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Polishness of Weak Topologies Generated by Gap and Excess Functionals

Ľubica Holá
Mathematical Institute, Slovak Academy of Sciences,
Štefánikovà 49 Bratislava, Slovakia.

Roberto Lucchetti
Dipartimento di Matematica,
Via C. Saldini, 50, 20133 Milano, Italia.
e-mail: rel@vmimat.mat.unimi.it

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Let (X, d) be a separable complete metric space and $CL(X)$ the family of all nonempty closed subsets of X. We show that the finite Hausdorff topology on $CL(X)$ is Polish. The finite Hausdorff topology is measurably compatible on $CL(X)$ [1], i.e. its Borel field coincides with the Effros sigma algebra. So Polishness of this topology can be useful for measurable multifunctions with values in $CL(X)$ equipped with the finite Hausdorff topology. Polishness of other weak topologies generated by a family of gap and excess funtionals is also proved.

1. Introduction

Let (X, d) be a separable complete metric space and $CL(X)$ the hyperspace of X, i.e. the space of all nonempty closed subsets of X. We are interested to find a class of Polish topologies on $CL(X)$. In [2] it is shown that the Wijsman topology generated by the metric d is Polish and in [6] is proved that also the Wijsman topology generated by a metric ϱ topologically equivalent to d is Polish.

We are going to show that, if Δ is a subfamily of $CLB(X)$ containing the singletons and separable with respect to the induced Hausdorff metric , then the weak topology τ_Δ^G generated by the family of gap functionals determined by Δ as well as the weak topology τ_Δ^{GE} generated by the family of gap and excess functionals determined by Δ are Polish. As a corollary we obtain that the finite Hausdorff topology for a separable complete metric space is Polish.

It is known [2] that the Borel fields determined by the above mentioned topologies τ_Δ^G and τ_Δ^{GE} are equal to the Effros sigma algebra, so Polishness of these topologies can be useful in the study of measurable multifunctions with values in $CL(X)$.

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2. Preliminaries

In the sequel (X,d) will be a metric space, $CL(X)$ the family of all nonempty closed subsets of X, $CLB(X)$ the family of all nonempty closed and bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X. The open (resp. closed) ball with center x and radius ε will be denoted by $S(x,\varepsilon)$ (resp. $B(x,\varepsilon)$). The distance of x from a nonempty set A is defined as

$$d(x,A) = \inf\{d(x,a) : \ a \in A\}.$$

The open (resp. closed) ε-enlargement of A is the set

$$S(A,\varepsilon) = \{x \in X : \ d(x,A) < \varepsilon\} \quad (B(A,\varepsilon) = \{x \in X : \ d(x,A) \le \varepsilon\}).$$

Given two nonempty sets $A,\ B$ we define the gap between them as

$$D(A,B) = \inf\{d(a,b) : \ a \in A, \ b \in B\}$$

and the excess of A over B as

$$e(A,B) = \sup\{d(a,B) : \ a \in A\}.$$

If Δ is a nonempty subfamily of $CL(X)$, then by τ_Δ^G we mean the weak topology on $CL(X)$ generated by the family of gap functionals

$$\{D(.,B) : \ B \in \Delta\}$$

and by τ_Δ^{GE} we mean the weak topology on $CL(X)$ generated by the family of gap and excess functionals

$$\{D(.,B) : \ B \in \Delta\} \cup \{e(.,B) : \ B \in \Delta\}.$$

If $\Delta = K(X)$ then $\tau_{K(X)}^{GE}$ is just the finite Hausdorff topology ([2]) which will be denoted by τ_{fH}. The finite Hausdorff topology when restricted to $K(X)$ agrees with the Hausdorff metric topology. Some properties as well as applications of the finite Hausdorff topology can be found in [3].

If Δ contains the singleton subsets of X, then τ_Δ^G and τ_Δ^{GE} are Hausdorff and admissible and they are unchanged if Δ is replaced by $\overline{\Delta}$, the closure of Δ in the Hausdorff metric [2, Chapter 4]. Of course τ_Δ^G and τ_Δ^{GE} are completely regular as weak topologies. We finish these preliminaries with the following Lemma,which will be used later. By $C(Y,Z)$ we mean the space of the continuous functions from Y to Z.

