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An Existence Result for a Class of Non Convex Problems of the Calculus of Variations

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We consider the functional

$$\int_\Omega [h(\gamma_K(\nabla u(x))) + u(x)]\, dx \qquad u(x) \in W_0^{1,1}(\Omega)$$

where γ_K is the gauge function of a convex set K and $h : [0,\infty[\,\to [0,\infty]$ is a possibly non convex function. In the case $K \subset \mathbb{R}^2$ is a closed polytope and $\Omega \subset \mathbb{R}^2$ is a bounded convex set we provide a sufficient condition for the existence of the minimum.
Besides, as a corollary, we give conditions on $\Omega \subset \mathbb{R}^2$ and $f : \mathbb{R}^2 \to [0,\infty]$ that are sufficient to the existence of a minimizer of

$$\int_\Omega [f(\nabla u(x)) + u(x)]\, dx \qquad u(x) \in W_0^{1,1}(\Omega).$$

1. Introduction

Cellina has recently proved an existence result for functionals of the type

$$\int_\Omega [h(\|\nabla u(x)\|) + u(x)]\, dx \qquad u(x) \in W_0^{1,1}(\Omega)$$

with no convexity assumptions (see [4]) on the function h. The first paper dealing with functionals of this type is a paper by Kawohl, Stara and Wittum [16] on a problem of shape optimization. They consider the case in which Ω is a two dimensional square and they prove that the minimum problem has no solutions.

It is well known that, when the convexity is not assumed, the limit of a minimizing sequence is not always a solution of the minimum problem. Then, in many cases, to obtain existence results one has to provide a construction yielding the solution.

Several authors [2], [6], [7], [8], [18], used this approach to study functionals depending only on the gradient. The technique they developed is the following: they solve the problem locally and, then, using covering arguments, they *build* a solution of the minimum problem. Simple examples show that this technique is not useful when the function depends both on ∇u and on u.

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The problem considered in [4] is the minimization problem stated above, where $h : [0,\infty[\to [0,\infty]$ is a lower semicontinuous function and Ω is any bounded open convex set of $\mathbb{R}^2$ with piecewise smooth boundary. The result presented in [4] states that if the set Ω is *not too large* with respect to a property of the function h, a solution to the problem does exist. In this case the solution is built *without* passing through a *covering argument.*

In this paper we make a first attempt to consider the more general functional

$$\int_\Omega [h(\gamma_K(\nabla u(x))) + u(x)]\, dx \qquad u(x) \in W_0^{1,1}(\Omega)$$

where γ_K is the gauge function of a convex set K.

We obtain an existence result in the case $K \subset \mathbb{R}^2$ is a polytope and $\Omega \subset \mathbb{R}^2$ is *not too large* with respect to a property that involves both the function h and the set K. This property is of the same type of the property presented in [4]. We want to underline that, due to the hypothesis on K, *no regularity assumption* is required on the boundary of Ω.

Besides, as a Corollary, we present an analogous existence result for the functional

$$\int_\Omega [f(\nabla u(x)) + u(x)]dx \qquad u(x) \in W_0^{1,1}(\Omega)$$

where $f : \mathbb{R}^2 \to [0,+\infty]$ is a lower semicontinuous function that vanishes on the boundary of a polytope K.

2. Preliminaries, notations and basic assumptions

Given a set $\mathcal{A}$ we denote by $C(\mathcal{A})$ its complement, by $\operatorname{int}(\mathcal{A})$ its interior, by $\overline{\mathcal{A}}$ its closure, and by $\partial\mathcal{A}$ its boundary. Given a convex set $C \subset \mathbb{R}^n$, we denote by C° the polar set of C, by $\operatorname{extr} C$ the set of the extremal points of C, by $\operatorname{ri}(C)$ the relative interior of C. The gauge function of C will be denoted by $\gamma_C(\cdot)$.

For every locally lipschitz convex function $f : \mathbb{R}^n \to \mathbb{R}$, let $\partial f(x)$ be the subgradient of f at x.

Following [1] and [9], given a point $x \in \mathbb{R}^n$ we set $d_C(x) = \inf\{\, |x-y| \;:\; y \in C\,\}$. We define the tangent cone to C at x as

$$T_C(x) = \{\, v \in \mathbb{R}^n \;:\; \lim_{t\to 0^+} \frac{d_C(x+tv) - d_C(x)}{t} = 0\,\} \tag{2.1}$$

and the normal cone to C at x as

$$N_C(x) = \{\, y \in \mathbb{R}^n \;:\; \langle y, v\rangle \le 0 \;\forall\, v \in T_C(x)\,\}. \tag{2.2}$$

The sets $T_C(x)$ and $N_C(x)$ are closed convex cones in $\mathbb{R}^n$ and $T_C(x) \cap N_C(x) = \{0\}$. In addition, for C is convex, $N_C(x)$ coincides with the cone of normals to C at x in the sense of convex analysis, namely

$$N_C(x) = \{\, \xi \in \mathbb{R}^n \;:\; \langle y - x, \xi\rangle \le 0 \;\forall y \in C\,\} \tag{2.3}$$

(see [9, proposition 2.4.4]).

We consider the problem

$$\textit{minimize} \qquad \int_\Omega [h(\gamma_K(\nabla u(x))) + u(x)]\, dx \qquad u(x) \in W_0^{1,1}(\Omega). \tag{$\mathcal{P}$}$$

Let us suppose that Ω is any bounded, open convex set contained in $\mathbb{R}^2$, K is a closed polytope of $\mathbb{R}^2$ such that $0 \in \operatorname{int}(K)$ and, according to the notations introduced above, $\gamma_K : \mathbb{R}^2 \to \mathbb{R}$ is the gauge function associated to K.

As in [4] the map $h : [0, +\infty) \to [0, +\infty]$ is a non–negative lower semicontinuous extended valued function with minimum value 0. Moreover we suppose that $\sup\{r \geq 0 : h(r) = 0\}$ is finite and we denote it by ρ. Let A be the set of supporting linear functions at ρ, i.e. $A = \{a \in \mathbb{R} : h(s) \geq a(s - \rho),\ \text{for every } s \geq 0\}$. We recall that $0 \in A$ and let $\Lambda = \sup A$.

We define

$$v(x) = \inf_{y \in \partial\Omega} \sup_{x^* \in -K} \langle x - y, x^* \rangle \tag{2.4}$$

and the width of Ω w.r.t. K to be $W_{(\Omega,K)} = \sup_{x \in \Omega} v(x)$.

