overline{R}, c \in \overline{R}),\end{aligned}$$

so $\overline{*}$ satisfies condition ($\beta$). Dually, interchanging sup and inf, we obtain that if $*$ satisfies ($\beta$), then $\overline{*}$ satisfies ($\alpha$). Hence, if $\overline{*}$ satisfies ($\beta$) (respectively, ($\alpha$)) then $* = \overline{\overline{*}}$ satisfies ($\alpha$) (respectively, ($\beta$)).
b) By (2.17) and (2.5) we have

$$a\overline{*}c \leq b \Leftrightarrow -(-a * c) \leq b \Leftrightarrow -a * c \geq -b \Leftrightarrow -b *_l c \leq -a.$$

If $*$ is also commutative, then, by (2.23) and (2.10), we have (2.24).
c) If $*$ is commutative, then

$$a\overline{*}c = -(-a * c) = -(c * -a) = -c\overline{*} - a \qquad\qquad (a, c \in \overline{R}).$$

Dually, *if $*$ is anti-commutative, then $\overline{*}$ is commutative.* Hence, if $\overline{*}$ is anti-commutative, then $* = \overline{\overline{*}}$ is commutative. □

Since $\overline{*}$ is defined for any binary operation $*$ (not necessarily satisfying condition ($\alpha$)), we may consider the conjugate of the epi-hypo-inverse of a binary operation $*$ satisfying condition ($\alpha$), i.e., the binary operation

$$\begin{aligned}a\overline{*_l}c &= -(-a *_l c) = -\min\{b' \in \overline{R} \mid -a \leq b' * c\} = \\ &= \max\{-b' \in \overline{R} \mid a \geq -(b' * c)\} = \max\{b \in \overline{R} \mid a \geq -(-b * c)\} = \\ &= \max\{b \in \overline{R} \mid a \geq b\overline{*}c\} \qquad\qquad (a, c \in \overline{R}).\end{aligned} \tag{2.26}$$

**Remark 2.16.** By (2.12) and (2.21) (applied to $*_l$ instead of $*$), for any binary operation $*$ on $\overline{R}$ we have

$$\overline{(*-)_l} = \overline{*_l -} = (\overline{*_l}) - . \tag{2.27}$$

**Theorem 2.17.** *If a binary operation $*$ on $\overline{R}$ satisfies condition ($\alpha$), then so does the binary operation $\overline{*_l}$, and we have*

$$(\overline{*_l})_l = \overline{*}, \tag{2.28}$$

$$\overline{(\overline{*_l})_l} = *. \tag{2.29}$$

**Proof.** By proposition 2.11, $*_l$ satisfies condition $(\beta)$. Hence, by proposition 2.15, $\overline{*_l}$ satisfies condition $(\alpha)$, and therefore $(\overline{*_l})_l$ is well defined. Then, by (2.6) (applied to $\overline{*_l}$), (2.5) and (2.17), we obtain

$$\begin{aligned} a(\overline{*_l})_l c &= \min\{b' \in \overline{R} \mid a \leq b' \overline{*_l} c\} = \min\{b' \in \overline{R} \mid a \leq -(-b' *_l c)\} = \\ &= -\max\{b \in \overline{R} \mid -a \geq b *_l c\} = -\max\{b \in \overline{R} \mid b \leq -a * c\} = \\ &= -(-a * c) = a \overline{*} c \qquad\qquad (a, c \in \overline{R}), \end{aligned}$$

which proves (2.28). Finally, (2.28) and (2.29) are equivalent (by (2.20)). □

**Remark 2.18.** By the first part of theorem 2.17, applied to $\overline{*_l}$, and by (2.29), *if $\overline{*_l}$ satisfies condition $(\alpha)$, then so does $*$.*

**Example 2.3** (continued). If $* = \dot{+}$, so $*_l = \underset{\cdot}{+}-$, then $\overline{*_l} = \dot{+}$ (since $-(-a\underset{\cdot}{+}-c) = a\dot{+}c$). Hence, by theorem 2.17, $(\overline{*_l})_l = \overline{*} = \underset{\cdot}{+}-$, $\overline{(\overline{*_l})_l} = * = \dot{+}$.

**Example 2.4** (continued). If $* = \vee$, so $*_l = \top$, then $\overline{*_l} = \perp-$ (since $-(-a\top c) = a\perp -c$). Hence, by theorem 2.17, $(\overline{*_l})_l = \overline{*} = \wedge-$, $\overline{(\overline{*_l})_l} = * = \vee$.

**Example 2.5** (continued). If $* = \perp$, so $*_l = \wedge$, then $\overline{*_l} = \vee-$ (since $-(-\alpha\wedge c) = a\vee -c$). Hence, by theorem 2.17, $(\overline{*_l})_l = \overline{*} = \top-$, $\overline{(\overline{*_l})_l} = * = \perp$.

Starting with $\overline{*_l}$ instead of $*$, from example 2.4 and theorem 2.17 above we obtain

**Example 2.19.** If $* = \perp- = \overline{\vee_l}$, then $*_l = \wedge-$, $\overline{*_l} = \vee$. Hence, $(\overline{*_l})_l = \top$, $\overline{(\overline{*_l})_l} = \perp-$.

**Remark 2.20.** a) Example 2.19 differs from example 2.5 only by the minus signs, but we shall use it in Section 4. Similarly, starting with $* = \vee- = \overline{\perp_l}$, one obtains an example which differs from example 2.4 only by the minus sign (however, we shall not use it in the sequel).
b) The following table summarizes examples 2.3–2.5 and 2.19 above:

| | Example 2.3 | Example 2.4 | Example 2.5 | Example 2.19 |
|---|---|---|---|---|
| $*$ | $\dot{+}$ | $\vee$ | $\perp$ | $\perp-$ |
| $*_l$ | $\underset{\cdot}{+}-$ | $\top$ | $\wedge$ | $\wedge-$ |
| $\overline{*_l}$ | $\dot{+}$ | $\perp-$ | $\vee-$ | $\vee$ |
| $(\overline{*_l})_l = \overline{*}$ | $\underset{\cdot}{+}-$ | $\wedge-$ | $\top-$ | $\top$ |

Each binary operation $*$ on $\overline{R}$ can be extended to $\overline{R}^X$, where $X$ is any set, as follows.

**Definition 2.21.** For any $f, h \in \overline{R}^X$, let

$$(f * h)(x) = f(x) * h(x) \qquad (x \in X). \tag{2.30}$$

## 3. $*$-dualities and their duals

**Definition 3.1.** Let $X$ and $W$ be two sets and let $*$ be a binary operation on $\overline{R}$. A duality $\Delta : \overline{R}^X \to \overline{R}^W$ (see (1.1)) will be called a *$*$-duality*, if

$$(f * d)^\Delta = f^\Delta \overline{*} d \qquad (f \in \overline{R}^X, d \in \overline{R}), \tag{3.1}$$

where we identify each $d \in \overline{R}$ with the constant function $h_d \in \overline{R}^X$ defined by $h_d(x) = d \quad (x \in X)$.

