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The Viscosity Subdifferential of the Sum of Two Functions in Banach Spaces I: First Order Case

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We present a formula for the viscosity subdifferential of the sum of two uniformly continuous functions on smooth Banach spaces. This formula is deduced from a new variational principle with constraints. We obtain as a consequence a weak form of Preiss' theorem for uniformly continuous functions. We use these results to give simple proofs of some uniqueness results of viscosity solutions of Hamilton-Jacobi equations and we show how singlevaluedness of the associated Hamilton-Jacobi operators is related to the geometry of Banach spaces.

1. Introduction

The aim of this paper is to investigate subdifferential calculus for lower semicontinuous functions and to show that this calculus sheds a new light on the proof of uniqueness of viscosity solutions of Hamilton-Jacobi equations.

In section 2, we give a formula for the viscosity subdifferential of the sum of two uniformly continuous functions on spaces which admit a $\mathcal{C}^1$ Lipschitz bump function. Our result extends former work of A. Ioffe [1] and M. Fabian ([2], [3]) on fuzzy calculus and trustworthiness. We shall present two different proofs of this result. The first one, valid under some more restrictive assumption on X, is deduced from a constrained smooth variational principle. The second proof is a direct one and shows actually that our results remain true if we assume one of the function locally uniformly continuous and the other one lower semicontinuous (A. Ioffe and M. Fabian assumed that one of the function was Lipschitz continuous and the other one lower semicontinuous). Both proofs rely on the smooth variational principle of R. Deville, G. Godefroy and V. Zizler [4] which is an extension of the smooth variational principle of J. Borwein and D. Preiss [5].

In section 3, we apply these results to Hamilton-Jacobi equations in infinite dimensions.

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We prove how the formula for the viscosity subdifferential of the sum of two (locally) uniformly continuous functions allows us to give a comprehensive proof of uniqueness of viscosity solutions of some Hamilton-Jacobi equations. We also prove a Rademacher-Preiss type theorem for uniformly continuous functions in spaces which admit a smooth bump function. As a consequence, we show how the singlevaluedness of the Hamilton-Jacobi operator associated to a uniformly continuous Hamiltonian is related to the existence of a smooth bump function on X.
A formula for the second order viscosity subdifferential of the sum of two lower semicontinuous functions is available only in finite dimensions. Very little is known about the second order subdifferential of the sum of two functions in infinite dimensions. We do not include here the second order case, since it involves different techniques. We shall treat this topic in part II.

We now introduce our definitions. Throughout this paper, unless otherwise stated the notion of differentiability r efer to Fréchet differentiability. Let E be a Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. As usual, we denote by $\mathrm{dom}(f)$ the *effective* domain of f:

$$\mathrm{dom}(f) := \{x \in E\,;\, f(x) < +\infty\}$$

Recall that whenever $x \in \mathrm{dom}(f)$ and f is convex, the subdifferential of f at x is the set:

$$D^- f(x) = \{p \in E^*; f - p \text{ has a local minimum at } x\}$$

If $x \in \mathrm{dom}(f)$ and f is not supposed convex, we define:

$$D^- f(x) = \{\, \varphi'(x)\,;\, \varphi : E \longrightarrow \mathbb{R} \text{ is } \mathcal{C}^1 \text{ and } f - \varphi \text{ has a local minimum at } x\}$$

It is not difficult to check that whenever f is convex, these two definitions coincide. We say that f is subdifferentiable at x if $D^- f(x) \neq \emptyset$ (for $x \notin \mathrm{dom}(f)$, we define $D^- f(x) = \emptyset$). We can define in a similar way the superdifferential of f at x:

$$D^+ f(x) = \{\, \varphi'(x)\,;\, \varphi : E \longrightarrow \mathbb{R} \text{ is } \mathcal{C}^1 \text{ and } f - \varphi \text{ has a local maximum at } x\}$$

A first difficulty is that even in finite dimensions, the sets $D^- f(x)$ and $D^+ f(x)$ may be empty at many points in $\mathrm{dom}(f)$. In fact, the sets $\mathrm{dom}\, D^- f = \{x \in X;\ f \text{ is subdifferentiable at } x\}$ and $\mathrm{dom}\, D^+ f = \{x \in X;\ f \text{ is superdifferentiable at } x\}$ need not be residual in X as shown by the following:

Fact 1.1. *If $x \in \mathrm{dom}\, D^- f \bigcap \mathrm{dom}\, D^+ f$, then f is differentiable at x. Consequently, if $f : X \longrightarrow \mathbb{R}$ is a continuous nowhere differentiable function on X (such functions do exist even when $X = \mathbb{R}$), then $\mathrm{dom}\, D^- f \bigcap \mathrm{dom}\, D^+ f = \emptyset$.*

Indeed, assume that $x \in \mathrm{dom}\, D^- f \bigcap \mathrm{dom}\, D^+ f$. So there exists $\varphi_1, \varphi_2 : X \longrightarrow \mathbb{R}$ such that $f - \varphi_1$ has a local maximum at x and $f - \varphi_2$ has a local minimum at x. Consequently, $\varphi_1 - \varphi_2$ has a local minimum at x and so $\varphi_1'(x) = \varphi_2'(x)$. Let us denote by p this common value. From the inequalities:

$$\begin{aligned} \frac{\varphi_2(x+h) - \varphi_2(x) - \langle p, h\rangle}{\|h\|} &\leq \frac{f(x+h) - f(x) - \langle p, h\rangle}{\|h\|} \\ &\leq \frac{\varphi_1(x+h) - \varphi_1(x) - \langle p, h\rangle}{\|h\|} \end{aligned}$$

we deduce that f is differentiable at x.

In some infinite dimensionsional Banach spaces, a Lipschitz function may be nowhere subdifferentiable and nowhere superdifferentiable. Take for instance the function $f : \ell^1(N) \longrightarrow \mathbb{R}$ defined by $f((x_n)) = \sum_{n\in\mathbb{N}} |x_{2n}| - \sum_{n\in\mathbb{N}} |x_{2n+1}|$.