By the hypothesis on K, the set $\operatorname{extr} K$ contains finitely many vectors. We fix one of them and we denote it by k_1; we denote the others by k_i, assuming that the index i is increasing when we move counter-clockwise from k_1. Then $\operatorname{extr} K = \{k_1, \ldots, k_n\}$. To simplify the notations, we define $k_{n+1} = k_1$. For $i \in \{1, \ldots, n\}$ let l_i be the set $\{k \in \partial K : \exists \lambda \in [0, 1] \text{ such that } k = \lambda k_i + (1 - \lambda) k_{i+1}\}$.

The Corollary 19.2.2 [20] imply that $\operatorname{extr} K^\circ$ contains exactly n vectors. Applying Corollary 23.5.3 [20] we can check that, for every x belonging to the interior of the convex cone generated by k_i and k_{i+1}, $\partial\gamma_K(x)$ contains exactly one vector, we denote it by ξ_i, and $\xi_i \in \operatorname{extr} K^\circ$. With respect to the notations introduced we have that $\xi_n = \partial\gamma_K(x)$ for every x in the interior of the convex cone generated by k_n and k_1. As before, we set $\xi_{n+1} = \xi_1$. For $i \in \{1, \ldots, n\}$ let ζ_i be the set $\{\xi \in \partial K^\circ : \exists \lambda \in [0, 1] \text{ such that } \xi = \lambda \xi_i + (1 - \lambda)\xi_{i+1}\}$. Using again Corollary 23.5.3 [20] we have that for every $x = \mu k_i$, $\mu > 0$, $\partial\gamma_K(x) = \zeta_{i-1}$.

We list some properties that will be useful in the following.

(a) $N_{K^\circ}(\xi_i)$ is the closed convex cone generated by k_i and k_{i+1};
(b) $T_{K^\circ}(\xi_i)$ is the closed convex cone generated by $(\xi_{i-1} - \xi_i)$ and $(\xi_{i+1} - \xi_i)$;
(c) for every $j \in \{1, \ldots, n\}$, $(\xi_j - \xi_i) \in T_{K^\circ}(\xi_i)$;
(d) for every $\xi \in \operatorname{ri}(\zeta_j)$, $N_{K^\circ}(\xi) = \{\lambda k_{j+1}; \lambda \geq 0\}$;
(e) for every $\xi \in \operatorname{ri}(\zeta_j)$, $T_{K^\circ}(\xi) = \{x \in \mathbb{R}^2 : \langle k_{j+1}, x \rangle \leq 0\}$;
(f) for every $\xi \in \operatorname{ri}(\zeta_j)$, for every $i \in \{1, \ldots, n\}$, $(\xi_i - \xi) \in T_{K^\circ}(\xi)$;
(g) for every $i \in \{1, \ldots, n\}$, $\langle k_i, \xi_{i-1} \rangle = \langle k_i, \xi_i \rangle > 0$.

3. Preliminary results

In this section we study the properties of the function $v(x)$ defined by (2.4). These properties will be useful to prove the existence theorem of the following section.

In [18] it has been proved that the function defined in (2.4) belongs to the Sobolev space $W_0^{1,\infty}(\Omega, \mathbb{R})$ and satisfies $\nabla v(x) \in \partial(-K)$ for almost every $x \in \Omega$.

Here we remark that, for every convex set C the function $\sup_{x^* \in C} \langle \cdot, x^* \rangle$, defined in $\mathbb{R}^n$, is the conjugate of the indicator function of C. This function is also said the support function of the set C. When C is a closed convex set containing the origin, Theorem 14.5 [20] says that this function is the gauge function of C°, then $\sup_{x^*\in C}\langle\cdot,x^*\rangle = \gamma_{C^\circ}(\cdot)$ and we have that $\{x\in\mathbb{R}^n : \gamma_{C^\circ}(x)\le\rho\} = \rho C^\circ$.

From now on, we suppose that $K\subset\mathbb{R}^2$ is a closed polytope, $0\in\operatorname{int}K$, and that $\Omega\subset\mathbb{R}^2$ is an open bounded convex set.

Lemma 3.1. *Let x be a point in Ω such that $v(x)=c$. Then $x+cK^\circ\subset\overline{\Omega}$.*
Moreover there exist $y\in\partial\Omega$ and $\xi_i\in\operatorname{extr}K^\circ$ such that $\gamma_{-K}(x-y)=v(x)=c$ and $x=y-c\xi_i$.

Proof. The boundedness assumption on K implies that $0\in\operatorname{int}(-K^\circ)$ (Corollary 14.5.1 [20]), and then $\gamma_{-K^\circ}(\cdot)$ is a convex function finite on $\mathbb{R}^2$. For this reason we can say that $\gamma_{-K^\circ}(\cdot)$ is continuous and that there exists $y\in\partial\Omega$ such that $\gamma_{-K^\circ}(x-y)=\inf_{z\in\partial\Omega}\gamma_{-K^\circ}(x-z)=c$. It is equivalent to say that, for any $z\in\partial\Omega$, $x-z\in\mathcal{C}\{y\in\mathbb{R}^n:\gamma_{-K^\circ}(y)<c\}$ and there exists $y\in\partial\Omega$ such that $x-y\in\partial\{z\in\mathbb{R}^n:\gamma_{-K}(z)\le c\}$, i.e. $y\in\partial(x+cK^\circ)$ and $x+cK^\circ\subset\overline{\Omega}$. Now, arguing by contradiction, let us suppose that for any $y\in\partial\Omega$ such that $\gamma_{-K}(x-y)=v(x)=c$ we have that $y\notin\operatorname{extr}(x+cK^\circ)$. If this is the case we have that there exist $z_1,z_2\in\operatorname{extr}(x+cK^\circ)\subset\Omega$ and $\lambda\in(0,1)$ such that $y=\lambda z_1+(1-\lambda)z_2$. By the convexity of Ω, we get $y\in\Omega$, that is a contradiction. □

Lemma 3.2. *For every $c>0$, we have that $\{v(x)\ge c\}=\Omega\cap(\cap_{i=1,\dots,n}(\overline{\Omega}-c\xi_i))$.*

Proof. When $x\in\Omega\cap(\cap_{i=1,\dots,n}(\overline{\Omega}-c\xi_i))$ we have that, for every $i\in\{1,\dots,n\}$, there exists $y_i\in\overline{\Omega}$ such that $x=y_i-c\xi_i$. Remarking that $\overline{\operatorname{co}}\{x+c\xi_i; i=1,\dots,n\}=x+cK^\circ$, we have that $x+cK^\circ\subset\overline{\Omega}$ and then $\min_{z\in\partial\Omega}\gamma_{-K^\circ}(x-z)=v(x)\ge c$.

