Bounds for Anti-Distance

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Given two matrices $A, B$, we give an upper and a lower bound for the anti-distance $\sup_{U} \|A - U^*BU\|_\infty$ where $U$ runs over the set of unitary matrices.

Let $\mathbb{M}_n$ be the space of $n \times n$ complex matrices, and $\mathbb{U}_n$ the group of unitary matrices. For each $B \in \mathbb{M}_n$, the set $\{U^*BU : U \in \mathbb{U}_n\}$ is the unitary similarity orbit through $B$. Thus $\sup_{U \in \mathbb{U}_n} \|A - U^*BU\|_\infty$ is considered as the anti-distance from $A$ to the orbit with respect to the spectral norm $\cdot \|\cdot\|_\infty$. In this note we give an upper and a lower bound for the anti-distance. Recall that

$$\|A\|_\infty \equiv \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |y^*Ax|,$$

where $y^*$ is the conjugate transposed (row) vector of $y$.

**Theorem 1.**

$$\sup_{U \in \mathbb{U}_n} \|A - U^*BU\|_\infty \leq \sqrt{2}\|A \otimes I - I \otimes B^T\|_\infty. \quad (1)$$

**Proof.** Since for any $U \in \mathbb{U}_n$

$$\|A \otimes I - I \otimes B^T\|_\infty = \|A \otimes I - I \otimes (U^*BU)^T\|_\infty$$

it suffices to prove

$$\|A - B\|_\infty \leq \sqrt{2}\|A \otimes I - I \otimes B^T\|_\infty, \quad (2)$$

or equivalently for any (column) vectors $x, y$

$$|y^*(A - B)x| \leq \sqrt{2}\|A \otimes I - I \otimes B^T\|_\infty \cdot \|x\| \cdot \|y\|. \quad (3)$$

Remark that

$$y^*(A - B)x = \text{tr}((Ax) \cdot y^* - x \cdot (B^*y)^*).$$

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Since the matrix \((Ax - y^*(B^*y)^*\) is of rank \(\leq 2\), the number of its non-zero singular values \(s_1 \geq s_2 \geq \cdots \geq s_n(\geq 0)\) is 2 at most. Therefore

\[
|y^*(A - B)x| \leq \|Ax - y^*(B^*y)^*\|_1 \\
= s_1 + s_2 \\
\leq \sqrt{2}(s_1^2 + s_2^2)^{1/2} \\
= \sqrt{2}\|(Ax) \cdot y^* - x \cdot (B^*y)^*\|_2,
\]  

(3)

where \(\| \cdot \|_1\) and \(\| \cdot \|_2\) denote the trace norm and the Hilbert–Schmidt norm respectively;

\[
\|X\|_1 \equiv \text{tr}(|X|) \quad \text{and} \quad \|X\|_2 \equiv (\text{tr}(|X^2|))^{1/2},
\]

with \(|X|\), the positive semidefinite square root of \(X^*X\).

Consider the linear map \(\Delta_{A,B}\) on the Hilbert space \((\mathbb{M}_n, \| \cdot \|_2)\), defined by

\[
\Delta_{A,B}(X) = AX - XB.
\]  

(4)

Then clearly we have

\[
(Ax) \cdot y^* - x \cdot (B^*y)^* = \Delta_{A,B}(x \cdot y^*).\]  

(5)

On the other hand, since \(\|A \otimes I - I \otimes B^T\|_\infty\) coincides with the operator (spectral) norm of \(\Delta_{A,B}\) on \((\mathbb{M}_n, \| \cdot \|_2)\), it follows from (3) and (5) that

\[
|y^*(A - B)x| \leq \sqrt{2}\|A \otimes I - I \otimes B^T\|_\infty \cdot \|x \cdot y^*\|_2 \\
= \sqrt{2}\|A \otimes I - I \otimes B^T\|_\infty \cdot \|x\| \cdot \|y\|.
\]

The following special case was proved by Omladic and Semrl [2] by a different method.

**Corollary 2.** If \(A, B\) are normal matrices with eigenvalues \(\{\alpha_1, \cdots, \alpha_n\}\) and \(\{\beta_1, \cdots, \beta_n\}\) respectively, then

\[
\sup_{U \in \mathbb{U}_n} \|A - U^*BU\|_\infty \leq \sqrt{2}\max_{i,j}|\alpha_i - \beta_j|.
\]  

(6)

**Proof.** Remark that for general \(A, B\), the set of eigenvalues of \(A \otimes I - I \otimes B^T\) coincides with \(\{\alpha_i - \beta_j : i, j = 1, 2, \cdots, n\}\). If \(A\) and \(B\) are normal, so is \(A \otimes I - I \otimes B^T\), hence

\[
\|A \otimes I - I \otimes B^T\|_\infty = \max_{i,j}|\alpha_i - \beta_j|.
\]  

(7)

Now the assertion follows from (1) and (7).

\[\square\]

Again in view of (7), for normal \(A, B\) the inequality

\[
\|A \otimes I - I \otimes B^T\|_\infty \leq \sup_{U \in \mathbb{U}_n} \|A - U^*BU\|_\infty
\]

is quite trivial. Let us show that this holds in general.
Theorem 3.
\[ \| A \otimes I - I \otimes B^T \|_\infty \leq \sup_{U \in U_n} \| A - U^* B U \|_\infty. \] (8)

Proof. Let us consider the operator \( \Delta_{A,B} \), defined by (4), on each of the Schatten \( p \)-space \((M_n, \| \cdot \|_p)\) \((1 \leq p \leq \infty)\), where
\[ \| X \|_p \equiv (\text{tr}(|X|^p))^{1/p}. \]

The operator norm of \( \Delta_{A,B} \) on \((M_n, \| \cdot \|_p)\) will be denoted by \( \| \Delta_{A,B} \|_{p \rightarrow p} \). We used already the fact
\[ \| \Delta_{A,B} \|_{2 \rightarrow 2} = \| A \otimes I - I \otimes B^T \|_\infty. \] (9)

Since \((M_n, \| \cdot \|_p)_{1 \leq p \leq \infty}\) is an interpolation scale (see [1] Chap. III), it follows from the general theory that
\[ \| \Delta_{A,B} \|_{2 \rightarrow 2} \leq \sqrt{\| \Delta_{A,B} \|_{1 \rightarrow 1} \cdot \| \Delta_{A,B} \|_{\infty \rightarrow \infty}}. \] (10)

By (9) and (10), for (8) it suffices to prove that
\[ \| \Delta_{A,B} \|_{1 \rightarrow 1} = \| \Delta_{A,B} \|_{\infty \rightarrow \infty} = \sup_{U \in U_n} \| A - U^* B U \|_\infty. \] (11)

First, since the unit ball of \((M_n, \| \cdot \|_\infty)\) is the convex hull of \(U_n\),
\[ \| \Delta_{A,B} \|_{\infty \rightarrow \infty} = \sup_{U \in U_n} \| U A - B U \|_\infty \]
\[ = \sup_{U \in U_n} \| A - U^* B U \|_\infty. \]

Next
\[ \sup_{U \in U_n} \| U A - B U \|_\infty = \sup_{\| T \|_1 \leq 1} \sup_{U \in U_n} |\text{tr}(U A - T B U)| \]
\[ = \sup_{\| T \|_1 \leq 1} \sup_{U \in U_n} |\text{tr}(U \cdot \Delta_{A,B}(T))| \]
\[ = \sup_{\| T \|_1 \leq 1} \| \Delta_{A,B}(T) \|_1 = \| \Delta_{A,B} \|_{1 \rightarrow 1}. \]

These establish (11) and complete the proof. \(\square\)

References