Throughout this paper, a *bump* b on X is a function from X into $\mathbb{R}$ non identically equal to zero, with bounded support. A key geometrical assumption on X in this paper will be the existence of a smooth bump function on X. To motivate this assumption, let us observe:

Fact 1.2. *Let $f : X \longrightarrow \mathbb{R}$ be a (Lipschitz continuous) function on X satisfying $f(x) > 0$ for all $x \in X$ and $\lim_{\|x\|\to\infty} f(x) = 0$. If there exists a $\mathcal{C}^1$ function φ such that $f - \varphi$ has a global minimum attained at some point x_0, then there exists a $\mathcal{C}^1$-bump function on X.*

Indeed, suppose that there exists $x_0 \in X$ and a $\mathcal{C}^1$-function $\varphi : X \longrightarrow \mathbb{R}$ such that $f - \varphi$ attains its minimum at x_0. Without loss of generality, we can assume that $\varphi(x_0) = f(x_0)$. Let $M > 0$ be such that $f(x) \leq \frac{f(x_0)}{2}$ whenever $\|x\| \geq M$. Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$-function such that $\alpha(f(x_0)) = 1$ and $\alpha(t) = 0$ whenever $t \leq \frac{f(x_0)}{2}$. The function $\alpha \circ \varphi$ is a $\mathcal{C}^1$-function on X such that $\alpha \circ \varphi(x_0) = 1$ and $\alpha \circ \varphi(x) = 0$ if $\|x\| \geq M$, so it is a $\mathcal{C}^1$-bump on X.

The following smooth variational principle is proved in [4] (see also [7]) and is a converse of Fact 1.2. We recall that the first smooth variational principle was obtained by J. Borwein and D. Preiss [5], and was then extended by R. Deville, G. Godefroy and V. Zizler in [4] with a very simple proof using the Baire category theorem.

We use in the next statement the following definition: a function $F : X \longrightarrow \mathbb{R}$ attains a strong minimum at $x_0 \in X$ if, by definition, $F(x_0) = \inf\{F(x)\,;\ x \in X\}$ and every minimizing sequence (y_n) in X (i.e. (y_n) satisfies $\lim_n F(y_n) = x_0$) converges to x_0.

Theorem 1.3. *Let X be a Banach space, $f : X \longrightarrow \mathbb{R}$ be a lower semicontinuous, bounded below function such that* $\mathrm{dom}(f) \neq \emptyset$ *and $\varepsilon > 0$. Assume that there exists a $\mathcal{C}^1$ Lipschitz continuous bump function b on X. Then there exists a $\mathcal{C}^1$-function g on X such that:*

(a) *$f + g$ has a strong minimum at some point $x_0 \in X$,*
(b) *$\|g\|_\infty = sup\{|g(x)|\,;\ x \in X\} < \varepsilon$ and $\|g'\|_\infty = sup\{\|g'(x)\|_\infty\,;\ x \in X\} < \varepsilon$*

Moreover, we have the following localization property: there exists a constant $c > 0$ (depending only on the space X) such that whenever $y \in X$ satisfies $f(y) \leq \inf_X f + c\varepsilon^2$, then the point x_0 can be chosen such that $\|y - x_0\| < \varepsilon$.

Let us point out that M. Fabian, P. Hajek and J. Vanderwerf were able in [6] to remove the assumption "b Lipschitz continuous" in Theorem 1.3.
A combination of Fact 1.2 and Theorem 1.3 yield:

Corollary 1.4. *Let X be a Banach space. The following conditions are equivalent:*

(1) *There exists a $\mathcal{C}^1$-bump function on X.*

(2) *For every lower semicontinuous, bounded below real valued function f on X, the set of all points $x \in X$ for which there exists a $\mathcal{C}^1$ function $\varphi : X \longrightarrow \mathbb{R}$ such that $f - \varphi$ has a global minimum at x is dense in X.*
(3) *For every Lipschitz continuous, bounded below real valued function f on X, there exists a $\mathcal{C}^1$ function $\varphi : X \longrightarrow \mathbb{R}$ such that $f - \varphi$ has a global minimum on X.*

Indeed, (1) implies (2) is a consequence of the variant of the smooth variational principle of Fabian, Hajek and Vanderwerff and is proved in [6]. Obviously (2) implies (3) while (3) implies (1) by Fact 1.2.

2. The viscosity subdifferential of the sum of two uniformly continuous functions in infinite dimension

In this section, we shall prove the following formula which extends a result of A. Ioffe ([1], [8]):

Theorem 2.1. *Let X be a Banach space such that there exists a $\mathcal{C}^1$ Lipschitz bump function on X. Let u_1, u_2 be two real valued functions defined on X such that u_1 is lower semicontinuous and u_2 is uniformly continuous. Suppose that $x_0 \in X$ and $p \in D^-(u_1+u_2)(x_0)$ are given. Then, for each $\varepsilon > 0$, there exists x_1, $x_2 \in X$, $p_1 \in D^-u_1(x_1)$ and $p_2 \in D^-u_2(x_2)$ such that:*

(1) $\|x_1 - x_0\| < \varepsilon$ *and* $\|x_2 - x_0\| < \varepsilon$
(2) $|u_1(x_1) - u_1(x_0)| < \varepsilon$ *and* $|u_2(x_2) - u_2(x_0)| < \varepsilon$
(3) $\|p_1 + p_2 - p\| < \varepsilon$

Let us recall that according to Rademacher's Theorem, every Lipschitz continuous function in $\mathbb{R}^n$ is differentiable almost everywhere. D. Preiss [9] has recently extended this result to an infinite dimensional setting. He proved that if X is an Asplund space, then every locally Lipschitz continuous real valued function defined on X is differentiable on a dense subset of X. We recall that a Banach space is an Asplund space if every convex continuous function on X is Fréchet-differentiable on a dense subset of X. It is well known that if there exists on X a $\mathcal{C}^1$ bump function, then X is an Asplund space. The converse, which is an open problem in general, is true if X is separable (see [10], [11] for recent developments on this problem and for references). The following result can be seen as a weak form of Preiss theorem for uniformly continuous functions.