Lemma 2.1. *Let (Y,e) be a separable metric space and (Z,f) a metric space. Then the uniformity $\mathcal{U}$ of the uniform convergence on compact subsets of Y on every equicontinuous family $\mathcal{F} \subset C(Y,Z)$ has a countable base, and so it is metrizable.*

Proof. For every compact subset $K \subset Y$ and $\varepsilon > 0$ denote by

$$U_{K,\varepsilon} = \{(g,h) \in \mathcal{F} \times \mathcal{F} : \ f(g(y),h(y)) < \varepsilon \text{ for every } y \in K\}.$$

Let E be a countable dense set in Y and $\mathcal{E}$ the family of the finite subsets of E. Then the family

$$\{U_{B,\frac{1}{n}}:\ B \in \mathcal{E},\ n \in Z^+\}$$

is a base of $\mathcal{U}$. (Let $K \subset Y$ be a compact set in Y and $\varepsilon > 0$. The compactness of K and the equicontinuity of $\mathcal{F}$ imply that there is $\delta > 0$ such that whenever $e(x,y) < \delta$ and $x \in K$ then $f(g(x), g(y)) < \frac{\varepsilon}{3}$ for every $g \in \mathcal{F}$. Let $\{x_1, ..., x_n\}$ be a finite set in K such that $K \subset S(\{x_1, ..., x_n\}, \frac{\delta}{2})$. For every $i \in \{1, 2, ..., n\}$ let $e_i \in E \cap S(x_i, \frac{\delta}{2})$. Then $K \subset S(\{e_1, ..., e_n\}, \delta)$. It easy to verify that $U_{\{e_1,...,e_n\},\frac{1}{k}} \subset U_{K,\varepsilon}$ where $k \in Z^+$ is such that $\frac{1}{k} < \frac{\varepsilon}{3}$.) □

3. Weak topologies generated by gap functionals

Let (X, d) be a metric space and Δ a nonempty subfamily of $CL(X)$. In this section we will study the weak topology τ^G_Δ on $CL(X)$. It is known that if $\Delta = CL(X)$ we obtain the proximal topology [5], if $\Delta = CLB(X)$ we obtain the bounded proximal topology [4] and if Δ is the family of all nonempty closed bounded and convex sets in a normed linear space we have the slice topology [2].
We start with the following proposition which generalizes some results from [5] and [4].

Proposition 3.1. *Let (X, d) be a metric space and Δ a subfamily of $CL(X)$ containing the singletons. If $(CL(X), \tau^G_\Delta)$ is first countable, then X is separable.*

Proof. Notice that the family of τ^G_Δ neighbourhoods of X is the same as in the lower Vietoris topology. So by Proposition 4.3 in [8] we have that X is separable. □

Theorem 3.2. *Let (X, d) be a metric space and Δ a subfamily of $CL(X)$ containing the singletons. The following are equivalent:*

(1) *$(CL(X), \tau^G_\Delta)$ is metrizable;*
(2) *$(CL(X), \tau^G_\Delta)$ is second countable;*
(3) *There is a countable subfamily $\Sigma \subset \Delta$ such that $\tau^G_\Delta = \tau^G_\Sigma$ on $CL(X)$.*

Proof. (1) $\Rightarrow$ (2) It is sufficient to prove that $(CL(X), \tau^G_\Delta)$ is separable. Let E be a countable dense set in X (see Proposition 3.1). It is easy to verify that the finite subsets of E form a dense set in $(CL(X), \tau^G_\Delta)$.
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1) $(CL(X), \tau^G_\Delta)$ is completely regular, Hausdorff and has a countable base. So $(CL(X), \tau^G_\Delta)$ is metrizable. □

We are now going to prove the following result:

Theorem 3.3. *Let (X, d) be a complete metric space. Let $\Delta \subset CL(X)$ be separable with respect to the induced Hausdorff metric H. Suppose moreover Δ contains the singletons. Then $(CL(X), \tau^G_\Delta)$ is a Polish space.*

The proof of Theorem 3.3 is based on the following remarks and lemmas.

Suppose $\Delta \subset CL(X)$ contains the singletons. A net $\{A_\sigma : \sigma \in \Omega\}$ in $CL(X)$ τ_Δ^G-converges to $A \in CL(X)$ if and only if for every $B \in \Delta$ the net $\{D(A_\sigma, B) : \sigma \in \Omega\}$ converges to $D(A, B)$. Consider Δ as a metric space equipped with the induced Hausdorff metric H. Under the identification $A \leftrightarrow D(A, .)$, where $D(A, .)$ is defined on Δ, we can consider $(CL(X), \tau_\Delta^G)$ as a topological subspace of $C_p(\Delta, R)$, the space of the continuous real valued functions defined on Δ equipped with the topology of pointwise convergence. For every $A, B, C \in CL(X)$ we have

$$|D(A, B) - D(A, C)| \leq H(B, C),$$

so for every $A \in CL(X)$ $D(A, .)$ is a Lipschitz continuous function with constant 1 ([2], Proposition 3.2.5.)