On the other hand if we suppose that $v(x)\ge c$ we have that $\gamma_{-K^\circ}(x-z)\ge c$ for every $z\in\partial\Omega$ and then $x+cK^\circ\subset\overline{\Omega}$. If we choose $y_i=x+c\xi_i\in\operatorname{extr}(x+cK^\circ)$ we get that $x\in\Omega\cap(\cap_{i=1,\dots,n}(\overline{\Omega}-c\xi_i))$. □

Let us define the following subsets of $\partial\Omega$:

$$\begin{aligned} I_i &= \{y\in\partial\Omega : k_i\in N_\Omega(y)\} \\ J_i &= \{y\in\partial\Omega\setminus(\cup_{j=1,\dots,n}I_j) : \exists\lambda\in(0,1) \text{ such that } \lambda k_i+(1-\lambda)k_{i+1}\in N_\Omega(y)\} \end{aligned}$$

Definition 3.3. We fix y and z on $\partial\Omega$, $y\ne z$. $\partial\Omega$ is divided in two arcs. We say that $x\ne y$, $x\ne z$ is *between* y and z, and we write $y\prec x\prec z$, if x belongs to the arc that can be covered moving counter-clockwise from y to z.

Proposition 3.4. *The following properties hold true.*

(i) *For every $k\in\partial K$ there exists $y\in\partial\Omega$ such that $k\in N_\Omega(y)$.*
(ii) $\cup_{i=1,\dots,n}(I_i\cup J_i)=\partial\Omega$.
(iii) *For every $i\in\{1,\dots,n\}$ the set I_i is nonempty; moreover either I_i containes exactly one point or it is a line segment.*
(iv) *When $I_i\cap I_{i+1}\ne\emptyset$, then $J_i=\emptyset$.*
(v) *When $I_i\cap I_{i+1}=\emptyset$, we have $J_i=\{x\in\partial\Omega : \forall y\in I_i, \forall z\in I_{i+1}$ it is $y\prec x\prec z\}$.*

Proof. (i) Ω is a bounded open set, then for any $k \in \partial K$ there exists a real number β such that $\langle k, x\rangle < \beta$ for every $x \in \Omega$. Let β^* be the infimum of the set $\{\beta \in \mathbb{R} : \langle k, x\rangle < \beta$ for every $x \in \Omega\}$. The halfspace $H_{\beta^*} = \{a \in \mathbb{R}^n : \langle k, a\rangle > \beta^*\}$ is a convex open set such that $H_{\beta^*} \cap \Omega = \emptyset$. Then the hyperplane $\partial H_{\beta^*}\{a \in \mathbb{R}^n : \langle k, a\rangle = \beta^*\}$ separates properly H_{β^*} and Ω. Moreover we have that there exists $y \in \partial\Omega$ such that $\langle k, y\rangle = \beta^*$, otherwise we can get a contradiction with the definition of β^*. Hence $k \in N_\Omega(y)$.

(ii) It is sufficient to remark that, by the assumption $0 \in \operatorname{int} K$, for every $v \in \mathbb{R}^2 \setminus \{0\}$ there exist $\lambda_v > 0$ and $k \in \partial K$ such that $\lambda_v k = v$.

(iii) I_i is nonempty by virtue of *(i)*. Let us suppose that x_1 and x_2 are two different points in I_i. For every $\lambda \in [0,1]$ we have

$$\begin{aligned}\langle x - (\lambda x_1 + (1-\lambda)x_2), k_i\rangle &= \lambda\langle x - x_1, k_i\rangle + (1-\lambda)\langle x - x_2, k_i\rangle \\ &\leq \max\{\langle x - x_j, k_i\rangle,\ j = 1,2\}.\end{aligned}$$

We remark that, by the definition of normal cone, the last term is less or equal to zero for every $x \in \Omega$, hence $k_i \in N_\Omega(\lambda x_1 + (1-\lambda)x_2)$. Recalling that, for every $y \in \Omega$, $N_\Omega(y) = \{0\}$ we get that, for every $\lambda \in (0,1)$, $\lambda x_1 + (1-\lambda)x_2 \in \partial\Omega$.

(iv) As a trivial consequence of the part *(iii)* of this proposition we have that whenever $I_i \cap I_{i+1} \neq \emptyset$ there exists only one point $y \in I_i \cap I_{i+1}$. By the convexity of the cone $N_\Omega(y)$, for every $\lambda \in (0,1)$, we have $\lambda k_i + (1-\lambda)k_{i+1} \in N_\Omega(y)$. Let λ be in $(0,1)$ and let us suppose that there exists a point $z \neq y$, $z \in \partial\Omega$, such that $\lambda k_i + (1-\lambda)k_{i+1} \in N_\Omega(z)$. Arguing as above we get that the line segment joining y and z is contained in $\partial\Omega$. Moreover we have that

$$0 = \langle \lambda k_i + (1-\lambda)k_{i+1}, z - y\rangle = \lambda\langle k_i, z - y\rangle + (1-\lambda)\langle k_{i+1}, z - y\rangle. \tag{3.1}$$

The last term in (3.1) is less or equal to zero because both k_i and k_{i+1} are in $N_\Omega(y)$. For the same reason if we have $\lambda\langle k_i, z - y\rangle + (1-\lambda)\langle k_{i+1}, z - y\rangle = 0$ we get $\langle k_i, z - y\rangle = \langle k_{i+1}, z - y\rangle = 0$ and then there exists $\mu \in \mathbb{R} \setminus \{0\}$ such that $k_i = \mu k_{i+1}$. This contradicts the fact that $0 \in \operatorname{int} K$.

(v) We fix $y \in I_i$ and $z \in I_{i+1}$. As a first case, we suppose that $\{x \in \partial\Omega : \forall y \in I_i, \forall z \in I_{i+1}$ it is $y \prec x \prec z\} = \{\lambda y + (1-\lambda)z; \lambda \in (0,1)\}$. It is easy to see that there exists $k \in \partial K$ such that for every x in the set considered, $N_\Omega(x) = \{\lambda k; \lambda \geq 0\}$, $k \in N_\Omega(y)$, $k \in N_\Omega(z)$ and the cone $N_\Omega(y) \cup N_\Omega(z)$ contains the convex cone generated by k_i and k_{i+1}. Then $x \in J_i$. In the other case we can proceed as follows. The line joining y and z divides $\mathbb{R}^2$ in two halfplanes. Let H be the one that does not contain x. We define $C = \overline{\text{co}}((H \cap \Omega) \cup x)$. It is $C \subset \Omega$ and $x \in C \cap \overline{\Omega}$ It follows immediately by (2.3) that $N_\Omega(x) \subset N_C(x)$. Moreover, if μ, ν are the vectors in ∂K that generate $N_C(x)$, it is not difficult to check that $\mu \in N_C(y)$ and $\nu \in N_C(z)$, and that μ and ν are contained in the convex cone generated by k_i and k_{i+1}. It remains only to prove that every $x \in J_i$ is *between* y and z for every $y \in I_i$ and $z \in I_{i+1}$. Repeating the same arguments used above, we see that it can not happen that there exists $j \in \{1, \ldots, n\}$, $j \neq i$, such that $y \prec x \prec z$ for $y \in I_j$ and $z \in I_{j+1}$. □