Corollary 2.2. *Let X be a Banach space such that there exists a $\mathcal{C}^1$ Lipschitz bump function on X. Let u be a uniformly continuous function defined on X. Then for every $x \in X$ and every $\varepsilon > 0$, there exists x_1, $x_2 \in X$, $p^- \in D^-u(x_1)$ and $p^+ \in D^+u(x_2)$ such that:*

(1) $\|x_1 - x\| < \varepsilon$ and $\|x_2 - x\| < \varepsilon$
(2) $\|p^- - p^+\| < \varepsilon$

In order to prove Corollary 2.2, it is enough to apply Theorem 2.1 with $u_1 = u$ and $u_2 = -u$, and to observe that $D^-(-u)(x_2) = -D^+u(x_2)$. Let us here stress the fact that Preiss' result is considerably harder to prove.

Remark 2.3. (1) If the function u of Corollary 2.2 is nowhere differentiable, then the points x_1 and x_2 are necessarily different. It is unknown whether one can take $p^- = p^+$ when the function u in Corollary 2.2 is an arbitrary uniformly continuous function (this problem is related to the problem of the singlevaluedness of the Hamilton-Jacobi operator, see Remark 3.6). Let us notice here that the answer to this question is yes if dim $X = 1$. Indeed, when $X = \mathbb{R}$, for $x \in \mathbb{R}$, we have two possibilities: either there is an open interval I containing x such that u is of bounded variation on I. In this case, u is differentiable at some point x_0 of I and we can choose $x_1 = x_2 = x_0$ and $p^- = p^+ = u'(x_0)$. Otherwise, for every open interval I containing x, u is not of bounded variation on I. So for every such interval I, $\bigcup\{D^-u(y); y \in I\} = \mathbb{R}$ and $\bigcup\{D^+u(y); y \in I\} = \mathbb{R}$ and it is then certainly possible to have $p^+ = p^-$.
(2) It is possible in Theorem 2.1 and Corollary 2.2 to replace the assumption "uniformly continuous" by "locally uniformly continuous" (see the second proof of Theorem 2.1 below). We recall that a function $u : X \longrightarrow \mathbb{R}$ is locally uniformly continuous if for every $x \in X$, there exists a neighbourhood V of x such that u is uniformly continuous on V. However, we were unable to prove a formula for the subdifferential of the sum of two lower semicontinuous functions in infinite dimensions and to prove a Preiss type theorem for functions which are only continuous.

We shall give two different proofs of Theorem 2.1. The first one uses a constrained variational principle and requires some further assumptions. The second one is more technical but yields Theorem 2.1 in full generality.
In order to proceed with the first proof, let us recall that a norm $\|\,.\,\|$ on a Banach space E is locally uniformly rotund if for all x in the unit sphere of E and for all sequences (x_n) in the unit sphere of E, $\lim\limits_n \|x + x_n\| = 2$ implies $\lim\limits_n \|x - x_n\| = 0$. Let us consider the following assumptions:

(1) E has an equivalent norm whose dual norm is locally uniformly rotund in X^*.

(2) E has an equivalent Fréchet differentiable norm.

(3) E admits a $\mathcal{C}^1$ Lipschitz bump function.

It is well known that (1) implies (2) and (2) implies (3). It has been shown by M. Talagrand [12] that (2) does not imply (1) and by R. Haydon [13], [14] that (3) does not imply (2). However, assertions (1), (2) and (3) are equivalent for separable Banach spaces. The following result can be seen as a variational principle with constraint.

Theorem 2.4. *Let E be a Banach space and Δ be a closed linear subspace of E. Assume that there is an equivalent norm on E whose dual norm is locally uniformly rotund. Let $u : E \longrightarrow \mathbb{R}$ be uniformly continuous and assume that $x_0 \in \Delta$ satisfies $u(x_0) = \inf\{u(x)\,;\, x \in \Delta\}$. Then for every $\varepsilon > 0$, there exists $z \in E$ and $p \in D^-u(z)$ such that:*

(1) $\|z - x_0\| < \varepsilon$
(2) $|u(z) - u(x_0)| < \varepsilon$
(3) $\|R_\Delta(p)\| < \varepsilon$ *where $R_\Delta : E^* \longrightarrow \Delta^*$ is the restriction mapping.*

Let us observe that if $u(x) = -d(x, \Delta)$, where $d(x, \Delta) = \inf\{\|x - y\|\,;\, y \in \Delta\}$, and if $\Delta \neq E$, then u is not differentiable at any point of Δ. Since u is concave and continuous,

$D^+u(x) \neq \emptyset$ for all $x \in \Delta$. It follows from Fact 1.1 that $D^-u(x) = \emptyset$ for all $x \in \Delta$. So, in general, the point z has to be chosen outside Δ.
We now deduce from Theorem 2.4 the following particular case of Theorem 2.1.