**Example 3.2.** Let $* = \dot{+}$. Then $\overline{*} = \dot{+}-$ (see example 2.3), and thus condition (3.1) means that we have (1.3), i.e., that $\Delta : \overline{R}^X \to \overline{R}^W$ is a conjugation.

**Example 3.3.** Let $* = \vee$. Then $\overline{*} = \wedge-$ (see example 2.4), and thus condition (3.1) means that we have (1.6), i.e., that $\Delta : \overline{R}^X \to \overline{R}^W$ is a $\vee$-duality.

**Example 3.4.** Let $* = \perp$. Then $\overline{*} = \top-$ (see example 2.5), and thus condition (3.1) means that we have (1.7), i.e., that $\Delta : \overline{R}^X \to \overline{R}^W$ is a $\perp$-duality.

**Remark 3.5.** If $\Delta$ is a $*$-duality, then it is also a $(*-)$-duality. Indeed, by (2.11), (3.1) and (2.21), we have

$$(f * -d)^\Delta = (f * (-d))^\Delta = f^\Delta \overline{*} - d = f^\Delta \overline{*-} d \qquad (f \in \overline{R}^X, d \in \overline{R}).$$

**Proposition 3.6.** *Let $X$ and $W$ be two sets and let $*$ be a commutative binary operation on $\overline{R}$, satisfying condition $(\alpha)$. Then, a duality $\Delta : \overline{R}^X \to \overline{R}^W$ is a $*$-duality if (and only if) we have* (3.1) *for all $d \in R$.*

**Proof.** Assume that $\Delta : \overline{R}^X \to \overline{R}^W$ is a duality, satisfying (3.1) for all $d \in R$. Then, by the commutativity of $*$, (2.2) and (1.1) for $I = \emptyset$ (with the conventions (1.2)), we have

$$(f * +\infty)^\Delta = (+\infty * f)^\Delta = +\infty^\Delta = -\infty \qquad (f \in \overline{R}^X). \tag{3.2}$$

On the other hand, since $\overline{*}$ satisfies condition $(\beta)$ (by proposition 2.15 a)) we have, by (2.25) and (2.14) (applied to $\overline{*}$),

$$f^\Delta \overline{*} + \infty = -\infty \overline{*} - f^\Delta = -\infty \qquad (f \in \overline{R}^X). \tag{3.3}$$

Thus, by (3.2) and (3.3), we have (3.1) for $d = +\infty$.
Finally, by the commutativity of $*$, (2.1), (1.1), (3.1) for all $d \in R$, (2.25) and (2.13) for $\overline{*}$ (by proposition 2.15 a)), we obtain

$$\begin{aligned}(f * -\infty)^\Delta &= (-\infty * f)^\Delta = ((\inf_{d\in R} d) * f)^\Delta = (\inf_{d\in R}(d * f))^\Delta = \\ &= (\inf_{d\in R}(f * d))^\Delta = \sup_{d\in R}(f * d)^\Delta = \sup_{d\in R}(f^\Delta \overline{*} d) = \sup_{d\in R}(-d \overline{*} - f^\Delta) = \\ &= (\sup_{d\in R}(-d))\overline{*} - f^\Delta = +\infty \overline{*} - f^\Delta = f^\Delta \overline{*} - \infty \qquad (f \in \overline{R}^X),\end{aligned}$$

so $\Delta$ satisfies (3.1) also for $d = -\infty$. $\square$

**Theorem 3.7.** *Let $X$ and $W$ be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying condition $(\alpha)$.*
(a) *If $\Delta\colon \overline{R}^X \to \overline{R}^W$ is a $*$-duality, then its dual $\Delta'\colon \overline{R}^W \to \overline{R}^X$ is a $\overline{*_l}$-duality.*
(b) *If $\Delta\colon \overline{R}^X \to \overline{R}^W$ is a $\overline{*_l}$-duality, then its dual $\Delta'\colon \overline{R}^W \to \overline{R}^X$ is a $*$-duality.*

**Proof.** a) If $\Delta$ is a $*$-duality, then so is $\Delta'$ (see Section 1), and, by (1.10), (2.23), (3.1), (1.11) and (2.5), we have

$$\begin{aligned}(g\overline{*_l}d)^{\Delta'} &= \inf_{h^\Delta \le g\overline{*_l}d} h = \inf_{h^\Delta \le -(-g*_l d)} h = \inf_{-h^\Delta \ge -g*_l d} h = \\ &= \inf_{g \ge h^\Delta \overline{*} d} h = \inf_{g \ge (h*d)^\Delta} h = \inf_{g^{\Delta'} \le h*d} h = \\ &= \inf_{g^{\Delta'} *_l d \le h} h = g^{\Delta'} *_l d = g^{\Delta'}\overline{\overline{*_l}}d \qquad (g \in \overline{R}^W, d \in \overline{R}).\end{aligned}$$

b) If $\Delta$ is a $\overline{*_l}$-duality, then, by part a) (applied to $\overline{*_l}$ instead of $*$) and (2.29), $\Delta'$ is a $*$-duality. □

**Corollary 3.8.** *Let $X$ and $W$ be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying condition $(\alpha)$. Then*
(a) *Every $*$-duality is the dual of a $\overline{*_l}$-duality.*
(b) *Every $\overline{*_l}$-duality is the dual of a $*$-duality.*

**Proof.** a) If $\Delta$ is a $*$-duality, then $\Delta = (\Delta')'$ (see Section 1), where $\Delta'$ is a $\overline{*_l}$-duality (by theorem 3.7 a)).
b) if $\Delta$ is a $\overline{*_l}$-duality, then $\Delta = (\Delta')'$, where $\Delta'$ is a $*$-duality (by theorem 3.7 b)). □

**Corollary 3.9.** *Let $X$, $W$ and $*$ be as above. Then*
(a) *A duality $\Delta\colon \overline{R}^X \to \overline{R}^W$ is a $*$-duality if and only if $\Delta'$ is a $\overline{*_l}$-duality.*
(b) *A duality $\Delta\colon \overline{R}^X \to \overline{R}^W$ is a $\overline{*_l}$-duality if and only if $\Delta'$ is a $*$-duality.*

**Remark 3.10.** a) For $* = \dot{+}$, $\vee$, or $\perp$, theorem 3.7 yields again that the dual of a conjugation, or $\vee$-duality, or $\perp$-duality, is a conjugation, or a $\perp$-duality, or a $\vee$-duality, respectively (see Section 1). Similar remarks can be made for corollaries 3.8 and 3.9.
b) One can generalize definition 3.1 as follows. Let $K$ be a family of non-decreasing upper semi-continuous functions $k\colon \overline{R} \to \overline{R}$. A duality $\Delta\colon \overline{R}^X \to \overline{R}^W$ is a *$K$-duality*, if

$$(k \circ f)^\Delta = -k \circ (-f^\Delta) \qquad (k \in K, f \in \overline{R}^X). \tag{3.4}$$