Definition 3.5. For every $x \in \Omega$ we define the following set:

$$\Pi(x) = \{y \in \partial\Omega : \text{ if } v(x) = c \text{ it is } y \in \operatorname{extr}(x + cK^\circ)\}.$$

Remark 3.6. Thanks to Lemma 3.1 and to the definition of the function $v(\cdot)$, $\Pi(x)$ is well defined for every $x \in \Omega$.

Lemma 3.7. *Let $x \in \Omega$ be such that $v(x) = c$ and let $y \in \partial\Omega$ be such that $y \in \Pi(x)$ and $y = x - c\xi_i$. Then there exists $\lambda \in [0, 1]$ such that $\lambda k_i + (1 - \lambda)k_{i+1} \in N_\Omega(y)$.*

Proof. By Lemma 3.1 we have that $x + cK^\circ \subset \overline{\Omega}$ and that $y = x + c\xi_i \in (x + cK^\circ) \cap \overline{\Omega}$. Then $N_\Omega(y) \subset N_{x+cK^\circ}(x + c\xi_i) = N_{K^\circ}(\xi_i)$ and, by *(a)* of Section 2, we get the proof. $\square$

Proposition 3.8. *The following properties hold for every $i \in \{1, \dots, n\}$ such that J_i is nonempty and for every $j \in \{1, \dots, n\}$ for which I_j has nonempty relative interior.*

(i) *Let $x \in \Omega$ and $y \in J_i$ be such that $v(x) = c$, $y \in \Pi(x)$ and $x = y - c\xi_i$. Then, for every $b \in (0, c)$, $z = y - b\xi_i$ is such that $\Pi(z) = y$.*

(ii) *Let ξ be an arbitrarily fixed vector in ζ_{j-1}, let $x \in \Omega$ and $y \in \mathrm{ri}(I_j)$ be such that $v(x) = c$, $x = y - c\xi$ and $\{y - c(\xi - \xi_{j-1}), y - c(\xi - \xi_j)\} \subset \Pi(x)$. Then, for every $b \in (0, c)$, $z = y - b\xi$ is such that $\Pi(z) = \{y - b(\xi - \xi_{j-1}), y - b(\xi - \xi_j)\}$.*

(iii) *For every $y \in J_i$ there exists $c > 0$ such that $\Pi(y - c\xi_i) = y$.*

(iv) *For every $y \in \mathrm{ri}(I_j)$ and for every $\xi \in \zeta_{j-1}$ there exists $c > 0$ such that $\Pi(y - c\xi) = \{y - c(\xi - \xi_{j-1}), y - c(\xi - \xi_j)\}$.*

Moreover, for every $i \in \{1, \dots, n\}$ and for every $y \in I_i \setminus \mathrm{ri}(I_i)$,

(v) *there exists $x \in \Omega$, such that $y \in \Pi(x)$ if and only if $v(x) = c$ and there exist $z \in I_i$ and $\xi \in \zeta_{i-1}$ such that $z \neq y$, $x = z - c\xi$ and either $y = z + c(\xi_{i-1} - \xi)$ or $y = z + c(\xi_i - \xi)$.*

Proof. (i) We recall that by the hypothesys on x we have that $x + cK^\circ \subset \overline{\Omega}$. Remarking that $\mathrm{extr}(x + cK^\circ) = \{y - c(\xi_i - \xi_j); j = 1, \dots, n\}$ we have that, for every $j \in \{1, \dots, n\}$, $y - c(\xi_i - \xi_j) \in \overline{\Omega}$. By (v) of Proposition 3.4, we have that $N_\Omega(y)$ is contained in $\mathrm{int}(N_{K^\circ}(\xi_i))$ and then $\mathrm{int}(T_\Omega(y))$ contains $T_{K^\circ}(\xi_i)$. For this reason and also by property (c) stated in Section 2, we can say that, for every $\lambda \in (0, 1)$ and for every $j \neq i$, $\lambda y + (1 - \lambda)(y - c(\xi_i - \xi_j)) \in \Omega$. Now, choosing $\lambda \in (0, 1)$ such that $b = (1 - \lambda)c$, we have
$y - b\xi_i = \lambda y + (1 - \lambda)(y - c\xi_i)$,
$\mathrm{extr}(y - b\xi_i + bK^\circ) = \{y - b(\xi_i - \xi_j); j = 1, \dots, n\}$,
$y - b(\xi_i - \xi_j) = \lambda y + (1 - \lambda)(y - c(\xi_i - \xi_j)) \in \Omega$ for every $j \neq i$,
and this concludes the proof.

(ii) In this case we observe that
$\mathrm{extr}(y - c\xi + cK^\circ) = \{y - c(\xi - \xi_i); i = 1, \dots, n\}$,
$y - c(\xi - \xi_j)$ and $y - c(\xi - \xi_{j+1})$ belong to I_j;
$y - c(\xi - \xi_i) \in \overline{\Omega} \setminus I_i$ for $i \notin \{j, j + 1\}$;
Then, arguing as above, keeping in mind the property (e) of Section 2, for every $b \in (0, c)$, we have that $y - b(\xi - \xi_i) \in \Omega$ for $i \notin \{j, j + 1\}$ and $y - b(\xi - \xi_i) \in I_i$, for $i \in \{j, j + 1\}$.

(iii) As observed in (i), for every $y \in J_i$, $\mathrm{int}(T_\Omega(y))$ containes $T_{K^\circ}(\xi_i)$. Then, by (c) of Section 2, by the convexity and the boudedness of Ω we can define, for every $j \in \{1, \dots, n\}$, $\lambda_j = \frac{1}{2} \sup\{\lambda \geq 0 : y - \lambda(\xi_i - \xi_j) \in \overline{\Omega}\}$. Now, choosing $c = \min\{\lambda_j; j = 1, \dots, n\}$ we have $\mathrm{extr}(y - c\xi_i + cK^\circ) \setminus \{y\} = \{y - c(\xi_i - \xi_j); j \neq i\} \subset \Omega$.