Proposition 2.5. *Let X be a Banach space such that X has an equivalent norm whose dual norm is locally uniformly rotund. Let u_1, u_2 be two real valued uniformly continuous functions defined on X. Suppose that $x_0 \in X$ and $p \in D^-(u_1+u_2)(x_0)$ are given. Then, for each $\varepsilon > 0$, there exists x_1, $x_2 \in X$, $p_1 \in D^-u_1(x_1)$ and $p_2 \in D^-u_2(x_2)$ such that:*
(1) $\|x_1 - x_0\| < \varepsilon$ *and* $\|x_2 - x_0\| < \varepsilon$
(2) $|u_1(x_1) - u_1(x_0)| < \varepsilon$ *and* $|u_2(x_2) - u_2(x_0)| < \varepsilon$
(3) $\|p_1 + p_2 - p\| < \varepsilon$

Let us set $E = X \times X$. If $\|\,.\,\|_X$ is a norm on X such that the dual norm is locally uniformly rotund, then $\|(x,y)\|_E^2 = \|x\|_X^2 + \|y\|_X^2$ defines an equivalent norm on E such that the dual norm is locally uniformly rotund. The set $\Delta := \{(x,x)\,;\, x \in X\}$ is a closed linear subspace of E. Since $p \in D^-(u_1+u_2)(x_0)$, there exists a $\mathcal{C}^1$ function $\varphi : X \longrightarrow \mathbb{R}$ such that $u_1 + u_2 - \varphi$ has a local minimum at x_0 and $\varphi'(x_0) = p$.

Claim. *We can assume that φ is globally Lipschitz and $u_1+u_2-\varphi$ has a global minimum at x_0.*

Proof of the claim: Since $v := u_1 + u_2$ is uniformly continuous on X, there exists $K_1 > 0$ such that for all $x \in X$:

$$v(x) \geq v(x_0) - 1 - K_1\|x - x_0\|$$

where $\|\,.\,\|$ is an equivalent Fréchet differentiable norm on X. On the other hand, since φ is $\mathcal{C}^1$, there exists $\varepsilon > 0$ such that φ is Lipschitz continuous of constant K_2 on the ball $B(x_0, \varepsilon)$ centered at x_0 of radius ε. Taking if necessary a smaller ε, we can assume without loss of generality that the restriction of $u_1 + u_2 - \varphi$ to $B(x_0, \varepsilon)$ has a global minimum at x_0. Let b be a $\mathcal{C}^1$ Lipschitz bump function on X such that $b(0) > 0$. Let $\eta : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$ Lipschitz increasing function on $\mathbb{R}$ such that $\eta(t) = 0$ if $t \leq 0$ and $\eta(t) = 1$ if $t \geq \frac{b(o)}{2}$. The function $a : X \longrightarrow \mathbb{R}$ defined by $a(x) = \eta \circ b(K_3(x - x_0))$ is a $\mathcal{C}^1$ Lipschitz bump on X such that $0 \leq a(x) \leq 1$ for all $x \in X$, $a(x) = 1$ for all x in some neighbourhood of x_0 and $a(x) = 0$ whenever $\|x - x_0\| \geq \varepsilon$ if the constant K_3 has been chosen large enough. The function:

$$\psi(x) = a(x)(\varphi(x) - \varphi(x_0)) + (1 - a(x))(-1 - K_1\|x - x_0\|)$$

is a $\mathcal{C}^1$ globally Lipschitz function on X and $v - \psi$ has a global minimum at x_0. Indeed:

$$\begin{aligned}(v - \psi)(x) = &\ a(x)(v(x) - \varphi(x) + \varphi(x_0)) + (1 - a(x))(v(x) + 1 + K_1\|x - x_0\|)\\ \geq &\ a(x)v(x_0) + (1 - a(x))v(x_0) = v(x_0) = (v - \psi)(x_0)\end{aligned}$$

We now conclude the proof of Proposition 2.5. The function u defined by $u(x,y) = u_1(x) + u_2(y) - \varphi(x)$ is uniformly continuous on E and

$$\begin{aligned}u(x_0,x_0) = u_1(x_0) + u_2(x_0) - \varphi(x_0) &= \inf\{u_1(x) + u_2(x) - \varphi(x)\,;\, x \in X\}\\ &= \inf\{u(x,x)\,;\,(x,x) \in \Delta\}\end{aligned}$$

Fix $\varepsilon > 0$. By continuity we may choose $\varepsilon_0 < \varepsilon$ such that $\|\varphi'(x) - \varphi'(x_0)\| < \frac{\varepsilon}{2}$, $|u_1(x) - u_1(x_0)| < \varepsilon$ and $|u_2(x) - u_2(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \varepsilon/2$. Applying Theorem 2.4, there exist $z = (x_1, x_2) \in E$ and $q = (q_1, q_2) \in D^- u(z) \subset X^* \times X^*$ such that:

(1) $\|(x_1, x_2) - (x_0, x_0)\|_E < \varepsilon_0$
(2) $\|R_\Delta(q)\| < \varepsilon_0$

Condition (1) implies that $\|x_1 - x_0\|_X < \varepsilon$ and $\|x_2 - x_0\|_X < \varepsilon$. Since $q \in D^- u(z)$, there exists a $\mathcal{C}^1$ function $w : X \times X \longrightarrow \mathbb{R}$, such that $u - w$ attains its minimum at (x_1, x_2) and $w'(x_1, x_2) = (q_1, q_2)$. If we fix $x = x_1$, we see that the function $y \in X \longrightarrow u_1(x_1) + u_2(y) - \varphi(x_1) - w(x_1, y)$ attains its minimum at x_2, so $p_2 := q_2 \in D^- u_2(x_2)$. Similarly, if we fix $y = x_2$, we see that the function $x \in X \longrightarrow u_1(x) + u_2(x_2) - \varphi(x) - w(x, x_2)$ attains its minimum at x_1, so $p_1 := \varphi'(x_1) + q_1 \in D^- u_1(x_1)$. Observe that for $x \in X$, $(q_1, q_2)(x, x) = (q_1 + q_2)(x)$, so the condition $\|R_\Delta(p)\| < \varepsilon_0$ means $\|q_1 + q_2\| < \varepsilon_0$. Consequently, using the fact that $\|x_1 - x_0\| < \varepsilon_0$ one obtains:

$$\begin{aligned} \|p_1 + p_2 - p\| &= \|q_1 + q_2 + \varphi'(x_1) - \varphi'(x_0)\| \\ &\leq \|q_1 + q_2\| + \|\varphi'(x_1) - \varphi'(x_0)\| \\ &< \varepsilon_0 + \frac{\varepsilon}{2} \quad \leq \quad \varepsilon \end{aligned}$$