Then, in particular, $\Delta$ *is a $*$-duality if and only if it is a $K_0$-duality, where*

$$K_0 = \{k_c \mid c \in \overline{R}\}, \tag{3.5}$$

with $k_c$ of (2.3). Indeed, by (2.3) and (2.30), we have

$$(k_c \circ f)(x) = k_c(f(x)) = f(x) * c = (f * c)(x) \qquad (f \in \overline{R}^X, c \in \overline{R}, x \in X),$$

whence

$$(k_c \circ f)^{\Delta} = (f * c)^{\Delta} \qquad\qquad (f \in \overline{R}^X, c \in \overline{R}), \tag{3.6}$$

and, on the other hand, by (2.3), (2.17) and (2.30), we have

$$\begin{aligned}(-k_c \circ (-f^{\Delta}))(w) = -k_c(-f^{\Delta}(w)) = -(-f^{\Delta}(w) * c) = f^{\Delta}(w) \overline{*} c = \\ = (f^{\Delta} \overline{*} c)(w) \quad (f \in \overline{R}^X, c \in R, w \in W);\end{aligned} \tag{3.7}$$

thus, (3.4) (for $K_0$ of (3.5)) is equivalent to (3.1). □

## 4. Representations of $*$-dualities and their duals, with the aid of coupling functions

**Definition 4.1.** Let $*$ be a binary operation on $\overline{R}$. An element $e \in \overline{R}$ is called
(a) a *left neutral element for* $*$, if

$$e * c = c \qquad\qquad (c \in \overline{R}); \tag{4.1}$$

(b) a *right neutral element for* $*$, if

$$c * e = c \qquad\qquad (c \in \overline{R}); \tag{4.2}$$

(c) a *neutral element for* $*$, if it is both a left and a right neutral element for $*$.
Note that a neutral element is necessarily unique.

**Example 4.2.** Let $* = \dot{+}$. Then $e = 0$ is the neutral element for $*$.

**Example 4.3.** Let $* = \vee$. Then $e = -\infty$ is the neutral element for $*$.

**Example 4.4.** Let $* = \perp$. Then $e = +\infty$ is the (unique) right neutral element for $*$ (by [7], formula (1.15)), but there exists no left neutral element for $*$.

**Definition 4.5.** Let $X$ be a set and let $*$ be a binary operation on $\overline{R}$, which admits a left (or right) neutral element $e$. Then, for any subset $S$ of $X$, the *generalized indicator function of $S$ (with respect to $e$)* is the function $\chi_S: X \to \{e, +\infty\}$ defined by

$$\chi_s(y) = \begin{cases} e & \text{if } y \in S \\ +\infty & \text{if } y \in X \setminus S. \end{cases} \tag{4.3}$$

**Example 4.2** (continued). If $* = \dot{+}$, so $e = 0$, then $\chi_S$ is the usual indicator function of $S$.

**Example 4.3** (continued). If $* = \vee$, so $e = -\infty$, then $\chi_S$ is the "representation function" of $S$, introduced by Flachs and Pollatschek [5].

**Lemma 4.6.** *Let $X$ be a set and let $*$ be a binary operation on $\overline{R}$, satisfying* (2.2) *and admitting a left neutral element $e$. Then, for any function $f \in \overline{R}^X$ we have*

$$f = \inf_{x \in X} \{\chi_{\{x\}} * f(x)\}, \tag{4.4}$$

*where $\chi_{\{x\}}$ is the generalized indicator function of the singleton $\{x\}$.*

**Proof.** By (4.3), (4.1) and (2.2) we have, for any $x, y \in X$,

$$\chi_{\{x\}}(y) * f(x) = \begin{cases} e * f(x) = f(x) = f(y) & \text{if } x = y \\ +\infty * f(x) = +\infty & \text{if } x \neq y, \end{cases}$$

whence

$$\inf_{x \in X} \{\chi_{\{x\}}(y) * f(x)\} = \inf\{f(y), +\infty\} = f(y) \qquad (y \in X).$$

□

We recall (see [11]) that if $X$ and $W$ are two sets, then every function $\psi: X \times W \to \overline{R}$ is called a *coupling function.*

**Theorem 4.7.** *Let $X$ and $W$ be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying* (2.2) *and admitting a left neutral element $e$. Then for each $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$ there exists a coupling function $\psi: X \times W \to \overline{R}$, for example,*

$$\psi(x, w) = (\chi_{\{x\}})^{\Delta}(w) \qquad (x \in X, w \in W), \tag{4.5}$$

*such that we have*

$$f^{\Delta}(w) = \sup_{x \in X} \{\psi(x, w) \overline{*} f(x)\} \qquad (f \in \overline{R}^X, w \in W). \tag{4.6}$$

*Moreover, if $*$ is also commutative, then $\psi$ of* (4.5) *is unique (i.e., the unique coupling function for which we have* (4.6)*).*

**Proof.** By lemma 4.6 and definition 3.1, for any $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$ we have

$$f^{\Delta} = (\inf_{x \in X} \{\chi_{\{x\}} * f(x)\})^{\Delta} = \sup_{x \in X} \{(\chi_{\{x\}})^{\Delta} \overline{*} f(x)\} \qquad (f \in \overline{R}^X),$$

i.e., (4.6), with $\psi$ of (4.5).
Moreover, if $e$ is a neutral element for $*$ and $x \in X$, then, applying (4.6) to $f = \chi_{\{x\}}$ and using (2.17), (4.3), (4.2) and (2.2), we obtain

$$\begin{aligned}(\chi_{\{x\}})^{\Delta}(w) &= \sup_{x' \in X} \{\psi(x', w) \overline{*} \chi_{\{x\}}(x')\} = \sup_{x' \in X} \{-\psi(x', w) * \chi_{\{x\}}(x'))\} = \\ &= -(-\psi(x, w)) = \psi(x, w) \qquad (w \in W).\end{aligned}$$

□

**Remark 4.8.** a) If $*$ is a binary operation on $\overline{R}$, satisfying (2.2) and admitting a left neutral element $e$, then, by (4.3), (4.1) and (2.2), for any $x, y \in X$ and any $d \in \overline{R}$ we have

$$\chi_{\{x\}}(y) * d = \begin{cases} e * d = d & \text{if } x = y \\ +\infty * d = +\infty & \text{if } x \neq y. \end{cases} \tag{4.7}$$

Now, by part of [6], theorem 4.7, for any duality $\Delta: \overline{R}^X \to \overline{R}^W$ we have

$$f^{\Delta}(w) = \sup_{x \in X} G_{\Delta}(x, w, f(x)) \qquad (f \in \overline{R}^X, w \in W), \tag{4.8}$$

where $G_{\Delta}: X \times W \times \overline{R} \to \overline{R}$ is the function defined by

$$G_{\Delta}(x, w, d) = (\varphi_{x,d})^{\Delta}(w) \qquad (x \in X, w \in W, d \in \overline{R}), \tag{4.9}$$

with $\varphi_{x,d}: X \to \overline{R}$ defined by

$$\varphi_{x,d}(y) = \begin{cases} d & \text{if } x = y \\ +\infty & \text{if } x \neq y. \end{cases} \tag{4.10}$$