(iv) Let us consider first the case in which $\xi \in \mathrm{ri}(\zeta_j)$. By Lemma 3.1, we have $N_\Omega(y) = N_{K^\circ}(\xi)$ and $T_\Omega(y) = T_{K^\circ}(\xi)$. Now, we define, for every $i \in \{1,\dots,n\}$, $\lambda_i = \frac{1}{2}\sup\{\lambda \geq 0 : y - \lambda(\xi - \xi_i) \in \overline{\Omega}\}$ and $c = \min\{\lambda_i; i = 1,\dots,n\}$. Hence we get $\mathrm{extr}(y - c\xi + cK^\circ) \setminus \{y - c(\xi - \xi_j), y - c(\xi - \xi_{j+1}\} = \{y - c(\xi - \xi_i); i \neq j, i \neq j+1\} \subset \Omega$ and $\{y - c(\xi - \xi_j), y - c(\xi - \xi_{j+1}\} \subset I_j$. If $\xi = \xi_j$, recalling that $N_\Omega(y) = k_{j+1} \subset N_{K^\circ}(\xi_j)$ we can proceed exactly as in the case studied above substituting ξ_j to ξ. The last case $\xi = \xi_{j+1}$ can be treated analogously.

(v) First of all we notice that one of the two implications is obviously true. For the other one we remark that, by Lemma 3.7, if $y \in I_i$ and $y \in \Pi(x)$ it is that either $x = y - c\xi_{i-1}$ or $x = y - c\xi_i$. Without loss of generality we can assume that $x = y - c\xi_{i-1}$. We have that $x + cK^\circ \subset \overline{\Omega}$ and let us suppose that $x + c\xi_i \in \Omega$. Then, there exists $\tilde{c} > c$ such that $x + \tilde{c}\xi_i \in \Omega$ and, by (g) of Section 2, we get

$$\langle k_i, (x + \tilde{c}\xi_i) - (x + c\xi_{i+1})\rangle = \langle k_i, (\tilde{c} - c)\xi_i - c(\xi_i - \xi_{i+1})\rangle = (\tilde{c} - c)\langle k_i, \xi_i\rangle > 0.$$

This contradicts the fact that $k_i \in N_\Omega(y)$. Then we can conclude that $x + c\xi_i \in \partial\Omega$. With the same argument we can prove that, for every $\xi \in \zeta_{i-1}$, $x + c\xi \in \partial\Omega$. Then the line segment joining y and $x + c\xi_i$ is contained in I_i. To conclude the proof it is sufficient to fix $\xi \in \zeta_{i-1}$ and $z = x + c\xi$. □

Definition 3.9. For every $y \in J_i$ we define

$$c(y) = \sup\{c > 0 : \Pi(y - c\xi_i) = y\}.$$

For every $i \in \{1,\dots,n\}$ such that I_i is a line segment, we fix a $\xi \in \zeta_{i-1}$ and for every $y \in \mathrm{ri}(I_i)$ we define

$$c(y) = \sup\{c > 0 : \Pi(y - c\xi) = \{y - c(\xi - \xi_{i-1}), y - c(\xi - \xi_i)\}\}.$$

Lemma 3.10. *For every $y \in J_i$ it is*

$$c(y) = \min_{j\in\{1,\dots,n\}} \sup\{\lambda \geq 0 : y - \lambda(\xi_i - \xi_j) \in \overline{\Omega}\}$$

and for every $y \in \mathrm{ri}(I_i)$ and for every $\xi \in \zeta_{i-1}$, it is

$$c(y) = \min_{j\in\{1,\dots,n\}} \sup\{\lambda \geq 0 : y - \lambda(\xi - \xi_j) \in \overline{\Omega}\}.$$

Proof. Let us consider the case $y \in J_i$, Let us define $\tilde{c} = \min_{j\in\{1,\dots,n\}} \sup\{\lambda > 0 : y - \lambda(\xi_i - \xi_j) \in \Omega\}$. If $\tilde{c} < c(y)$ there exists $j \neq i$ such that $y - \tilde{c}(\xi_i - \xi_j) \in \partial\Omega$. Then $\{y, y - \tilde{c}(\xi_i - \xi_j)\} \subset \Pi(y)$ a contradiction with the definition of $c(y)$. On the other hand if $\tilde{c} > c(y)$, recalling that $(\xi_i - \xi_j)$ are in the interior of $T_\Omega(y)$, we get the contradiction $y - c(\xi_i - \xi_j) \in \Omega$ for every $j \neq i$ and for every $c \in (c(y), \tilde{c})$, i.e. $y = \Pi(y - c\xi_i)$. In the case $y \in \mathrm{ri}(I_i)$, if $\tilde{c} < c(y)$ for every $c \in (\tilde{c}, c(y))$ there exists j such that $y - c(\xi - \xi_j) \notin \overline{\Omega}$ and then $v(y - c\xi) \neq c$. If $\tilde{c} > c(y)$, for every $c \in (c(y), \tilde{c})$, we have $y - c(\xi - \xi_j) \in \Omega$ for every $j \notin \{i, i+1\}$ and $y - c(\xi - \xi_j) \in \mathrm{ri}(I_i)$ for $j \in \{i, i+1\}$. Hence $\{y - c(\xi - \xi_i), y - c(\xi - \xi_{i+1})\} = \Pi(y - c\xi)$, a contradiction. □

Remark 3.11. As immediate consequences of Lemma 3.10 and Proposition 3.8 we have the following properties.

(1) $0 < c(y) \leq W_{(\Omega,K)}$ for every $y \in J_i$ and for every $y \in \mathrm{ri}(I_i)$.

(2) $y \in \Pi(y - c(y)\xi_i)$ for every $y \in J_i$; for every $y \in \mathrm{ri}(I_i)$ and for every $\xi \in \zeta_{i-1}$ we have $\{y - c(y)(\xi - \xi_{i-1}, y - c(y)(\xi - \xi_i\} \in \Pi(y - c(y)\xi)$.

(3) For every $y \in J_i$ there exist $z \in \partial\Omega$, $z \neq y$, and $j \neq i$ such that $z \in \Pi(y - c(y)\xi_i)$ and $z = y - c(y)(\xi_i - \xi_j)$. Moreover $c(z) = c(y)$. Analogously, for every $y \in \mathrm{ri}(I_i)$, for every $\xi \in \zeta_{i-1}$, there exist $z \in \partial\Omega$, $z \notin \mathrm{ri}(I_i)$, and $j \in \{1, \ldots, n\}$ such that $z \in \Pi(y - c(y)\xi)$, $z = y - c(y)(\xi - \xi_j)$ and $c(z) = c(y)$.