This completes the proof of Proposition 2.5. □

Proof of Theorem 2.4. Fix an equivalent norm $\|\,.\,\|$ on E such that the dual norm of $\|\,.\,\|$ is locally uniformly rotund. For $x \in E$, define

$$d(x, \Delta) = \inf \{ \|x - y\| \,;\, y \in \Delta \}$$

$d(x, \Delta)$ is the quotient norm of the coset of x in E/Δ. Its dual norm is the restriction of the dual norm of $\|\,.\,\|$ to $\Delta^\perp$, so it is locally uniformly rotund. Consequently, using Smulyan test and the chain rule, the map $\varphi : x \in E \longrightarrow (d^2(x, \Delta) + 1)^{1/2}$ is Fréchet-differentiable. Moreover, if $h \in \Delta$ and $t \in \mathbb{R}$, then $\varphi(x + th) = \varphi(x)$. Therefore:

$$\langle \varphi'(x), h \rangle \quad = \quad \lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t} \quad = \quad 0$$

for $h \in \Delta$, so the restriction of $\varphi'(x)$ to Δ is equal to zero. Let us fix $\varepsilon > 0$ and let $c > 0$ be given by Theorem 1.3. By uniform continuity, choose $\alpha > 0$ such that $|u(x) - u(y)| \leq c\varepsilon^2$ whenever $\|x - y\| < \alpha$. Consequently, if $d(x, \Delta) < \alpha$, there exists $y \in \Delta$ such that $\|x - y\| < \alpha$ and

$$u(x) \quad \geq \quad u(y) - c\varepsilon^2 \quad \geq \quad u(x_0) - c\varepsilon^2 \tag{2.1}$$

Using again the uniform continuity of u, there exists $K > 0$ such that for all $x, y \in E$

$$|u(x) - u(y)| \quad \leq \quad 1 + K\|x - y\|$$

For $x \in E$, there exists $y \in \Delta$ such that $\|x - y\| \leq 2d(x, \Delta)$. Consequently

$$u(x) \quad \geq \quad u(y) - 1 - 2Kd(x, \Delta) \quad \geq \quad u(x_0) - 1 - 2Kd(x, \Delta) \tag{2.2}$$

We now choose $\lambda > 1$ such that $(\lambda - 2K)\alpha \geq 2$ and define $w(x) = u(x) + \lambda\varphi(x)$. The function w is the sum of two uniformly continuous functions and we claim that w is bounded below. Indeed, if $d(x, \Delta) < \alpha$, we deduce from (2.1):

$$w(x) \geq u(x_0) - c\varepsilon^2 + \lambda\varphi(x) \geq u(x_0) - c\varepsilon^2 + 1$$

and if $d(x, \Delta) \geq \alpha$, we deduce from (2.2) and from the choice of λ:

$$\begin{aligned} w(x) &\geq u(x_0) - 1 - 2Kd(x, \Delta) + \lambda d(x, \Delta) \\ &\geq u(x_0) - 1 + (\lambda - 2K)\alpha \quad \geq \quad u(x_0) + 1 \end{aligned}$$

Therefore

$$w(x_0) \; = \; u(x_0) + 1 \; \leq \; \inf_X w + c\varepsilon^2$$

By the smooth variational principle, we can find a $\mathcal{C}^1$ Lipschitz bounded function $g : E \longrightarrow \mathbb{R}$ such that:

(a) $\|g\|_\infty < \varepsilon$ and $\|g'\| < \varepsilon$,

(b) $w + g$ attains its minimum at some point $z \in E$ such that $\|z - x_0\| < \varepsilon$.

We have $p := -g'(z) - \lambda\varphi'(z) \in D^- u(z)$ and for all $h \in \Delta$,

$$\begin{aligned} |\langle p, h\rangle| \; &\leq \; |\langle g'(z), h\rangle| + \lambda|\langle\varphi'(z), h\rangle| \\ &\leq \; \|g'(z)\| \, \|h\| \; \leq \; \|g'\|_\infty \|h\| \end{aligned}$$

This shows that $\|R_\Delta(p)\| \; \leq \; \|g'\|_\infty \; < \; \varepsilon$. So we have checked conditions (1), (2) and (3) of Theorem 2.4. $\square$

Proof of Theorem 2.1. Let X be a Banach space such that there exists a $\mathcal{C}^1$ Lipschitz bump function on X. Let u_1, u_2 be two real valued functions defined on X such that u_1 is lower semicontinuous and u_2 is locally uniformly continuous. Finally, fix $\varepsilon > 0$ and $p \in D^-(u_1 + u_2)(x_0)$.

Let $\varphi : X \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$ function such that $u_1 + u_2 - \varphi$ has a local minimum at x_0 and $\varphi'(x_0) = p$. There exists $r_0 > 0$ such that the restriction of $u_1 + u_2 - \varphi$ to the ball $B(x_0, r_0)$ has a global minimum at x_0. According to the smooth variational principle applied to $f = 0$, there exists a $\mathcal{C}^1$ function g on X such that g attains its strong minimum on X. By translation, we can assume that g attains its minimum at x_0. Replacing φ by $\varphi - g$, we can assume without loss of generality that the restriction of $u_1 + u_2 - \varphi$ to $B(x_0, r_0)$ attains its strong minimum at x_0 ($\varphi'(x_0)$ has not been changed).