But, by (4.10) and (4.7), we have

$$\varphi_{x,d} = \chi_{\{x\}} * d \qquad (x \in X, d \in \overline{R}), \tag{4.11}$$

and hence, if $\Delta$ is a $*$-duality, then, by (4.9), (4.11) and (3.1),

$$G_{\Delta}(x, w, d) = (\chi_{\{x\}} * d)^{\Delta}(w) = (\chi_{\{x\}})^{\Delta}(w) \overline{*} d \qquad (x \in X, w \in W, d \in \overline{R}), \tag{4.12}$$

which, together with (4.8), yields again (4.6), with $\psi$ of (4.5).
b) One can also prove that if $*$ and $\Delta$ are as in a) above, then $\psi: X \times W \to \overline{R}$ of (4.5) is the unique coupling function satiisfying, for any index set $I$,

$$-\psi(x, w) * \inf_{i \in I} d_i = \inf_{i \in I} \{-\psi(x, w) * d_i\} \qquad (x \in X, w \in W, \{d_i\}_{i \in I} \subseteq \overline{R}). \tag{4.13}$$

c) One cannot omit the assumption of commutativity of $*$ in the uniqueness part of theorem 4.7, as shown by the following example: Let $X$ be a non-empty set, let $W = X$ and let $*$ be the binary operation on $\overline{R}$ defined by

$$a * b = \begin{cases} b & \text{if } a < +\infty \\ +\infty & \text{if } a = +\infty. \end{cases}$$

Then $*$ satisfies (2.2) (moreover, it is easy to see that $*$ satisfies even condition ($\alpha$)) and each $e \in R \cup \{-\infty\}$ is a left neutral element for $*$. Furthermore, the mapping $\Delta: \overline{R}^X \to \overline{R}^W$ defined by

$$f^{\Delta} = -f \qquad (f \in \overline{R}^X)$$

is obviously a $*$-duality (even for any binary operation $*$ on $\overline{R}$) and, for any coupling function $\psi: X \times W \to \overline{R}$ such that

$$\psi(x, w) = -\infty \quad \text{if and only if} \quad x \neq w,$$

we have

$$f^{\Delta}(w) = -f(w) = -\inf_{x \in X} \{-\psi(w, x) * f(x)\} = \sup_{x \in X} \{\psi(x, w) \overline{*} f(x)\} \quad (f \in \overline{R}^X, w \in W).$$

In the converse direction to theorem 4.7, we have

**Theorem 4.9.** *Let $X$ and $W$ be two sets, $*$ a binary operation on $\overline{R}$, $\psi: X \times W \to \overline{R}$ a coupling function and $\Delta: \overline{R}^X \to \overline{R}^W$ the mapping defined by* (4.6).
a) *If $*$ is commutative and satisfies condition* $(\alpha)$, *then $\Delta$ is a duality.*
b) *If $*$ is associative, then $\Delta$ satisfies* (3.1).
*Hence, if $*$ is commutative, associative and satisfies condition $(\alpha)$, then $\Delta$ is a $*$-duality.*

**Proof.** a) For any $\{f_i\}_{i\in I} \subseteq \overline{R}^X$ and $w \in W$ we have, by (4.6), (2.17), the commutativity of $*$, and (2.1),

$$\begin{aligned}
(\inf_{i\in I} f_i)^{\Delta}(w) &= \sup_{x\in X}\{\psi(x,w)\overline{*}\inf_{i\in I} f_i(x)\} = \sup_{x\in X}\{-\{-\psi(x,w) * \inf_{i\in I} f_i(x)\}\} = \\
&= \sup_{x\in X}\{-\{\inf_{i\in I} f_i(x) * -\psi(x,w)\}\} = -\inf_{x\in X}\{\inf_{i\in I} f_i(x) * -\psi(x,w)\} = \\
&= -\inf_{x\in X}\{\inf_{i\in I}\{f_i(x) * -\psi(x,w)\}\} = -\inf_{i\in I}\{\inf_{x\in X}\{f_i(x) * -\psi(x,w)\}\} = \\
&= \sup_{i\in I}\{-\inf_{x\in X}\{f_i(x) * -\psi(x,w)\}\} = \sup_{i\in I}\sup_{x\in X}\{-(f_i(x) * -\psi(x,w))\} = \\
&= \sup_{i\in I}\sup_{x\in X}\{-(-\psi(x,w) * f_i(x))\} = \sup_{i\in I}\{\psi(x,w)\overline{*}f_i(x)\} = \sup_{i\in I} f_i^{\Delta}(w).
\end{aligned}$$

b) For any $f \in \overline{R}^X$, $d \in \overline{R}$ and $w \in W$ we have, by (4.6), (2.30), (2.17) and the associativity of $*$,

$$\begin{aligned}
(f * d)^{\Delta}(w) &= \sup_{x\in X}\{\psi(x,w)\overline{*}(f(x) * d)\} = \sup_{x\in X}\{-\{-\psi(x,w) * (f(x) * d)\}\} = \\
&= \sup_{x\in X}\{-((-\psi(x,w) * f(x)) * d)\} = \sup_{x\in X}\{(\psi(x,w)\overline{*}f(x))\overline{*}d\} = \\
&= f^{\Delta}(w)\overline{*}d = (f^{\Delta}\overline{*}d)(w).
\end{aligned}$$

$\square$

**Remark 4.10.** One can also prove that $\Delta$ of (4.6) is a duality whenever $\psi$ satisfies (4.13) (even if $*$ is not commutative or does not satisfy condition $(\alpha)$).

**Proposition 4.11.** *Under the assumptions of* theorem 4.9 a), *we have*

$$f^{\Delta}(w) = \min_{\substack{d\in\overline{R} \\ -d *_l -\psi(\cdot,w)\le f}} d \qquad (f \in \overline{R}^X, w \in W). \tag{4.14}$$

**Proof.** By (4.6) and (2.24), for any $f \in \overline{R}^X$ and $w \in W$ we have

$$f^{\Delta}(w) = \min_{\substack{d\in\overline{R} \\ f^{\Delta}(w)\le d}} d = \min_{\substack{d\in\overline{R} \\ \psi(\cdot,w)\overline{*}f\le d}} d = \min_{\substack{d\in\overline{R} \\ -d *_l -\psi(\cdot,w)\le f}} d.$$

$\square$

**Remark 4.12.** For $* = \dot{+}$ and, respectively, $* = \vee$, proposition 4.11 yields [14], proposition 3.1 and, respectively, [7], corollary 2.2.

**Definition 4.13.** We shall say that a binary operation $*$ on $\overline{R}$ satisfies *condition* $(r)$, if $*$ is commutative, associative and admits a neutral element $e$.

From theorems 4.7 and 4.9, we obtain

**Theorem 4.14.** *Let $X$ and $W$ be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying conditions* $(\alpha)$ *and* $(r)$. *For a mapping $\Delta: \overline{R}^X \to \overline{R}^W$, the following statements are equivalent:*
$1°$. *$\Delta$ is a $*$-duality.*
$2°$. *There exists a coupling function $\psi: X \times W \to \overline{R}$, such that we have* (4.6).
*Moreover, in this case $\psi$ of $2°$ is unique, namely, it is the function* (4.5).