Lemma 3.4. *Let (X, d) be a metric space. Suppose Δ is a subfamily of $CL(X)$ separable with respect to the induced Hausdorff metric H and containing the singletons. Then $\overline{CL(X)}$ in $C_p(\Delta, R)$ is Polish, where $\overline{CL(X)}$ is the closure of $CL(X)$ in $C_p(\Delta, R)$.*

Proof. The family $\{D(A, .) : A \in CL(X)\}$ of the continuous functions defined on (Δ, H) is equicontinuous, so $\overline{CL(X)}$ is equicontinuous. By Lemma 2.5.2 in [2] $\overline{CL(X)}$ is second countable. On $\overline{CL(X)}$ the topology of pointwise convergence and the topology of uniform convergence on compact sets coincide. By Lemma 2.1 the uniformity of uniform convergence on compact sets on $\overline{CL(X)}$ is metrizable and of course it is complete, since R is complete. □

Lemma 3.5. *Let (X, d) be a metric space. Suppose Δ is a subfamily of $CL(X)$ containing the singletons. Let $F \in \overline{CL(X)} \subset C_p(\Delta, R)$. Put $A = \{x \in X : F(\{x\}) = 0\}$. Then $F = D(A, .)$ if and only if $A \neq \emptyset$ and for every $B \in \Delta$, $D(A, B) \leq F(B)$.*

Proof. $\Rightarrow$ This part is easy.
$\Leftarrow$ Let $B \in \Delta$. We show that $F(B) \leq D(A, B)$. Let $\varepsilon > 0$. Pick $a \in A$ with $D(a, B) < D(A, B) + \frac{\varepsilon}{3}$. $F \in \overline{CL(X)}$, so there is $C \in CL(X)$ such that

$$|F(B) - D(C, B)| < \frac{\varepsilon}{3} \text{ and } |F(\{a\}) - D(C, \{a\})| < \frac{\varepsilon}{3}.$$

So

$$F(B) < D(C, B) + \frac{\varepsilon}{3} < D(C, \{a\}) + D(\{a\}, B) + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + D(A, B) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Since $\varepsilon > 0$ was arbitrary, we have $F(B) \leq D(A, B)$. □

Proof of Theorem 3.3. The assumptions of Theorem 3.3 imply that X is separable. Let $\{x_1, x_2, ..., x_n, ...\}$ be a countable dense set in X and $\{B_1, B_2, ..., B_n, ...\}$ a countable dense set in (Δ, H). For every $n, k \in Z^+$ define

$$\Omega^1(n, k) = \{F \in \overline{CL(X)} : \{x \in X : F(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset \text{ and}$$

$$d(x_k, \{x \in X: \ F(\{x\}) < \frac{1}{2^{n+1}}\}) < F(\{x_k\}) + \frac{1}{2^n}\};$$

$$\Omega^2(n,k) = \{F \in \overline{CL(X)}: \ \{x \in X: \ F(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset \ \text{ and}$$

$$D(\{x \in X: \ F(\{x\}) < \frac{1}{2^{n+1}}\}, B_k) < F(B_k) + \frac{1}{2^n}\}.$$

We show that $CL(X) = \bigcap_{n,k}(\Omega^1(n,k) \cap \Omega^2(n,k))$.

The inclusion $CL(X) \subset \bigcap_{n,k}(\Omega^1(n,k) \cap \Omega^2(n,k))$ is clear. Now let

$$F \in \bigcap_{n,k}(\Omega^1(n,k) \cap \Omega^2(n,k)).$$

To prove that $F \in CL(X)$, it is sufficient by Lemma 3.5 to show that

$$A = \{x \in X: \ F(\{x\}) = 0\}$$

is nonempty, and that for each $B \in \Delta$ we have $D(A,B) \leq F(B)$. Clearly

$$F \in \bigcap_{n,k} \Omega^1(n,k) \quad \text{and} \quad F_{|X} \in \overline{CL(X)} \ \text{ in } \ C_p(X,R).$$

From the proof of Theorem 2.5.4 in [2] we can see that

$$A = \{x \in X: \ (F_{|X})(x) = 0\} \neq \emptyset \ \text{ and}$$

$$F(\{x\}) = d(A,x) = D(A,x) \ \text{ for every } \ x \in X.$$

Now we prove that also for every $B \in \Delta$ $D(A,B) \leq F(B)$. First, let $B = B_l$ for some l. Suppose $D(A,B_l) > 0$. Let $\varepsilon > 0$. Let $n \in Z^+$ be such that

$$\frac{1}{2^{n-1}} < \min\{\varepsilon, D(A,B_l)\}. \ \text{ Then}$$

$$\begin{aligned} D(A,B_l) \ &\leq D(S(A,\frac{1}{2^{n+1}}),B_l) + H(A,S(A,\frac{1}{2^{n+1}})) \\ &< F(B_l) + \frac{1}{2^n} + \frac{1}{2^{n+1}} < F(B_l) + \varepsilon, \end{aligned}$$

since $F \in \Omega^2(n,k)$ and $F_{|X} = d(A,.)$, so

$$S(A,\frac{1}{2^{n+1}}) = \{x \in X: \ d(A,x) < \frac{1}{2^{n+1}}\} = \{x \in X: \ F(\{x\}) < \frac{1}{2^{n+1}}\}.$$

Since $\varepsilon > 0$ was arbitrary, we have $D(A,B_l) \leq F(B_l)$.