Choose $0 < r < \inf\{\varepsilon, r_0\}$ such that $u_1 - \varphi$ is bounded below on the closed ball $B(x_0, r)$ centered at x_0 and of radius r, u_2 is uniformly continuous on $B(x_0, r)$ and

$$\begin{array}{lll} |\varphi(y) - \varphi(x_0)| < \varepsilon/3 & \|\varphi'(y) - \varphi'(x_0)\| < \varepsilon/3 & \\ u_1(y) - u_1(x_0) > -\varepsilon & \text{and} \qquad |u_2(y) - u_2(x_0)| < \varepsilon & \\ \text{whenever} \quad \|y - x_0\| < r & & \end{array} \tag{2.3}$$

Since the restriction of $u_1 + u_2 - \varphi$ to $B(x_0, r)$ has a strong minimum at x_0 there exists $0 < r_1 < r$ such that:

$$\|y - x_0\| < r/2 \text{ whenever } (u_1 + u_2 - \varphi)(y) \leq (u_1 + u_2 - \varphi)(x_0) + r_1 \text{ and } \|y - x_0\| \leq r \tag{2.4}$$

Finally, using the fact that u_2 is uniformly continuous on $B(x_0,r)$, there exists $0 < r_2 < r/2$ such that:

$$|u_2(y) - u_2(z)| < r_1/3 \quad \text{whenever} \quad y, z \in B(x_0,r) \quad \text{and} \quad \|y - z\| < r_2 \tag{2.5}$$

By a construction of Leduc (see for instance [4] Proposition II-5-1), there exists a function $d : X \longrightarrow \mathbb{R}$ satisfying:

(a) d is Lipschitz continuous and $\mathcal{C}^1$ on $X\backslash\{0\}$
(b) $d(x) \geq \|x\|$

For each $n \geq 1$ we set:

$$w_n(x,y) = \begin{cases} u_1(x) - \varphi(x) + u_2(y) + nd^2(x-y) & \text{if} \quad \|x - x_0\| \leq r \text{ and } \|y - x_0\| \leq r \\ +\infty & \text{otherwise} \end{cases}$$

If $b : X \longrightarrow \mathbb{R}$ is a $\mathcal{C}^1$ Lipschitz continuous bump on X, then b is bounded and $B(x,y) = b(x)b(y)$ defines a $\mathcal{C}^1$ Lipschitz continuous bump on $X \times X$. The function w_n is lower semicontinuous and bounded below on $X \times X$. According to the smooth variational principle, there exists a $\mathcal{C}^1$ function $g : X \times X \longrightarrow \mathbb{R}$ such that $\|g\|_\infty < r_1/3$, $\|g'\|_\infty < \varepsilon/3$ and $w_n + g$ attains its minimum at some point $(x_{1,n}, x_{2,n}) \in B(x_0,r) \times B(x_0,r)$. Consequently, for all $z \in X \quad (w_n + g)(z,z) \geq (w_n + g)(x_{1,n}, x_{2,n})$.
In particular, when $z = x_0$:

$$\begin{aligned} n\|x_{1,n} - x_{2,n}\|^2 &\leq nd^2(x_{1,n} - x_{2,n}) \\ &\leq u_1(x_0) + u_2(x_0) - \varphi(x_0) - u_1(x_{1,n}) - u_2(x_{2,n}) + \varphi(x_{1,n}) + 2r_1/3 \end{aligned} \tag{2.6}$$

The right hand side of the above inequality is bounded above for $(x_{1,n}, x_{2,n}) \in B(x_0,r) \times B(x_0,r)$, so if we choose n large enough, we have

$$\|x_{1,n} - x_{2,n}\| < r_2 \tag{2.7}$$

So, by (2.5), $|u_2(x_{1,n}) - u_2(x_{2,n})| < r_1/3$ and, using again (2.6):

$$u_1(x_{1,n}) + u_2(x_{1,n}) - \varphi(x_{1,n}) \leq u_1(x_0) + u_2(x_0) - \varphi(x_0) + r_1$$

By (2.4), $\|x_{1,n} - x_0\| < r/2$, and using (2.7), $\|x_{2,n} - x_0\| < r$. Let us denote $g'(x_{1,n}, x_{2,n}) := (q_1, q_2)$. By hypothesis, $\|q_1\| < \varepsilon$ and $\|q_2\| < \varepsilon$. If we fix $x = x_{1,n}$, we see that the function $y \in X \longrightarrow u_1(x_{1,n}) + u_2(y) - \varphi(x_{1,n}) + nd^2(x_{1,n} - y) + g(x_{1,n}, y)$ has a local minimum at $x_{2,n}$, so $p_2 := 2nd(x_{1,n} - x_{2,n})d'(x_{1,n} - x_{2,n}) - q_2 \in D^- u_2(x_{2,n})$. Similarly, if we fix $y = x_{2,n}$, we see that the function $x \in X \longrightarrow u_1(x) + u_2(x_{2,n}) - \varphi(x) + nd^2(x - x_{2,n}) + g(x, x_{2,n})$ has a local minimum at $x_{1,n}$, so $p_1 := \varphi'(x_{1,n}) - 2nd(x_{1,n} - x_{2,n})d'(x_{1,n} - x_{2,n}) - q_1 \in D^- u_1(x_{1,n})$. Consequently,

$$\begin{aligned} \|p_1 + p_2 - p\| &= \| -q_1 - q_2 + \varphi'(x_{1,n}) - \varphi'(x_0)\| \\ &\leq \|q_1\| + \|q_2\| + \|\varphi'(x_{1,n}) - \varphi'(x_0)\| \quad \leq \quad \varepsilon \end{aligned}$$

Since $r < \varepsilon$, it follows from the above discussion that $\|x_{1,n} - x_0\| < \varepsilon$ and $\|x_{2,n} - x_0\| < \varepsilon$. Using (2.3), we also have $|u_2(x_{2,n}) - u_2(x_0)| < \varepsilon$ and $u_1(x_{1,n}) - u_1(x_0) > -\varepsilon$. Finally, it follows from (2.6) that $u_1(x_{1,n})-u_1(x_0) \leq u_2(x_0)-u_2(x_{2,n})+\varphi(x_{1,n})-\varphi(x_0)+2r_1/3 \leq 2\varepsilon$, and this completes the proof. $\square$