Lemma 3.12. *The function $c(\cdot)$ is continuous.*

Proof. For every $y \in \partial\Omega$ and $a \in \mathbb{R}^2$, $a \neq 0$ we can define the width of Ω in y in the direction a to be $w(y, a) = \sup\{\lambda > 0 : y - \lambda a \in \overline{\Omega}\}$. By the continuity of $\partial\Omega$ it is straightforward that $w(y, a)$ is continuous in the natural topology induced on $\partial\Omega$ by $\mathbb{R}^2$. Recalling the characterization of $c(y)$ we get the proof. □

To conclude this section we remark that, thanks to the properties proved above, we can say that the set Ω can be regarded as the union of a certain number of sets in which the function $v(\cdot)$ can be computed in a more convenient way.

In fact, for every $i \in \{1, \ldots, n\}$ such that $J_i \neq \emptyset$ and for every $j \in \{1, \ldots, n\}$ such that $\mathrm{ri}(I_j) \neq \emptyset$, we can define, respectively

$$\begin{aligned} \Omega_i &= \{x \in \Omega : \exists y \in J_i \text{ and } 0 < c \leq c(y) \text{ such that } x = y - c\xi_i\} \\ \mathcal{O}_j &= \{x \in \Omega : \forall \xi \in \zeta_{j-1} \exists y \in I_j \text{ and } 0 < c \leq c(y) \text{ such that } x = y - c\xi\} \end{aligned}$$

and then we get

$$\Omega = (\cup_i \Omega_i) \cup (\cup_j \mathcal{O}_j).$$

Now, if $y \in J_i$ and y is a point of differentiability for $\partial\Omega$, by the definition of J_i, there exists $\lambda \in]0, 1[$ such that $\lambda k_i + (1 - \lambda)k_{i+1} \in N_\Omega(y)$ and, by the results proved in this section, for every $0 < c < c(y)$ we have that $\nabla v(x) = -(\lambda k_i + (1 - \lambda)k_{i+1})$. Analogously, for every $y \in I_j$ and for every $0 < c < c(y)$, we get $\nabla v(x) = -k_i$.

4. Existence theorem

We have the following existence theorem:

Theorem 4.1. *Let Ω be an open bounded convex set contained in $\mathbb{R}^2$. Let $K \subset \mathbb{R}^2$ be a closed polytope such that $0 \in \mathbb{R}^2$. Let h satisfy the hypothesis stated in* Section 2. *Let ρ, Λ and $W_{(\Omega,K)}$ be defined as before. If $W_{(\Omega,K)} \leq \Lambda$, the function*

$$u(x) = -\rho \inf_{y \in \partial\Omega} \sup_{x^* \in -K} \langle x - y, x^* \rangle$$

is a solution to the problem $(\mathcal{P})$.

Proof. (a) First of all we remark that, for every $k \in \partial\rho K$, for every vector $v \in \mathbb{R}^2$ and for every $p \in \partial\gamma_K(k)$ we have

$$\begin{aligned} h(\gamma_K(k + v)) &= h(\gamma_K(k) + \gamma_K(k + v) - \gamma_K(k)) \\ &\geq h(\gamma_K(k)) + \alpha(\gamma_K(k + v) - \gamma_K(k)) \\ &\geq h(\gamma_K(k)) + \alpha\langle p, v\rangle \end{aligned}$$

where $\alpha \in [0, \Lambda]$. Now, recalling the properties of $\partial\gamma_K(\cdot)$ stated in Section 2, we can consider the restriction of $\partial\gamma_K(\cdot)$ to ∂K and we fix an arbirary selection $p(\cdot)$ of this multifunction. By the very definition of the function $u(\cdot)$, for every $\rho \geq 0$,and for almost every $x \in \Omega$, we have $\nabla u(x) = -\rho\nabla v(x)$ and $\nabla u(x) \in \partial\rho K$. Then we can define $p(\nabla u(x)) = p(-\nabla v(x))$. For every function $\eta(\cdot) \in W_0^{1,1}(\Omega)$ and for every function $\alpha(x) \in L^\infty(\Omega)$, $0 \leq \alpha \leq \Lambda$,we have

$$\begin{aligned}\int_\Omega &[h(\gamma_K(\nabla u(x) + \nabla\eta(x))) + u(x) + \eta(x)]\,dx \\ &\geq \int_\Omega [h(\gamma_K(\nabla u(x)) + u(x)]\,dx + \int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\,dx.\end{aligned}$$

If we prove that for every selection $p(\cdot)$ and for every function $\eta(\cdot) \in W_0^{1,1}(\Omega)$ there exists a function $\alpha(x) \in L^\infty(\Omega)$, $0 \leq \alpha \leq \Lambda$, such that

$$\int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\,dx = 0 \tag{4.1}$$

we have proved that the function $u(x)$ is a minimum of the functional considered. Using standard arguments on mollifiers, it is sufficient to show that (4.1) holds true for every $\eta(\cdot) \in \mathcal{C}_0^\infty$.

(b) For every $i \in \{1, \ldots, n\}$ such that J_i is nonempty there exists a point $O_i \in J_i$ such that ξ_i belongs to $N_\Omega(O_i)$. Let ν_i be a vector normal to ξ_i, with norm equal to 1, and we consider the pair of coordinate axis with origin in O_i and directions defined by (ν_i) and $(-\xi_i)$. There exist an open interval $]a_i, b_i[$ and a non-negative lipschitzean convex function $\Phi_i :]a_i, b_i[\to \mathbb{R}^2$ such that $\{(s, \Phi_i(s)); s \in]a_i, b_i[\} = J_i$. We will use the notation $c(s) = c((s, \Phi_i(s)))$ and we recall that the function $c(s)$ is continuous on $]a_i, b_i[$ and admits finite limits both for $s \to a_i$ and for $s \to b_i$. We define $S_i = \{(s, c) : s \in]a_i, b_i[$ and $0 < c \leq c(s)\}$ and, for every $\epsilon \geq 0$, $S_i^\epsilon = \{(s, c) : s \in]a_i, b_i[$ and $0 < c < c(s) - \epsilon\}$. We will denote by $g_i : \mathbb{R}^2 \to \mathbb{R}^2$ the function that describes this change of variables. We have that $g_i(\Omega_i) = S_i$ and denote by Ω_i^ϵ the set such that $g_i(\Omega_i^\epsilon) = S_i^\epsilon$