**Remark 4.15.** a) One can prove that the equivalence $1° \Leftrightarrow 2°$ also holds for an associative binary operation $*$ on $\overline{R}$ satisfying (2.2) and having a neutral element (instead of satisfying conditions $(\alpha)$ and $(r)$). Under these assumptions, if $\Delta$ is a $*$-duality, then $\psi$ of (4.5) is the unique coupling function satisfying (4.13) and such that we have (4.6).
b) By theorem 4.14 and a) above, for $*$ satisfying conditions $(\alpha)$ and $(r)$ (or alternatively, being associative, satisfying (2.2) and having a neutral element), we have a one-to-one correspondence between $*$-dualities $\Delta: \overline{R}^X \to \overline{R}^W$ and coupling functions $\psi: X \times W \to \overline{R}$. We shall call $\Delta = \Delta(*, \psi)$ of (4.6) (respectively, $\psi = \psi_{\Delta,*}$ of (4.5)) *the $*$-duality associated to the coupling function $\psi$* (respectively, *the coupling function associated to the $*$-duality $\Delta$).*
c) In particular, for $* = \dot{+}$ and, respectively, $* = \vee$ (which satisfy conditions $(\alpha)$ and $(r)$), from theorem 4.14 we obtain again the results of [14] and [7] on the relations between conjugations, respectively, $\vee$-dualities, and coupling functions ([14], example 2.1 and theorem 3.1 and, respectively, [7], example 2.1 and theorem 2.1).
d) By (4.7), one can replace (4.4) of lemma 4.6 by

$$f = \inf_{(x,d) \in \mathrm{Epi} f} \{\chi_{\{x\}} * d\}, \tag{4.15}$$

where $\mathrm{Epi} f = \{(x, d) \in X \times R \mid f(x) \leq d\}$, the epigraph of $f$. Then, by the above arguments, using (4.15) and (1.1) with $I = \mathrm{Epi} f$ (which is $\emptyset$ for $f = +\infty$), we obtain, for any $*$-duality $\Delta$ and any $f \in \overline{R}^X$,

$$f^\Delta = \sup_{(x,d) \in \mathrm{Epi} f} \{(\chi_{\{x\}})^\Delta \overline{*} d\} = \sup_{(x,d) \in \mathrm{Epi} f} \{\psi_\Delta(x, w) \overline{*} d\}. \tag{4.16}$$

Let us consider now the dual mappings $\Delta'$ (defined by (1.10)).

**Theorem 4.16.** *Let $X$ and $W$ be two sets, $*$ a commutative binary operation on $\overline{R}$, satisfying condition* $(\alpha)$, *$\psi: X \times W \to \overline{R}$ a coupling function, and $\Delta: \overline{R}^X \to \overline{R}^W$ the mapping defined by* (4.6). *Then*

$$g^{\Delta'} = \sup_{w \in W} \{-g(w) *_l -\psi(x, w)\} \qquad (g \in \overline{R}^W, x \in X). \tag{4.17}$$

**Proof.** By (1.10), (4.6) and (2.24), we have

$$g^{\Delta'} = \inf_{h^{\Delta} \leq g} h = \inf_{\psi(\cdot,\cdot)\overline{*}h \leq g} h = \inf_{-g*_l-\psi(\cdot,\cdot)\leq h} h \qquad (g \in \overline{R}^W), \tag{4.18}$$

whence

$$g^{\Delta'} \geq \sup_{w\in W} \{-g *_l -\psi(\cdot, w)\} \qquad (g \in \overline{R}^W). \tag{4.19}$$

On the other hand, for any $g \in \overline{R}^W$, the function $h_g$ defined by

$$h_g(x) = \sup_{w\in W} \{-g(x) *_l -\psi(x, w)\} \qquad (x \in X),$$

belongs to the set $\{h \in \overline{R}^X \mid -g *_l -\psi(\cdot, \cdot) \leq h\}$, whence, by (4.18), we obtain $g^{\Delta'} \leq h_g$, which, together with (4.19), yields (4.17). □

Under the assumptions of theorem 4.16, $\Delta$ of (4.6) is a duality (by theorem 4.9), and hence so is $\Delta'$ of (4.17); however, we do not know whether $\Delta$ is a $*$-duality. In the next result we obtain the same conclusion (4.17), with different assumptions on $*$ and $\Delta$.

**Theorem 4.17.** *Let $X$ and $W$ be two sets, $*$ a binary operation on $\overline{R}$ satisfying condition $(\alpha)$ and admitting a left neutral element $e$, $\Delta: \overline{R}^X \to \overline{R}^W$ a $*$-duality and $\psi: X \times W \to \overline{R}$ the coupling function* (4.5). *Then we have* (4.17) *(and, by* theorem 4.7, *we have also* (4.6)*).*

**Proof.** By part of [6], theorem 3.5, for any duality $\Delta: \overline{R}^X \to \overline{R}^W$ we have

$$g^{\Delta'}(x) = \sup_{w\in W} G_{\Delta'}(w, x, g(w)) \qquad (g \in \overline{R}^W, x \in X), \tag{4.20}$$

where

$$G_{\Delta'}(w, x, b) = \min_{\substack{a\in\overline{R} \\ G_{\Delta}(x,w,a)\leq b}} a \qquad (w \in W, x \in X, b \in \overline{R}), \tag{4.21}$$

with $G_{\Delta}$ of (4.9), (4.10). But, since now $*$ satisfies (2.2) and admits a left neutral element $e$, and since $\Delta$ is a $*$-duality, we have (4.12) (see remark 4.8). Thus, by (4.21), (4.12), (2.24) and (4.5), we obtain

$$G_{\Delta'}(w, x, b) = -b *_l -\psi(x, w) \qquad (w \in W, x \in X, b \in \overline{R}), \tag{4.22}$$

which, together with (4.20), yields (4.17). □

**Theorem 4.18.** *Let $X$ and $W$ be two sets, $*$ a commutative binary operation on $\overline{R}$, and $\Delta: \overline{R}^X \to \overline{R}^W$ a duality for which there exist a unique coupling function $\psi_{\Delta,*}: X \times W \to \overline{R}$ such that*

$$f^{\Delta}(w) = \sup_{x\in X}\{\psi_{\Delta,*}(x,w)\overline{*}f(x)\} \qquad (f\in \overline{R}^X, w\in W), \tag{4.23}$$

*and a coupling function* $\psi: X\times W\to \overline{R}$ *such that* $\Delta': \overline{R}^W \to \overline{R}^X$ *satifies* (4.17)*. Then* $\psi$ *of* (4.17) *is unique, namely, we have*

$$\psi = \psi_{\Delta,*}. \tag{4.24}$$

**Proof.** By $\Delta = (\Delta')'$, (1.10) (applied to $\Delta'$ instead of $\Delta$), (4.17) and (2.24), we have

$$f^{\Delta}(w) = \inf_{\substack{g\in \overline{R}^W \\ g^{\Delta'}\le f}} g(w) = \inf_{\substack{g\in \overline{R}^W \\ \sup_{x\in X}\{\psi(x,\cdot)\overline{*}f(x)\}\le g}} g(w) =$$
$$= \sup_{x\in X}\{\psi(x,w)\overline{*}f(x)\} \qquad (f\in \overline{R}^X, x\in W),$$

which by our assumption of uniqueness of $\psi_{\Delta,*}$ in (4.23), implies (4.24). □