Remark 2.6. It is possible to prove a formula for the subdifferential of the sum for weaker forms of differentiability. If β is a bornology on X and φ is a real valued function on X, we say that φ is β-differentiable at $x_0 \in X$ with β-derivative $\varphi'(x_0) = p \in X^*$ if

$$\lim_{t \longrightarrow 0} t^{-1}\Big(f(x_0+th) - f(x_0) - \langle p, th\rangle\Big) = 0$$

uniformly for h in the elements of β. We denote by τ_β the topology on X^* of uniform convergence on the elements of β. When β is the class of all bounded subsets (resp. all singletons) of X, the β-differentiability coincides with the usual Fréchet-differentiability (resp. Gâteaux differentiability), and τ_β coincides with the norm (resp. weak*) topology on X^*. Finally, if f is a real valued function on X, the β-subdifferential of f at some point $x_0 \in X$ is the set:

$D^-_\beta f(x_0) = \{\varphi'(x_0); \varphi : X \longrightarrow \mathbb{R}$ is β-differentiable, $f - \varphi$ has a local minimum at x_0 and φ' is norm to τ_β continuous$\}$

The following result is a straightforward adaptation of Theorem 2.1 and we omit the proof.

Proposition 2.7. *Let X be a Banach space such that there exists a Lipschitz continuous and β-differentiable bump function b on X such that b' is norm to τ_β continuous. Let u_1, u_2 be two real valued functions defined on X such that u_1 is lower semicontinuous and u_2 is uniformly continuous. Suppose that $x_0 \in X$ and $p \in D^-_\beta(u_1+u_2)(x_0)$ are given. Then, for each $\varepsilon > 0$ and each τ_β neighbourhood V of p, there exist x_1, $x_2 \in X$, $p_1 \in D^-_\beta u_1(x_1)$ and $p_2 \in D^-_\beta u_2(x_2)$ such that:*

(1) $\|x_1 - x_0\| < \varepsilon$ *and* $\|x_2 - x_0\| < \varepsilon$
(2) $|u_1(x_1) - u_1(x_0)| < \varepsilon$ *and* $|u_2(x_2) - u_2(x_0)| < \varepsilon$
(3) $p_1 + p_2 \in V$

3. Application to Hamilton-Jacobi equations in infinite dimensions

Let X be a Banach space and $H : X \times X^* \longrightarrow \mathbb{R}$ be *uniformly continuous.* In this section, we are interested in the uniqueness of a bounded uniformly continuous viscosity solution $u : X \longrightarrow \mathbb{R}$ of the equation:

$$u + H(x, Du) = 0 \tag{3.1}$$

Let us first recall some definitions:

Definition 3.1. A function $u : X \longrightarrow \mathbb{R}$ is a viscosity subsolution of (3.1) if u is upper semicontinuous and, for every $x \in X$ and every $p \in D^+u(x)$:

$$u(x) + H(x,p) \leq 0$$

The function u is a viscosity supersolution of (3.1) if u is lower semicontinuous and, for every $x \in X$ and every $p \in D^-u(x)$:

$$u(x) + H(x,p) \geq 0$$

Finally, u is a viscosity solution of (3.1) if u is both a viscosity subsolution and a viscosity supersolution of (3.1).

Note that a viscosity solution of (3.1) is a continuous function on X. The notion of viscosity solution of (3.1) has been introduced by M. G. Crandall and P.-L. Lions in [15] (see also the recent paper of F. H. Clarke and Y. S. Ledyaev [16] for equivalent définitions). Uniqueness of viscosity solutions of Hamilton-Jacobi equations was first proved in finite dimensions in [15]. The theory of viscosity solutions of Hamilton-Jacobi equations in infinite dimensions has been developed by M. G. Crandall and P.-L. Lions in a series of papers [17]. Our purpose here is to show that the results of the former section yield a very simple proof of the uniqueness of a bounded uniformly continuous viscosity solution of (3.1).

Proposition 3.2. *Let u, v be two real valued bounded uniformly continuous functions defined on X. Assume that X admits a $\mathcal{C}^1$ Lipschitz bump function. If u is a viscosity subsolution of* (3.1) *and if v is a viscosity supersolution of* (3.1)*, then $u \leq v$.*

Actually, let us prove the following stronger result:

Proposition 3.3. *Assume that X admits a $\mathcal{C}^1$ Lipschitz bump function. Let u, v be two real valued bounded uniformly continuous functions defined on X, H_1 and H_2 be two uniformly continuous functions from $X \times X^*$ into $\mathbb{R}$. If u is a viscosity subsolution of $u + H_1(x, Du) = 0$ and v is a viscosity supersolution of $u + H_2(x, Du) = 0$, then*

$$\inf_X (v-u) \geq \inf_{X \times X^*} (H_1 - H_2)$$

Proof. Let us fix $\varepsilon > 0$. The function $v - u$ is uniformly continuous and bounded below. According to the smooth variational principle, there exists $x_0 \in X$ and $p \in D^-(v-u)(x_0)$ such that $\|p\| < \varepsilon$ and $(v-u)(x_0) < \inf_X (v-u) + \varepsilon$. Applying Theorem 2.1 with $u_1 = v$ and $u_2 = -u$, there exists $x_1, x_2 \in X$, $p_1 \in D^+u(x_1)$ and $p_2 \in D^-v(x_2)$ satisfying:

(1) $\|x_1 - x_0\| < \varepsilon$ and $\|x_2 - x_0\| < \varepsilon$
(2) $|u(x_1) - u(x_0)| < \varepsilon$ and $|v(x_2) - v(x_0)| < \varepsilon$
(3) $\|p_2 - p_1 - p\| < \varepsilon$