Now, for every $i \in \{1, \ldots, n\}$ such that $\mathrm{ri}(I_i)$ is non empty and for every $\xi \in \zeta_{i-1}$, we can fix a point $P_i \in I_i$ and consider a pair of coordinate axis with origin in P_i and directions defined by (ν) and $(-\xi)$, where ν is a vector, normal to ξ, with norm equal to 1. By (ii) of Proposition 3.4, there exist a closed interval $[c_i, d_i]$ and a linear function $\Psi_i : [c_i, d_i] \to \mathbb{R}^2$ such that $\{(s, \Psi_i(s)); s \in [c_i, d_i]\} = I_i$. As before we will use the notation $c(s) = c(s, \Psi_i(s))$ and we remark that the function $c(s)$ is continuous on $[c_i, d_i]$. We define $R_i = \{(s, c) : s \in [c_i, d_i]$ and $0 < c \leq c(s)\}$ and, for every $\epsilon \geq 0$, $R_i^\epsilon = \{(s, c) : s \in [c_i, d_i]$ and $0 < c < c(s) - \epsilon\}$. We will denote by $h_i : \mathbb{R}^2 \to \mathbb{R}^2$ the function that describes this change of variables. In this case we have that $h_i(\mathcal{O}_i) = R_i$ and we denote by $\mathcal{O}_i^\epsilon$ the set such that $h_i(\mathcal{O}_i^\epsilon) = R_i^\epsilon$.

The definition of $c(y)$, the properties (i) and (ii) in Proposition 3.8 and the Remarks 3.11 imply for every $\epsilon > 0$, the sets $\mathcal{O}_i^\epsilon$, Ω_i^ϵ satisfy the following properties

$$\mathcal{O}_i^\epsilon \cap \mathcal{O}_j^\epsilon = \emptyset \quad \forall i \neq j, \qquad \Omega_i^\epsilon \cap \Omega_j^\epsilon = \emptyset \quad \forall i \neq j, \qquad \mathcal{O}_i^\epsilon \cap \Omega_j^\epsilon = \emptyset \quad \forall i\ \forall j.$$

Moreover we have that $S_i \setminus S_i^\epsilon = \{(s,c) : s \in]a_i, b_i[\text{ and } c(y) - \epsilon \le c \le c(y)\}$, then $\mu(S_i \setminus S_i^\epsilon) = (b_i - a_i)\|\xi_i\|\epsilon$ and

$$\lim_{\epsilon \to 0} \mu(\Omega \setminus (\cup_{i=1,\dots,n}(\mathcal{O}_i^\epsilon \cup \Omega_j^\epsilon)) = 0. \tag{4.2}$$

By the properties proved for the function $v(x)$ we have that, on S_i, $v(g_i(s,c)) = c - \Phi_i(s)$. By the convexity of the function Φ_i there exists at most a countable collection of points $(s_n)_{n\in\mathbb{N}} \subset]a_i, b_i[$ in which the function Φ_i is not differentiable. It is $\mu(\{(s,c) : s = s_n \text{ and } 0 < c \le c(y)\}) = 0$, and then for every $(s,c) \in S_i$ such that $v(g_i(\cdot,\cdot))$ is differentiable in (s,c) and $(s,c) \notin \{(s,c) : s = s_n \text{ and } 0 < c < c(y)\}$ we have

$$\nabla v(g_i^{-1}(s,c)) = (-\Phi_i'(s), 1) \in -N_\Omega((s, \Phi_i(s))). \tag{4.3}$$

On R_i we have $v(h_i(s,c)) = c - \Psi_i(s)$ and then, for every point of differentiability of $v(h_i(\cdot))$,

$$\nabla v(h_i^{-1}(s,c)) = (-\Psi_i'(s), 1) \in -N_\Omega(s, \Psi_i(s)). \tag{4.4}$$

(c) We define the following functions

$$\beta_i(s,c) = \begin{cases} \Phi_i(s) + c(s) - c & \text{for } (s,c) \in S_i \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_i(s,c) = \begin{cases} \Psi_i(s) + c(s) - c & \text{for } (s,c) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

We claim that the function

$$\alpha(x) = \sum_{i=1,\dots,n} \beta_i(g_i(x))\chi_{\Omega_i}(x) + \delta_i(h_i(x))\chi_{\mathcal{O}_i}(x)$$

satisfies (4.1). We remark that $\alpha(\cdot)$ is measurable and, for almost every $x \in \Omega$, $0 \le \alpha(x) \le W_{\Omega,K}$. By (4.2) we have

$$\begin{aligned} &\int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\,dx \\ &= \lim_{\epsilon\to 0} \sum_{i=1,\dots,n} \int_{\Omega_i^\epsilon} [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\,dx \\ &\quad + \lim_{\epsilon\to 0} \sum_{i=1,\dots,n} \int_{\mathcal{O}_i^\epsilon} [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\,dx. \end{aligned} \tag{4.5}$$

Let us compute

$$\int_{\Omega_i^\epsilon} \eta(x)dx = \int_{a_i}^{b_i} \int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon} \eta(g_i^{-1}(s,c))\|\xi_i\|dsdc.$$

Integrating by parts and recalling that $\eta(g_i^{-1}(s,\Phi(s)))=0$, we obtain that the last term is equal to

$$\|\xi_i\|\int_{a_i}^{b_i}\Big[\epsilon\eta(g_i^{-1}(s,\Phi(s)+c(s)-\epsilon)) \\ -\int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon}(\Phi(s)+c(s)-c)\langle\xi_i,\nabla\eta(g_i^{-1}(s,c))\rangle dc\Big]\,ds.$$

Hence

$$\begin{aligned}
&\int_{\Omega_i^\epsilon}[\alpha_i(x)\langle p(\nabla u(x)),\nabla\eta(x)\rangle+\eta(x)dx] \\
&= \|\xi_i\|\int_{a_i}^{b_i}\int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon}\beta_i(s,c)\langle p(\nabla u(g_i^{-1}(s,c))),\nabla\eta(g_i^{-1}(s,c))\rangle dcds \\
&\quad -\|\xi_i\|\int_{a_i}^{b_i}\int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon}(\Phi(s)+c(s)-c)\langle\xi_i,\nabla\eta(g_i^{-1}(s,c))\rangle dcds \\
&\quad +\|\xi_i\|\int_{a_i}^{b_i}\epsilon\eta(g_i^{-1}(s,\Phi(s)+c(s)-\epsilon))dcds. \qquad (4.6)
\end{aligned}$$