Now let $B \in \Delta$. We show that $D(A,B) \leq F(B) + \varepsilon$ for every $\varepsilon > 0$. Let $\varepsilon > 0$. There is $B_l \in \Delta$ such that $H(B,B_l) < \frac{\varepsilon}{2}$. Then $|D(A,B_l) - D(A,B)| \leq H(B,B_l)$ and also $|F(B_l) - F(B)| \leq H(B,B_l)$, so

$$D(A,B) \leq D(A,B_l) + H(B,B_l) < F(B_l) + \frac{\varepsilon}{2} \leq F(B) + \varepsilon.$$

It is very easy to verify that for every $n,k \in Z^+$ $\Omega^1(n,k)$ and $\Omega^2(n,k)$ are open subsets of $\overline{CL(X)}$ in $C_p(\Delta, R)$. By Lemma 3.4 $\overline{CL(X)}$ is a Polish space, so also $(CL(X), \tau^G_\Delta)$ is a Polish space as a G_δ-subset of a Polish space. □

Theorem 3.6. *Let (X,d) be a separable complete metric space and Δ a subfamily of $CL(X)$ containing the singletons. If $(CL(X), \tau^G_\Delta)$ is metrizable, then it is Polish.*

Proof of Theorem 3.6. Theorem 3.2 and the metrizability of $(CL(X), \tau^G_\Delta)$ imply that there is a countable subfamily Σ of Δ such that $\tau^G_\Sigma = \tau^G_\Delta$. Let $\{x_1, x_2, ..., x_n, ...\}$ be a countable dense set in X. Put

$$\Gamma = \overline{\Sigma \cup \{\{x_i\} : \ i \in Z^+\}},$$

where the closure is taken with respect to the Hausdorff metric H. Then by Theorem 3.3 $(CL(X), \tau^G_\Gamma)$ is a Polish space and $\tau^G_\Gamma = \tau^G_\Delta$ on $CL(X)$. □

Observe that actually it is not necessary to suppose the separability of X (Proposition 3.1).

Remark 3.7. If (X,d) is a separable complete metric space then the Wijsman topology induced by the metric d on $CL(X)$ is Polish ([2]). So it is of interest to know conditions on Δ under which τ^G_Δ is different from the Wijsman topology. Results of this type can be found in [7].

Remark 3.8. By using Theorem 3.3 we can find another proof of Polishness of the slice topology. Let X be a Banach space with strongly separable dual. Let E be a countable dense set in X and E^* a countable dense set in X^*. Put

$$\mathcal{B} = \{B(e,q) : \ e \in E, \ q \in Q\}$$

where Q are positive rationals. The family $\mathcal{S}$ of slices of balls from $\mathcal{B}$ generated by elements from E^* and rationals is countable. Put $\Delta = \overline{\mathcal{S}}$, where the bar means the closure in the Hausdorff metric induced by the norm of X. By Theorem 3.3 we have that $(CL(X), \tau^G_\Delta)$ is Polish. The family $C(X)$ of the nonempty closed convex sets is closed in $(CL(X), \tau^G_\Delta)$, so $(C(X), \tau^G_\Delta)$ is Polish and τ^G_Δ on $C(X)$ is just the slice topology.

4. Weak topologies generated by gap and excess functionals

Let (X,d) be a metric space and Δ a nonempty subfamily of $CL(X)$. In this section we will study the weak topology τ^{GE}_Δ on $CL(X)$. If $\Delta = K(X)$ we have the finite Hausdorff topology τ^G_{fH} ([2]).

We start with the following Proposition:

Proposition 4.1. *Let (X,d) be a separable metric space and Δ a nonempty subfamily of $CL(X)$ which contains the singletons. The following are equivalent:*

(1) $(CL(X), \tau_\Delta^{GE})$ *is metrizable;*

(2) $(CL(X), \tau_\Delta^{GE})$ *is second countable;*

(3) *There is a countable subfamily $\Sigma \subset \Delta$ such that $\tau_\Sigma^{GE} = \tau_\Delta^{GE}$ on $CL(X)$.*