From theorems 4.7, 4.16 and 4.18 we obtain

**Theorem 4.19.** *Let $X$ and $W$ be two sets and let $*$ be a commutative binary operation on $\overline{R}$, satisfying condition $(\alpha)$ and admitting a neutral element $e$. Then, for each $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$ there exists a unique coupling function $\psi: X\times W\to \overline{R}$ such that we have*

$$g^{\Delta'}(x) = \sup_{w\in W}\{-g(w) *_l -\psi(x,w)\} \qquad (g\in \overline{R}^W, x\in X),$$

*namely*

$$\psi(x,w) = (\chi_{\{x\}})^{\Delta}(w) \qquad (x\in X, w\in W).$$

**Remark 4.20.** a) By theorem 4.7, the above $\psi$ coincides with the unique coupling function for which we have

$$f^{\Delta}(w) = \sup_{x\in X}\{\psi(x,w)\overline{*}f(x)\} \qquad (f\in \overline{R}^X, w\in W).$$

b) In particular, for $* = \dot{+}$ and $* = \vee$, from theorem 4.19 we obtain again the results of [14] and [7] on the representation of conjugations, $\vee$-dualities and their duals, with the aid of coupling functions.

Let us consider now, for a $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$, the "second dual" (called also the $\Delta'\Delta$-hull) $f^{\Delta\Delta'} = (f^{\Delta})^{\Delta'} \in \overline{R}^X$ of a function $f\in \overline{R}^X$.

**Theorem 4.21.** *Under the assumptions of* theorem 4.19, *for any $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$ we have*

$$f^{\Delta\Delta'}(x) = \sup_{w \in W} \{-f^{\Delta}(w) *_l -\psi(x,w)\} = \tag{4.25}$$
$$= \sup_{w \in W} \min_{\substack{b \in \overline{R} \\ \psi(x,w)\overline{*}b \leq f^{\Delta}(w)}} b \qquad (f \in \overline{R}^X, x \in X),$$

*with* $\psi: X \times W \to \overline{R}$ *of* (4.5).

**Proof.** The first equality follows from (4.17) applied to $g = f^{\Delta}$. Furthermore, by (2.6), the commutativity of $*$ and (2.17), for any $f \in \overline{R}^X$, $x \in X$ and $w \in W$ we have

$$-f^{\Delta}(w) *_l -\psi(x,w) = \min\{b \in \overline{R} \mid -f^{\Delta}(w) \leq b * -\psi(x,w) = -\psi(x,w) * b\} =$$
$$= \min\{b \in \overline{R} \mid f^{\Delta}(w) \geq -(-\psi(x,w) * b) = \psi(x,w)\overline{*}b\},$$

which yields the second equality in (4.25). □

**Theorem 4.22.** *Under the assumptions of* theorem 4.17, *for any* $*$*-duality* $\Delta: \overline{R}^X \to \overline{R}^W$ *we have*

$$f^{\Delta\Delta'}(x) = \sup_{\substack{w \in W, b \in \overline{R} \\ b *_l -\psi(\cdot,w) \leq f}} \{b *_l -\psi(x,w)\} \qquad (f \in \overline{R}^X, x \in X), \tag{4.26}$$

*with* $\psi: X \times W \to \overline{R}$ *of* (4.5).

**Proof.** By [6], theorem 3.6, for any duality $\Delta: \overline{R}^X \to \overline{R}^W$ we have

$$f^{\Delta\Delta'}(x) = \sup_{\substack{w \in W, b \in \overline{R} \\ G_{\Delta'}(w,\cdot,b) \leq f}} G_{\Delta'}(w,x,b) \qquad (f \in \overline{R}^X, x \in X), \tag{4.27}$$

with $G_{\Delta'}$ of (4.21), where $G_{\Delta}$ is that of (4.9), (4.10). But, by the above proof of theorem 4.17, we have now (4.22), which, together with (4.27), yields (4.26). □

**Remark 4.23.** a) Theorem 4.22 shows that, under the assumptions of theorem 4.19, for any $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$ the $\Delta'\Delta$-hull of $f$ coincides with the "$\Phi$-convex hull" of $f$, in the sense of [3], where

$$\Phi = \{b *_l -\psi(\cdot,w) \mid w \in W, b \in \overline{R}\}, \tag{4.28}$$

or, in other words, that for any $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$, the "elementary functions", in a sense similar to that of [11], are the functions $\gamma_{w,b} = b *_l -\psi(\cdot w) \in \overline{R}^X$ $(w \in W, b \in \overline{R})$.
b) In particular, for $* = \dot{+}$ and $* = \vee$, from theorems 4.21 and 4.22 we obtain again the main results of [14] and [7] on the representation of second conjugates and second $\vee$-duals of $f$, with the aid of coupling functions.

Let us observe now that *the above results can be "dualized" as follows:* Let $X$ and $W$ be two sets, $*$ a binary operation on $\overline{R}$, satisfying condition $(\alpha)$, and $\Delta: \overline{R}^X \to \overline{R}^W$ a $*$-duality. Then $\overline{*_l}$ satisfies condition $(\alpha)$ and $\Delta': \overline{R}^W \to \overline{R}^X$ is a $\overline{*_l}$-duality (by theorems 2.17 and 3.7 a)). Hence, replacing the assumptions of the above results by the same assumptions on $\overline{*_l}$ and using (2.20), (2.28), we obtain representations of $\Delta'$ and $\Delta = (\Delta')'$ with the aid of the coupling function $\psi': W \times X \to \overline{R}$ defined by

$$\psi'(w,x) = (\chi_{\{w\}})^{\Delta'}(x) \qquad (w \in W, x \in X), \tag{4.29}$$

or, equivalently, with the aid of the coupling function $\psi: X \times W \to \overline{R}$ defined by

$$\psi(x,w) = \psi'(w,x) = (\chi_{\{w\}})^{\Delta'}(x) \qquad (x \in X, w \in W). \tag{4.30}$$

For example, dualizing in this way theorem 4.7, we arrive at

**Theorem 4.24.** *Let $X$ and $W$ be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying condition $(\alpha)$ and such that $\overline{*_l}$ admits a left neutral element $e$, $\Delta: \overline{R}^X \to \overline{R}^W$ a $*$-duality, and $\psi: X \times W \to \overline{R}$ the coupling function* (4.30)*. Then we have*

$$g^{\Delta'}(x) = \sup_{w \in W} \{\psi(x,w) *_l g(w)\} \qquad (g \in \overline{R}^W, x \in X). \tag{4.31}$$

*Moreover, if $\overline{*_l}$ is also commutative, then $\psi$ of* (4.30) *is the only coupling function for which we have* (4.31).