Since u and v are viscosity subsolution and supersolution of $u + H_1(x, Du) = 0$ and $v + H_2(x, Dv) = 0$ respectively one has $u(x_1) + H_1(x_1, p_1) \leq 0$ and $v(x_2) + H_2(x_2, p_2) \geq 0$. Consequently

$$\begin{aligned} \inf_X (v-u) &> (v-u)(x_0) - \varepsilon \\ &> v(x_2) - u(x_1) - 3\varepsilon \\ &\geq H_1(x_1, p_1) - H_2(x_2, p_2) - 3\varepsilon \end{aligned}$$

Moreover, $\|x_1 - x_2\| \leq \|x_1 - x_0\| + \|x_0 - x_2\| < 2\varepsilon$ and $\|p_1 - p_2\| \leq \|p_2 - p_1 - p\| + \|p\| < 2\varepsilon$. Using the uniform continuity of H_1 and H_2 and sending ε to zero, we get:

$$\inf_X (v-u) \geq \inf_{X \times X^*} (H_1 - H_2)$$

□

Remark 3.4. (a) A formula for the viscosity subdifferential of the sum of two arbitrary lower semicontinuous functions on X is available only in finite dimensions. In this setting, it is even possible to obtain a formula for the second order subdifferential of the sum of two lower semicontinuous functions (This formula can be derived from the results presented in [18]. We shall discuss it with applications in part II, see also [19]). However, as noted by H. Ishii [20], it is possible to obtain comparison results between lower semicontinuous viscosity supersolutions of (3.1) and upper semicontinuous viscosity subsolutions of Hamilton-Jacobi equations in infinite dimensions.

(b) Our formula does not yield a proof of uniqueness in all cases. For instance, it is known that the equation $u + H(Du) = f$, where H is uniformly continuous on bounded sets of X^* and f is bounded uniformly continuous on X, has a unique bounded viscosity solution (see the book of Barles [21]), but this cannot be obtained by our formula.

We conclude this section by studying the problem of singlevaluedness of the Hamilton-Jacobi operator. Let X be a Banach space and $H : X^* \longrightarrow \mathbb{R}$ be a uniformly continuous function. We denote by $UC(X)$ the space of uniformly continuous functions defined on X. Consider the operator $A_H : UC(X) \longrightarrow UC(X)$ defined by:

$$A_H u = \{ f \in UC(X)\,;\, f = H(Du) \text{ in the viscosity sense}\}$$

The problem of the singlevaluedness of A_H in finite dimensions has been studied by Evans [22] and Frankowska [23]. We prove here:

Theorem 3.5. *Let X be a Banach space which admits a $\mathcal{C}^1$ Lipschitz bump function. Then, for every uniformly continuous function H defined on X^*, the associated Hamilton-Jacobi operator A_H is singlevalued.*
On the other hand, if X is not an Asplund space, then for every uniformly continuous function H on X^*, the associated Hamilton-Jacobi operator A_H is not singlevalued.

Proof. Let us assume that X admits a $\mathcal{C}^1$ Lipschitz bump function, that H is uniformly continuous on X^* and that $H(Du) = f$ and $H(Du) = g$ in the viscosity sense, where f and g are two continuous real vaued functions defined on X. We want to prove that $f = g$. Let $x \in X$ and $\varepsilon > 0$ be fixed. According to corollary II-2, there exists $p^- \in D^-u(x_1)$ and $p^+ \in D^+u(x_2)$ such that:
(1) $\|x_1 - x\| < \varepsilon$ and $\|x_2 - x\| < \varepsilon$
(2) $\|p^- - p^+\| < \varepsilon$
Since u is a viscosity subsolution of $H(Du) = f$, we have $H(p^+) \le f(x_2)$ and since u is a viscosity supersolution of $H(Du) = g$, we have $H(p^-) \ge g(x_1)$. Consequently, $g(x_1) - f(x_2) \le H(p^-) - H(p^+)$. Using (1), (2), the continuity of f and g at x and the uniform continuity of H, we obtain, as ε goes to 0:

$$g(x) - f(x) \le 0$$

Similarly, $g(x) - f(x) \ge 0$. This is true for all $x \in X$, so $f = g$.
Conversely, let us assume that X is not an Asplund space. According to [7], Theorem I-5-3, there exists an equivalent nowhere differentiable norm $u : X \longrightarrow \mathbb{R}$ such that $u(x) \le \|x\|$.

The function u is Lipschitz continuous on X and $D^+u(x) = \emptyset$ for all $x \in X$. Let us denote by $M = \inf\{|H(p)|\,;\, \|p\| \leq 1\}$. For every continuous function f on X such that $f \leq M$, we have $H(Du) = f$ in the viscosity sense: indeed, we trivially have $H(p) \leq f(x)$ for all $x \in X$ and all $p \in D^+u(x)$ (since there is nothing to prove!). Moreover, for all $x \in X$ and all $p \in D^-u(x)$ one has $\|p\| \leq 1$, so $H(p) \geq M \geq f(x)$. This proves that A_H is not singlevalued. □

Remark 3.6. H. Frankowska has proved that A_H is singlevalued when $\dim X = 1$ and H is only assumed continuous, and Evans has proved that A_H is singlevalued when $\dim X$ is finite and H is uniformly continuous. We recall that the problem of the singlevaluedness of A_H when $\dim X$ is finite and H is only supposed continuous is open. H. Frankowska has observed that the problem of finding a continuous Hamiltonian H such that the associated operator A_H is not singlevalued is equivalent to the problem of finding a uniformly continuous function on X such that the norm closures of $\bigcup\{D^-u(x)\,;\, x \in X\}$ and of $\bigcup\{D^+u(x)\,;\, x \in X\}$ do not intersect. (So A_H is singlevalued when $\dim X = 1$ and H is only assumed continuous follows from Remark 2.3 (1)).

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