Proof. (1) $\Rightarrow$ (2) It is sufficient to prove that $(CL(X), \tau_\Delta^{GE})$ is separable. Let E be a countable dense set in X. It is easy to verify that finite subsets of E form a dense set in $(CL(X), \tau_\Delta^{GE})$.
(2) $\Rightarrow$ (3) This is clear.
(3) $\Rightarrow$ (1) $(CL(X), \tau_\Delta^{GE})$ is completely regular (as a weak topology), has a countable base and it is Hausdorff. So $(CL(X), \tau_\Delta^{GE})$ is metrizable. □

Now we will formulate the main result of this part:

Theorem 4.2. *Let (X,d) be a separable complete metric space and Δ a subfamily of $CLB(X)$ which contains the singletons. If $(CL(X), \tau_\Delta^{GE})$ is metrizable, then $(CL(X), \tau_\Delta^{GE})$ is Polish.*

We will deduce the above Theorem from the following one:

Theorem 4.3. *Let (X,d) be a complete metric space and Δ a subfamily of $CLB(X)$ separable with respect to the induced Hausdorff metric H and containing the singletons. Then $(CL(X), \tau_\Delta^{GE})$ is a Polish space.*

The proof of theorem 4.3 is based on the following remarks and lemmas.
Suppose $\Delta \subset CL(X)$ contains the singletons. A net $\{A_\sigma : \sigma \in \Omega\}$ in $CL(X)$ τ_Δ^{GE}-converges to $A \in CL(X)$ if and only if for every $B \in \Delta$ the net $\{D(A_\sigma, B) : \sigma \in \Omega\}$ converges to $D(A,B)$ and the net $\{e(A_\sigma, B) : \sigma \in \Omega\}$ converges to $e(A,B)$.
Let u be the usual metric on $[0,\infty)$. By $[0,\infty]$ we mean the one-point compactification of $[0,\infty)$. So $[0,\infty]$ can be equipped with a separable complete metric f. Put $Y = [0,\infty) \times [0,\infty]$ and consider the box metric ϱ on Y generated by u and f.
Let Δ be equipped with the induced Hausdorff metric H. Under the identification

$$A \leftrightarrow (D(A,.), e(A,.))$$

(D(A,.) and e(A,.) are defined on Δ) we can consider $(CL(X), \tau_\Delta^{GE})$ as a topological subset of $C_p(\Delta, Y)$.

Lemma 4.4. *Let (X,d) be a metric space and Δ a subfamily of $CL(X)$ containing the singletons. Let Δ be equipped with H and Y with ϱ. Then $CL(X) \subset C_p(\Delta, Y)$ is equicontinuous.*

Proof. Let $B \in \Delta$ and $\varepsilon > 0$. There is $\delta > 0$, $\delta < \varepsilon$ such that whenever $u(x,y) < \delta$ $x, y \in [0,\infty)$ then $f(x,y) < \varepsilon$. By proposition 3.2.5. in [2] we can deduce that if $C \in \Delta$ is such that $H(B,C) < \delta$, then

$$\varrho((D(A,B), e(A,B)), (D(A,C), e(A,C)) < \varepsilon$$

for every $A \in CL(X)$. □

Lemma 4.5. *Let (X,d) be a metric space, Δ a subfamily of $CL(X)$ separable with respect to the induced Hausdorff metric and containing the singletons and let Y be equipped with the metric ϱ. Then $\overline{CL(X)}$ in $C_p(\Delta,Y)$ is Polish, where $\overline{CL(X)}$ is the closure of $CL(X)$ in $C_p(\Delta,Y)$.*

Proof. $\overline{CL(X)}$ in $C_p(\Delta,Y)$ is equicontinuous, since by Lemma 4.4 $CL(X)$ is equicontinuous. By Lemma 2.5.2 in [2] $\overline{CL(X)}$ is second countable. On $\overline{CL(X)}$ the topology of pointwise convergence and the topology of uniform convergence on compact sets coincide. By Lemma 2.1 the uniformity of uniform convergence on compact sets on $\overline{CL(X)}$ is metrizable and it is complete, since Y is complete. □

In what follows we will suppose that Δ is a subset of $CLB(X)$. So if $A \in CLB(X)$ then the function $e(A,.)$ defined on Δ has values in $[0,\infty)$ and if $A \in CL(X)$ is unbounded then $e(A,.)$ defined on Δ is identically equal to ∞. The following lemma describes the behaviour of the elements of $\overline{CL(X)}$.