Similarly, dualizing theorem 4.19 and using theorem 2.17, we arrive at

**Theorem 4.25.** *Let $X$ and $W$ be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying condition $(\alpha)$ and such that $\overline{*_l}$ is commutative and admits a neutral element $e$. Then for each $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$ there exists a unique coupling function $\psi: X \times W \to \overline{R}$, namely, $\psi$ of* (4.30)*, such that*

$$f^{\Delta}(w) = \sup_{x \in X} \{-f(x) \overline{*} - \psi(x,w)\} \qquad (f \in \overline{R}^X, w \in W). \tag{4.32}$$

*Moreover, the same $\psi$ is the unique coupling function for which we have* (4.31).

**Remark 4.26.** In particular, let $* = \perp-$. Then $*$ satisfies condition $(\alpha)$ (by remark 2.8d)) and $\overline{*_l} = \vee$ (see example 2.19), so $\overline{*_l}$ satisfies the assumptions of theorem 4.25. Also, $\overline{*} = \top$ and $*_l = \wedge-$ (see example 2.19). Hence, for $* = \perp-$, from theorem 4.25, combined with remark 3.5, we obtain again the results of [7] on the representation of $\perp$-dualities and their duals, with the aid of coupling functions. However, note that in [7] we have also obtained another expression for the coupling function $\psi$ of (4.30) (see [7], formula (3.9)), by exploiting the *special* properties of $\perp$ and $\top$, and this has also implied another expression for the coupling function $\psi$ occurring in theorem 4.7, i.e., for $\psi$ of (4.5) (see [7], formula (4.10)).

**Proposition 4.27.** *Under the assumptions of* theorem 4.25*, we have* (4.14).

**Proof.** By (4.32) and (2.23), for any $f \in \overline{R}^X$ and $w \in W$ we have

$$f^{\Delta}(w) = \min_{\substack{d \in \overline{R} \\ f^{\Delta}(w) \leq d}} d = \min_{\substack{d \in \overline{R} \\ -f\overline{*}-\psi(\cdot,w) \leq d}} d = \min_{\substack{d \in \overline{R} \\ -d*_l-\psi(\cdot,w) \leq f}} d.$$

□

**Remark 4.28.** For $* = \perp-$ we have $*_l = \wedge-$ (see example 2.19), so proposition 4.27, combined with remark 3.5, yields again [7], corollary 3.1.

Finally, let us consider the second duals $f^{\Delta\Delta'}$.

**Theorem 4.29.** *Under the assumptions of* theorem 4.25*, for any $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$ we have*

$$f^{\Delta\Delta'}(x) = \sup_{w \in W} \{\psi(x,w) *_l f^{\Delta}(w)\} = \sup_{w \in W} \min_{\substack{b \in \overline{R} \\ -b\overline{*}-\psi(x,w) \leq f^{\Delta}(w)}} b \qquad (f \in \overline{R}^X, x \in X), \tag{4.33}$$

*with $\psi: X \times W \to \overline{R}$ of* (4.30)*.*

**Proof.** The first equality follows from (4.31) applied to $g = f^{\Delta}$. Furthermore, by proposition 2.15 c) (applied to $\overline{*_l}$), $*_l = \overline{\overline{*_l}}$ is anti-commutative. Hence, by (2.6) and (2.19),

$$\begin{aligned} \psi(x,w) *_l f^{\Delta}(w) &= -f^{\Delta}(w) *_l -\psi(x,w) = \min\{b \in \overline{R} \mid -f^{\Delta}(w) \leq b * -\psi(x,w)\} = \\ &= \min\{b \in \overline{R} \mid -f^{\Delta}(w) \leq -(-b\overline{*} - \psi(x,w))\} = \min\{b \in \overline{R} \mid f^{\Delta}(w) \geq -b\overline{*} - \psi(x,w)\}, \end{aligned}$$

which yields the second equality in (4.33). □

**Theorem 4.30.** *Under the assumptions of* theorem 4.25*, for any $*$-duality $\Delta: \overline{R}^X \to \overline{R}^W$ we have*

$$f^{\Delta\Delta'}(x) = \sup_{\substack{w \in W, b \in \overline{R} \\ \psi(\cdot,w)*_l b \leq f}} \{\psi(x,w) *_l b\} \qquad (f \in \overline{R}^X, w \in W), \tag{4.34}$$

*with $\psi: X \times W \to \overline{R}$ of* (4.30)*.*

**Proof.** Since $\Delta'$ is a $\overline{*_l}$-duality (by theorem 3.7 a)), we have, by (4.12) (applied to $\Delta'$ and $\overline{*_l}$), (2.20) (for $*_l$) and (4.30),

$$G_{\Delta'}(w,x,b) = (\chi_{\{w\}})^{\Delta'}(x) *_l b = \psi(x,w) *_l b \qquad (w \in W, x \in X, b \in \overline{R}), \tag{4.35}$$

which, together with (4.27), yields (4.34). □

**Remark 4.31.** a) One can make an observation similar to remark 3.5, with $\Phi$ of (4.28) replaced by

$$\Phi = \{\psi(\cdot, w) *_l b \mid w \in W, b \in \overline{R}\}, \tag{4.36}$$

and with the "elementary functions" $\gamma_{w,b} = \psi(\cdot, w) *_l b \qquad (w \in W, b \in \overline{R})$.
b) In particular, for $* = \perp-$ we have $*_l = \wedge-$ (see example 2.19), and thus theorems 4.29 and 4.30, combined with remark 3.5, yield again the main results of [7] on the representation of second $\perp$-duals of $f$, with the aid of coupling functions.

## 5. Appendix: A unifying framework for the above results and those of [8]

Let us first recall some concepts from [8].
Let $A = (A, \leq, *)$ be a *complete totally ordered group*, i.e. (see e.g. [1], Ch. 14) a set endowed with a total order $\leq$ such that $(A, \leq)$ is a conditionally complete lattice (that is, every non-empty order-bounded subset of $A$ admits a supremum and an infimum in $A$) and with a binary operation $*$ for which $(A, *)$ is a group, such that all group translations are isotone; then, by a result of Iwasawa (see e.g. [1], Ch. 14, theorem 20), $*$ is commutative. In the paper [8], assuming that $A$ is not a singleton, we have adjoined to it a greatest element $+\infty$ and a least element $-\infty$, i.e., we have considered the set

$$\overline{A} = A \cup \{+\infty\} \cup \{-\infty\}, \tag{5.1}$$

with the order $\leq$ extended to $\overline{A}$ by

$$-\infty \leq a \leq +\infty \qquad (a \in \overline{A}), \tag{5.2}$$

and we have extended the binary operation $*$ on $A$ to two different binary operations $\dot{*}$ and $\underset{\cdot}{*}$ on $\overline{A}$ (called *upper* and *lower composition*, respectively), by the rules

$$a \dot{*} b = a \underset{\cdot}{*} b = a * b \qquad (a, b \in A), \tag{5.3}$$

$$+\infty \dot{*} a = a \dot{*} +\infty = +\infty \qquad (a \in \overline{A}), \tag{5.4}$$

$$-\infty \dot{*} a = a \dot{*} -\infty = -\infty \qquad (a \in A \cup \{-\infty\}), \tag{5.5}$$

$$+\infty \underset{\cdot}{*} a = a \underset{\cdot}{*} +\infty = +\infty \qquad (a \in A \cup \{+\infty\}), \tag{5.6}$$

$$-\infty \underset{\cdot}{*} a = a \underset{\cdot}{*} -\infty = -\infty \qquad (a \in \overline{A}). \tag{5.7}$$