By (4.3), recalling that $N_\Omega(s,\Phi(s))$ is strictly contained in the convex cone generated by k_i and k_{i+1}, there exists $\lambda\in(0,1)$ such that $\nabla v(g_i^{-1}(s,c))=-(\lambda k_i+(1-\lambda)k_{i+1})$ and then for every selection $p(\cdot)$ we have $p(\nabla u(s,c))=\xi_i$. Hence (4.6) is equal to

$$\|\xi_i\|\int_{a_i}^{b_i}\epsilon\eta(g_i^{-1}(s,\Phi(s)+c(s)-\epsilon))ds \\ +\|\xi_i\|\int_{a_i}^{b_i}\int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon}[\beta_i(s,c)-(\Phi(s)+c(s)-c)]\,\langle\xi_i,\nabla\eta(g_i^{-1}(s,c))\rangle dcds$$

and, by the definition of $\beta_i(s,c)$, it is equal to

$$\|\xi_i\|\int_{a_i}^{b_i}\epsilon\eta(g_i^{-1}(s,\Phi(s)+c(s)-\epsilon))ds.$$

If we want to compute the integral on $\mathcal{O}_i^\epsilon$, first of all we have to notice that $\nabla v(x)=-k_i$ for almost every $x\in\mathcal{O}_i^\epsilon$ and then $\partial\gamma_K(\nabla u(x))=\overline{\mathrm{co}}(k_i,k_{i+1})$. For every selection $p(\cdot)$, it is $p(\nabla u(x))=\xi\in\overline{\mathrm{co}}(k_i,k_{i+1})$. We can consider the coordinates introduced in (b) and, proceeding exactly as above, we get

$$\int_{\mathcal{O}_i^\epsilon}[\alpha(x)\langle p(\nabla u(x)),\nabla\eta(x)\rangle+\eta(x)]\,dx=\|\xi\|\int_{c_i}^{d_i}\epsilon\eta(h_i^{-1}(s,\Psi(s)+c(s)-\epsilon))ds.$$

Hence, by the hypothesis $\eta(\cdot)\in C_0^\infty(\Omega)$, by (4.5) and by the assumption on Ω, the conclusion follows. □

The following example is in the same spirit of the Example 2 in [4]. It shows that the condition $W_{\Omega,K}\le\Lambda$ can not be improved in the sense that if it is not fulfilled it may happen that the function $u(x)=-\rho v(x)$ is not a solution of the problem $(\mathcal{P})$.

Example 4.2. Let us consider the function

$$h(r) = \begin{cases} r & \text{if } 0 \le r \le 1 \\ +\infty & \text{if } r > 1. \end{cases}$$

In this case we have $\rho = 0$ and $\Lambda = 1$. Let $K \subset \mathbb{R}^2$ be the square $\{x = (x_1, x_2) : \max_{i=1,2} |x_i| \le 1\}$. The functional defined in such a way is weakly lower semicontinuous and has superlinear growth, then it always admits a solution.

Applying Theorem 4.1 we have that, for every Ω such that $W_{\Omega,K} \le 1$, the function $u(x) \equiv 0$ is a solution of the problem $(\mathcal{P})$. We show, now, that for every $\epsilon > 0$ there exists a set Ω, with $W_{\Omega,K} = 1 + \epsilon$, such that the function $u(x) \equiv 0$ is not a minimum.

We choose $\Omega = \{x = (x_1, x_2) : |x_1| \le a + \epsilon \text{ and } |x_2| \le 1 + \epsilon\}$ and $\Omega_0 = \{x = (x_1, x_2) : |x_1| \le a \text{ and } |x_2| \le 1\}$. Let us consider the negative function $w(x)$ that has gradient in norm equal to one and orthogonal to the sides of Ω on the strip $\Omega \setminus \Omega_0$ and gradient 0 on Ω_0. The values of the functional computed along the maps u and w are, respectively, 0 and $4(\epsilon + \frac{\epsilon^2}{2}(1 - a) + \frac{\epsilon^3}{3})$. It is easy to see that if a is sufficiently large with respect to ϵ the last value is strictly less than zero.

Now, let us consider the following problem

$$\int_\Omega [f(\nabla u(x)) + u(x)]\, dx \qquad u(\cdot) \in W_0^{1,1}(\Omega) \tag{$\mathcal{P}'$}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative lower semicontinuous function with minimum value 0. Let $K \subset \mathbb{R}^2$ be a closed polytope with $0 \in \operatorname{int}(K)$. We suppose that $f(k) = 0$ for every $k \in \partial K$ and $f(k) > 0$ for every $k \in C(K)$.

We can consider the family

$$\mathcal{H} = \{h : [0, +\infty) \to [0, +\infty] : h(\cdot) \text{ lower semicontinuous and } h(\gamma_K(k)) \le f(k)\ \forall k \in \mathbb{R}^2\},$$

and we can define

$$\tilde{h}(x) = \sup_{h \in \mathcal{H}} h(x).$$

We have that $\tilde{h}(\cdot) \in \mathcal{H}$ and $\tilde{h}(1) = 0$. We define $\tilde{\Lambda} = \sup\{a \in \mathbb{R} : \tilde{h}(s) \ge a(s - 1) \text{ for every } s \ge 0\}$ and $W_{(\Omega,K)} = \sup_{x \in \Omega} v(x)$, where $v(\cdot)$ is defined by (2.4). Then we have

Corollary 4.3. *Let Ω, K, f be defined as above. If $W_{(\Omega,K)} \le \tilde{\Lambda}$ the function*

$$u(x) = -\inf_{y \in \partial\Omega} \sup_{x^* \in -K} \langle x, x^* \rangle$$

is a solution of the problem $(\mathcal{P}')$.

Proof. It is sufficient to remark that for every $\eta(\cdot) \in W_0^{1,1}(\Omega)$, for every selection $p(\cdot)$ of the multifunction $\partial\gamma_k$ restricted to ∂K and for $\alpha(\cdot) \in L^\infty(\Omega)$, with $0 \le \alpha(x) \le \tilde{\Lambda}$, we have

$$
\begin{aligned}
&\int_\Omega [f(\nabla u(x) + \nabla \eta(x)) + u(x) + \eta(x)]\, dx \\
\geq\ & \int_\Omega \left[\tilde{h}(\gamma_K(\nabla u(x) + \nabla \eta(x))) + u(x) + \eta(x)\right] dx \\
\geq\ & \int_\Omega \left[\tilde{h}(\gamma_K(\nabla u(x))) + u(x)\right] dx + \int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla \eta(x)\rangle + \eta(x)]\, dx \\
=\ & \int_\Omega [f(\nabla u(x)) + u(x)]\, dx + \int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla \eta(x)\rangle + \eta(x)]\, dx.
\end{aligned}
$$

The construction of the function $\alpha(\cdot)$ given in the proof of Theorem 4.1 completes the proof. □

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