Lemma 4.6. *Let (X,d) be a metric space and Δ a subfamily of $CLB(X)$ containing the singletons. Let $F=(F_1,F_2) \in \overline{CL(X)}$ in $C_p(\Delta,Y)$. If there is $B \in \Delta$ with $F_2(B)=\infty$ then for every $K \in \Delta$ $F_2(K)=\infty$.*

Proof. Of course if $F \in CL(X)$ we are done. Otherwise suppose there is $K \in \Delta$ such that $F_2(K)<\infty$. Let $\varepsilon>0$. $F \in \overline{CL(X)}$, so there is $A \in CL(X)$ such that

$$e(A,K) \in (F_2(K)-\varepsilon, F_2(K)+\varepsilon) \quad \text{and} \quad e(A,B) > F_2(K)+\varepsilon+H(K,B).$$

Since $e(A,K)<\infty$ also $e(A,B)<\infty$ and by Proposition 3.2.5 in [2] we have

$$e(A,B) \leq e(A,K)+H(K,B).$$

So $e(A,B) < F_2(K)+\varepsilon+H(K,B)$, a contradiction. □

Lemma 4.7. *Let (X,d) be a metric space and Δ a subfamily of $CLB(X)$ containing the singletons. Let $F=(F_1,F_2) \in \overline{CL(X)}$ in $C_p(\Delta,Y)$ be such that $F_2(B)<\infty$ for every $B \in \Delta$. Then for every $\varepsilon>0$ the set $H=\{x \in X: \ F_1(\{x\})<\varepsilon\}$ is bounded.*

Proof. Let $\varepsilon>0$ and suppose that

$$H=\{x \in X: \ F_1(\{x\})<\varepsilon\}$$

is unbounded. Let $x_0 \in X$ be a fixed point of X. Let $M \in Z^+$ be such that $F_2(\{x_0\})<M$ and let $K \in Z^+$ be such that $KM > M+\varepsilon$. There is $a \in H$ such that $F_1(\{a\})<\varepsilon$ and $a \notin B(x_0, KM+2\varepsilon)$. $F \in \overline{CL(X)}$, so there is $B \in CL(X)$ such that

$$|D(B,\{a\})-F_1(\{a\})|<\varepsilon \text{ and also } |e(B,\{x_0\})-F_2(\{x_0\})|<\varepsilon.$$

Thus

$$e(B,\{x_0\}) < F_2(\{x_0\})+\varepsilon \text{ and } D(B,\{a\})<2\varepsilon.$$

Let $b \in B$ be such that $d(b,a) < 2\varepsilon$. We claim that $d(b,x_0) > KM$. Suppose $d(b,x_0) \leq KM$. Then

$$d(x_0,a) \leq d(b,x_0) + d(b,a) \leq KM + 2\varepsilon,$$

a contradiction. Thus $d(b,x_0) > KM$ and so $e(B,\{x_0\}) > KM$, a contradiction. □

Lemma 4.8. *Let (X,d) be a metric space and Δ a subfamily of $CLB(X)$ containing the singletons. Let $F = (F_1,F_2) \in \overline{CL(X)}$ in $C_p(\Delta,Y)$. Let $A = \{x \in X : \ F_1(\{x\}) = 0\}$.*

(1) *If for every $B \in \Delta$ $F_2(B) < \infty$, then $(F_1,F_2) = (D(A,.),e(A,.)) \Leftrightarrow A \neq \emptyset$, A is bounded and for each $K \in \Delta$, $D(A,K) \leq F_1(K)$ and $e(A,K) \geq F_2(K)$.*
(2) *If for every $B \in \Delta$ $F_2(B) = \infty$ then $(F_1,F_2) = (D(A,.),e(A,.)) \Leftrightarrow A \neq \emptyset$, A is unbounded and, for each $K \in \Delta$, $D(A,K) \leq F_1(K)$.*

Proof. 1) $\Rightarrow$ This implication is obvious.
$\Leftarrow$ Suppose $A \neq \emptyset$, A bounded and for each $K \in \Delta$

$$D(A,K) \leq F_1(K) \ \text{ and } \ e(A,K) \geq F_2(K).$$

We have to prove also the opposite inequalities. Take $K \in \Delta$ and $\varepsilon > 0$. There is $a \in A$ such that

$$d(a,K) > e(A,K) - \frac{\varepsilon}{3}.$$

$F \in \overline{CL(X)}$, so there is $B \in CL(X)$ such that

$$|F_2(K) - e(B,K)| < \frac{\varepsilon}{3} \ \text{ and also } \ |F_1(\{a\}) - D(B,\{a\})| < \frac{\varepsilon}{3}.$$

For every $C \in CL(X)$ $d(a,K) \leq e(C,K) + D(C,a)$. (Take $\eta > 0$. There is $c \in C$ such that $d(a,C) < D(C,a) + \eta$. Then $d(a,K) \leq d(c,K) + d(a,c) \leq e(C,K) + D(C,a) + \eta$.) We have

$$\begin{aligned} e(A,K) &< d(a,K) + \frac{\varepsilon}{3} < e(B,K) + D(B,a) + \frac{\varepsilon}{3} \\ &< F_2(K) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = F_2(K) + \varepsilon. \end{aligned}$$