Then, $\overline{A} = (\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ has been called (in [8]) *the canonical enlargement of* $(A, \leq, *)$. Furthermore, a mapping $M \colon \overline{A}^X \to \overline{A}^W$ has been called ([8], definition 2.3) a $*$-duality, if for any index set $I$ we have

$$(\inf_{i \in I} f_i)^M = \sup_{i \in I} f_i^M \qquad (\{f_i\}_{i \in I} \subseteq \overline{A}^X), \tag{5.8}$$

$$(f\dot{*}a)^M = f^M \underset{\cdot}{*} a^{-1} \qquad (f \in \overline{A}^X, a \in \overline{A}), \tag{5.9}$$

where inf, $\dot{*}$ (in $\overline{A}^X$) and sup, $\underset{\cdot}{*}$ (in $\overline{A}^W$) are defined pointwise, each $a \in \overline{A}$ is identified with the constant function $f_a(x) = a$ ($x \in X$), and if $a \in A$, then $a^{-1}$ denotes the inverse of $a$ in the Abelian group $(A, *)$, while the "inverses" of $a \in \overline{A} \backslash A$ are defined by $(+\infty)^{-1} = -\infty$, $(-\infty)^{-1} = +\infty$. In particular, clearly for $A = R$, with the usual total order $\leq$ on $R$ and with $* = +$, the usual addition on $R$, $\dot{*}$ and $\underset{\cdot}{*}$ are nothing else than the upper and lower additions (1.4), (1.5) on $\overline{R}$ and the $*$-dualities are the conjugations (1.1), (1.3).
Now we can give the following unifying framework for the results of the present paper and those of [8].

**Definition 5.1.** Let $(\overline{A}, \leq)$ be a complete chain (i.e., a complete lattice, where $\leq$ is a total order on $\overline{A}$), and let $s\colon (\overline{A}, \leq) \to (\overline{A}, \leq)$ be a bijective duality (i.e., a bijective mapping $s\colon \overline{A} \to \overline{A}$ such that $s(\inf_{i \in I} a_i) = \sup_{i \in I} s(a_i)$ for every index set $I$ and every family $\{a_i\}_{i \in I} \subseteq \overline{A}$). Given a binary operation $*$ on $\overline{A}$, we define a new binary operation $*^s$ on $\overline{A}$, called *the $s$-conjugate of* $*$, by

$$a *^s c = s(s^{-1}(a) * c) \qquad (a, c \in \overline{A}). \tag{5.10}$$

**Remark 5.2.** a) If $\overline{A} = \overline{R}$, endowed with the usual total order $\leq$ and if $s\colon (\overline{R}, \leq) \to (\overline{R}, \leq)$ is the mapping defined by

$$s(a) = -a \qquad (a \in \overline{R}), \tag{5.11}$$

then $s$ is a bijective duality and, by (5.10) and (2.17), for any binary operation $*$ on $\overline{R}$ we have

$$a *^s c = -(-a * c) = a \bar{*} c \qquad (a, c \in \overline{R}). \tag{5.12}$$

b) If $\overline{A} = (\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ is the canonical enlargement of a complete totally ordered group $A = (A, \leq, *)$, and if $s\colon (\overline{A}, \leq) \to (\overline{A}, \leq)$ is the mapping defined by

$$s(a) = a^{-1} \qquad (a \in \overline{A}), \tag{5.13}$$

then $s$ is a bijective duality (by [8], lemma 1.1) and, by (5.10) with $*$ being now the binary operation $\dot{*}$ of $(\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ and [8], lemma 1.3, we have

$$a \dot{*}^s c = (a^{-1} \dot{*} c)^{-1} = a \underset{\cdot}{*} c^{-1} \qquad (a, c \in \overline{A}). \tag{5.14}$$

**Definition 5.3.** Let $(\overline{A}, \leq)$, $s$ and $*$ be as in definition 5.1. A mapping $\Delta\colon \overline{A}^X \to \overline{A}^W$ is called a $(*, s)$-*duality*, if it is a duality (in the sense (1.1), with $\overline{R}$ replaced by $\overline{A}$) and if

$$(f * a)^\Delta = f^\Delta *^s a \qquad (f \in \overline{A}^X, a \in \overline{A}), \tag{5.15}$$

where each $a \in \overline{A}$ is identified with the constant function $f_a(x) = a$ $(x \in X)$ and where $*$ (in $\overline{A}^X$) and $*^s$ (in $\overline{A}^W$) are defined pointwise on $\overline{A}$.

**Remark 5.4.** a) If $(\overline{A}, \leq)$, $s$ and $*$ are as in remark 5.2a), then, by (5.15) and (5.12), $\Delta\colon \overline{R}^X \to \overline{R}^W$ is a $(*, s)$-duality if and only if it is a $*$-duality in the sense of definition 3.1.

b) If $(\overline{A}, \leq)$, $s$ and $*$ are as in remark 5.2b), then, by (5.15) and (5.14), $\Delta\colon \overline{A}^X \to \overline{A}^W$ is a $(*, s)$-duality if and only if it is a $*$-duality in the sense of [8] (i.e., in the sense of (5.8), (5.9) above, with $M = \Delta$).

By remarks 5.2a) and 5.4a) and by our assumptions on $(\overline{A}, \leq)$ and $s$, *some of the results of the present paper can be extended to results on* $(*, s)$*-dualities*, which, by remarks 5.2b) and 5.4b), encompass, as particular cases, also the results of [8] on $*$-dualities (in the sense of [8]); indeed, note that if $\overline{A} = (\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ is the canonical enlargement of a complete totally ordered group $A = (A, \leq, *)$, then, by [8], lemma 1.4, the binary operation $\dot{*}$ on $\overline{A}$ satisfies condition $(\alpha)$ (i.e., (2.1) with $*$ replaced by $\dot{*}$ and with inf taken in $\overline{A}$), so the "extended" proposition 2.6 and definition 2.7 (of $*_l$) can be applied to $(\overline{A}, \leq)$ and $*$ of remark 5.2b).

**Addendum.** Given a binary operation $*$ on $\overline{R}$ which satisfies condition $(\alpha)$, formula (2.4) of remark 2.2b) means that, for each $c \in \overline{R}$, the mapping $k_c\colon \overline{R} \to \overline{R}$ defined by (2.3) is a duality between $(\overline{R}, \leq)$ and $(\overline{R}, \geq)$. Note also that, by (1.10) and (2.7), for each $c \in \overline{R}$, the dual of this duality $k_c$ is the mapping $j_c$ defined by (2.9). Therefore, some relations between the operations $*$ and $*_l$ can be deduced from the theory of dualities (or, equivalently [21], of Galois connections) between complete lattices (see also Blyth and Janowitz [18], Baccelli, Cohen, Olsder and Quadrat [17]). For some earlier applications of lattice theory to generalized conjugation theory, see e.g. Dolecki [19], Volle [22], Penot and Volle [20].

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HIER :

**Leere Seite**
**210**