To prove $F_1(K) \leq D(A,K)$ we use the same idea as in the proof of Lemma 3.5.
2) $\Rightarrow$ This implication is obvious.
$\Leftarrow$ Suppose $A \neq \emptyset$, A is unbounded and for each $K \in \Delta$ $D(A,K) \leq F_1(K)$. With the same proof of the Lemma 3.5 we obtain $F_1(K) = D(A,K)$ for every $K \in \Delta$. The unboundedness of A implies that $e(A,K) = \infty$ for every $K \in \Delta$. So $F_2(K) = \infty = e(A,K)$ for every $K \in \Delta$. □

Proof of Theorem 4.3. Now we prove that $CL(X)$ is a G_δ-set in $\overline{CL(X)}$.
Let $\{x_1,x_2,...,x_n,...\}$ be a countable dense set in X and $\{B_1,B_2,...,B_n,...\}$ a countable

dense set in Δ. For every $n,k \in Z^+$ put

$$\begin{aligned}\Omega^1(n,k) = \{F \in \overline{CL(X)} :\ & \{x \in X :\ F_1(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset \text{ and} \\ & d(x_k, \{x \in X :\ F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1(\{x_k\}) + \frac{1}{2^n}\} \\ & \cap\ \{F \in \overline{CL(X)} :\ F_2(B_k) \in R\} \\ & \cap\ \{F \in \overline{CL(X)} :\ D(B_k, \{x \in X :\ F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1(B_k) + \frac{1}{2^n}\} \\ & \cap\ \{F \in \overline{CL(X)} :\ e(\{x \in X :\ F_1(\{x\} < \frac{1}{2^{n+1}}\}, B_k) > F_2(B_k) - \frac{1}{2^n}\}.\end{aligned}$$

It is easy to verify that if $A \in CL(X)$ is bounded, then

$$(D(A,.), e(A,.)) \in \bigcap_{n,k} \Omega^1(n,k).$$

For every $n,k \in Z^+$ $\Omega^1(n,k)$ is open in $\overline{CL(X)} \subset C_p(\Delta, Y)$. We prove only that the set

$$\begin{aligned}L = \{F \in \overline{CL(X)} :\ & \{x \in X :\ F_1(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset\} \\ & \cap \{F \in \overline{CL(X)} :\ F_2(B_k) \in R\} \\ & \cap \{F \in \overline{CL(X)} :\ e(\{x \in X :\ F_1(\{x\}) < \frac{1}{2^{n+1}}\}, B_k) \\ & > F_2(B_k) - \frac{1}{2^n}\}\end{aligned}$$

is open in $\overline{CL(X)}$.

Let $G \in L$. By Lemma 4.7 the set $\{x \in X :\ G_1(\{x\}) < \frac{1}{2^{n+1}}\}$ is bounded. Let $\varepsilon > 0$ be such that

$$e(\{x \in X :\ G_1(\{x\}) < \frac{1}{2^{n+1}}\}, B_k) - \varepsilon > G_2(B_k) - \frac{1}{2^n} + \varepsilon.$$

Let $x_0 \in X$ be such that

$$G_1(\{x_0\}) < \frac{1}{2^{n+1}} \text{and} d(x_0, B_k) > e(\{x \in X :\ G_1(\{x\}) < \frac{1}{2^{n+1}}\}, B_k) - \varepsilon.$$

Then the set

$$\begin{aligned}&\{F \in \overline{CL(X)} :\ F_1(\{x_0\}) < \frac{1}{2^{n+1}}\} \\ &\cap \{F \in \overline{CL(X)} :\ F_2(B_k) \in R\} \\ &\cap \{F \in \overline{CL(X)} :\ F_2(B_k) < G_2(B_k) + \varepsilon\}\end{aligned}$$

is an open neighbourhood of G which is contained in L.

Thus $\bigcap_{n,k} \Omega^1(n,k)$ is a G_δ-set. Now we show that $\bigcap_{n,k} \Omega^1(n,k) = CLB(X)$. Let

$$G \in \bigcap_{n,k} \Omega^1(n,k).$$

So

$$G \in \bigcap_{n,k} \{F \in \overline{CL(X)}:\ \{x \in X:\ F_1(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset\}$$
$$\cap \{F \in \overline{CL(X)}:\ d(x_k, \{x \in X:\ F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1\{x_k\} + \frac{1}{2^n}\}.$$

Following the proof of Theorem 2.5.4 in [2] we can see that $A = \{x \in X:\ G_1(\{x\}) = 0\} \neq \emptyset$ and $G_1(\{x\}) = d(x, A)$ for every $x \in X$. By Lemma 4.7 A is bounded. By Lemma 4.8 1) it is sufficient to prove that $D(A, K) \leq G_1(K)$ and $e(A, K) \geq G_2(K)$ for every $K \in \Delta$. But this is routine.
Now let x_0 be a fixed point from X. Put

$$\Omega^2(n,k) = \{F \in \overline{CL(X)}:\ \{x \in X:\ F_1(\{x\}) < \frac{1}{2^{n+1}}\} \neq \emptyset \text{ and } F_2(B_k) > n\}$$
$$\cap \{F \in \overline{CL(X)}:\ d(x_k, \{x \in X:\ F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1(\{x_k\}) + \frac{1}{2^n}\}$$
$$\cap \{F \in \overline{CL(X)}: D(B_k, \{x \in X:\ F_1(\{x\}) < \frac{1}{2^{n+1}}\}) < F_1(B_k) + \frac{1}{2^n}\}$$
$$\cap \{F \in \overline{CL(X)}: \{x \in X:\ F_1(\{x\}) < \frac{1}{2^{n+1}}\} \cap B(x_0, n)^c \neq \emptyset\}.$$

It is easy to verify that $\Omega^2(n,k)$ is open in $C_p(\Delta, Y)$ for every $n, k \in Z^+$. If $A \in CL(X)$ is unbounded, then $A \in \bigcap_{n,k} \Omega^2(n,k)$.
Now suppose that $(F_1, F_2) \in \bigcap_{n,k} \Omega^2(n,k)$. Again from the proof of Theorem 2.5.4 in [2]

$$A = \{x \in X:\ F_1(\{x\}) = 0\} \neq \emptyset.$$

We prove that A is unbounded. Let $n \in Z^+$. We show that $B(x_0, n)^c \cap A \neq \emptyset$. As

$$\{x \in X:\ F_1(\{x\}) < \frac{1}{2^{2n+1}}\} \cap B(x_0, 2n)^c \neq \emptyset,$$

there is $z \in X$ such that

$$F_1(\{z\}) < \frac{1}{2^{2n+1}} \quad \text{and} \quad d(z, x_0) > 2n.$$

So

$$F_1(\{z\}) = d(z, A) < \frac{1}{2^{2n+1}},$$

and there is $a \in A$ such that $d(z, a) < \frac{1}{2^{2n+1}}$. We claim that $d(a, x_0) > 2n - \frac{1}{2^{2n+1}}$. Suppose not. Then

$$d(z, x_0) \leq d(a, x_0) + d(z, a) \leq 2n - \frac{1}{2^{2n+1}} + \frac{1}{2^{2n+1}} = 2n,$$

a contradiction. From the above results we have

$$CL(X) = \bigcap_{n,k} \Omega^1(n,k) \cup \bigcap_{n,k} \Omega^2(n,k).$$

So $CL(X)$ is a G_δ-set in $\overline{CL(X)}$, i.e. $(CL(X), \tau_\Delta^{GE})$ is a Polish space. $\square$

Now let (X,d) be a separable completely metrizable space. We are going to show that if Δ is a subfamily of $CLB(X)$ containing the singletons and separable with respect to the induced Hausdorff metric, then τ_Δ^G and τ_Δ^{GE} are Polish. As a corollary we obtain that the finite Hausdorff topology for a separable completely metrizable space is Polish.

We prove only the case of τ_Δ^G; the proof for τ_Δ^{GE} is the same.

Let $(\tilde{X}, \tilde{d})$ be the completion of (X,d). Let $\{x_1, x_2, ..., x_n, ...\}$ be a countable dense set in X and $\{B_1, B_2, ..., B_n, ...\}$ a countable dense set in (Δ, H).

Put

$$\Delta_{\tilde{X}} = \{CL_{\tilde{X}} B_1, CL_{\tilde{X}} B_2, ..., CL_{\tilde{X}} B_n, ...\} \cup \{x : \ x \in \tilde{X}\}.$$

Define the map

$$L : \ (CL(X), \tau_\Delta^G) \to (CL(\tilde{X}), \tau_{\Delta_{\tilde{X}}}^G)$$

as follows: $L(A) = CL_{\tilde{X}} A$. It is easy to verify that L is a topological embedding. By Theorem 3.3 $(CL(\tilde{X}), \tau_{\Delta_{\tilde{X}}}^G)$ is a Polish space. By [6] $CL(X)$ is a G_δ-subset of $CL(\tilde{X})$ equipped with the Wijsman topology induced by $\tilde{d}$, so $CL(X)$ is a G_δ-subset of $CL(\tilde{X})$ equipped with $\tau_{\Delta_{\tilde{X}}}^G$, since the Wijsman topology on $CL(\tilde{X})$ induced by $\tilde{d}$ is weaker than $\tau_{\Delta_{\tilde{X}}}^G$.

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