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A New Subdifferential in Quasiconvex Analysis

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We introduce a new notion of subdifferential, which we call Q-subdifferential, for functions defined on subsets of normed spaces. The Q-subdifferential is a subset of the Greenbeg-Pierskalla quasi-subdifferential and is therefore useful in quasiconvex analysis.

Keywords: Greenberg-Pierskalla quasi-subdifferential, Fréchet subdifferential, quasiconvexity

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1. Introduction

Since the pioneering work of Greenberg and Pierskalla [8], who introduced the notion of quasi-subdifferential, there have been several attemps to define an appropiate notion of subdifferential in quasiconvex analysis. Crouzeix [5] introduced the tangential, which shares with the quasi-subdifferential the drawback of being a too large set; indeed, both are cones and therefore they give too little information on the function. Smaller sets are the lower subdifferential of Plastria [18] and its variants, the α-lower subdifferentials [15], but they are still unbounded. Trying to reproduce the pattern of convex analysis, where the support function of the subdifferential coincides with the directional derivative, the weak lower subdifferential was introduced in [14] in such a way that it partially fulfills this condition. But this set is even bigger than the lower subdifferential of Plastria. Thus, up to now, the problem of defining an appropriate subdifferential for quasiconvex functions remains open. By "appropriate" we mean that it should satisfy at least two properties: its nonemptiness on the domain of a function to imply quasiconvexity and, on the other hand, to be a small set.

The aim of this paper is to provide a new notion of subdifferential, suitable for quasiconvex

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functions. To this aim, we define what we call the Q-subdifferntial, whose nonemptiness on a dense subset of the domin of a lower semicontinuous function implies its quasiconvexity. So our sufficient condition for the quasiconvexity of f is different from those given in [1, 3, 4, 7, 9, 10, 11, 12, 16, 17], where the generalized convexity of f is assured by the generalized monotonicity of its directional derivatives or subdifferentials.

The Q-subdifferential has also the advantage, in comparison with other subdifferentials of quasiconvex analysis, of being a rather small set, as it is always contained in the Fréchet subdifferential.

The organization of this paper is as follows. In the second section we give some properties of the Q-subdifferential, including its relations with the Greenberg-Pierskalla quasi-subdifferential and the Fréchet subdifferential. In the third one we provide the relationship between the quasiconvexity of f and its Q-subdifferential.

2. Q-subdifferential: definition and properties

Let X be a normed space with topological dual X^*. Denote by $\langle .,. \rangle$ the canonical bilinear form between X^* and X. Let $f : X \to \bar{R} = R \cup \{\pm\infty\}$ be a given function and $x_0 \in X$ be such that $|f(x_0)| < +\infty$. We begin with a new notion of subdifferential.

Definition 2.1. The Q-subdifferential of f at x_0, denoted $\partial^Q f(x_0)$, is the set of all $x^* \in X^*$ such that

$$f(x) \geq f(x_0) + \varphi(\langle x^*, x - x_0 \rangle) \quad \forall x \in X, \tag{2.1}$$

where $\varphi : R \to \bar{R}$ is a function depending on x^* and having the following properties:

$$\varphi \text{ is non-decreasing} : t_1 \geq t_2 \Rightarrow \varphi(t_1) \geq \varphi(t_2), \tag{2.2}$$

$$\varphi(0) = 0, \tag{2.3}$$

$$\varphi'(0) := \lim_{t \to 0} \frac{\varphi(t) - \varphi(0)}{t} = 1. \tag{2.4}$$

Definition 2.2. The Greenberg-Pierskalla subdifferential of f at x_0, denoted $\partial^{GP} f(x_0)$, is defined by

$$\partial^{GP} f(x_0) = \{x^* \in X^* : \langle x^*, x - x_0 \rangle \geq 0 \Rightarrow f(x) \geq f(x_0)\}$$

(in [8], $\partial^{GP} f(x_0)$ is called the quasi-subdifferential of f at x_0).

A relationship between Def. 2.1 and Def. 2.2 is given in the next proposition, which shows that the problem of finding a Q-subdifferential in normed spaces reduces to that in the real line.

Proposition 2.3. *The following statements are equivalent:*

(i) $x^* \in \partial^Q f(x_0)$,

(ii) $x^* \in \partial^{GP} f(x_0)$ *and* $1 \in \partial^Q h(t_0)$, *where*

$$t_0 = \langle x^*, x_0 \rangle \tag{2.5}$$

and

$$h(t) = \inf \{ f(x) : \langle x^*, x \rangle \geq t \} . \tag{2.6}$$

Proof. (i) $\Rightarrow$ (ii). By taking $t_1 = \langle x^*, x - x_0 \rangle$ and $t_2 = 0$ we derive from (2.2) and (2.3) that

$$\langle x^*, x - x_0 \rangle \geq 0 \Rightarrow \varphi \left(\langle x^*, x - x_0 \rangle \right) \geq \varphi (0) = 0$$

$$\Rightarrow f(x) \geq f(x_0) \quad \text{(see (2.1))}.$$

Therefore, $x^* \in \partial^{GP} f(x_0)$; in other words,

$$f(x_0) = h(t_0). \tag{2.7}$$

Since, by assumption, $|f(x_0)| < +\infty$, we get $|h(t_0)| < +\infty$. Now let $t \in R$ and set $A_t = \{x \in X : \langle x^*, x \rangle \geq t\}$. If $A_t \neq \emptyset$ then, for $x \in A_t$, by the non-decreasing property of φ we have

$$\varphi \left(\langle x^*, x - x_0 \rangle \right) \geq \varphi(t - t_0),$$

which, by (2.1), implies

$$h(t) \geq h(t_0) + \varphi(t - t_0) = h(t_0) + \varphi(1(t - t_0)). \tag{2.8}$$

If $A_t = \emptyset$ (i.e., if $x^* = 0$ and $t > 0$) then (2.8) also holds because in this case $h(t) = +\infty$. This proves that $1 \in \partial^Q h(t_0)$, since $t \in R$ is arbitrarily chosen.

(ii) $\Rightarrow$ (i). Assume that (2.8) holds for all $t \in R$, where $\varphi : R \to \bar{R}$ is some suitable function with properties (2.2)–(2.4). Since $x^* \in \partial^{GP} f(x_0)$, (2.7) is valid. Obviously, for any $x \in X$, by taking $t = \langle x^*, x \rangle$ we get $x \in A_t$. Thus, combining (2.6) and (2.8) yields

$$f(x) \geq h(\langle x^*, x \rangle) \geq f(x_0) + \varphi(\langle x^*, x - x_0 \rangle).$$

□

Definition 2.4. The Fréchet subdifferential of f at x_0, denoted $\partial^F f(x_0)$, is defined by

$$\partial^F f(x_0) = \{x^* \in X^* : f(x) \geq f(x_0) + \langle x^*, x - x_0 \rangle + o(x - x_0) \quad \forall x \in X\}, \tag{2.9}$$

where $o(.) : X \to \bar{R}$ is some function such that

$$\lim_{x \to 0} \frac{o(x)}{\|x\|} = 0. \tag{2.10}$$

We now list some properties of Q-subdifferentials, whose easy proofs are omitted:

(1) $\partial f(x_0) \subset \partial^Q f(x_0) \subset \partial^F f(x_0)$,
where $\partial f(x_0)$ denotes subdifferential in the sense of convex analysis:

$$\partial f(x_0) = \{x^* \in X^* : f(x) \geq f(x_0) + \langle x^*, x - x_0 \rangle \quad \forall x \in X\}.$$

If f is convex then

$$\partial f(x_0) = \partial^Q f(x_0) = \partial^F f(x_0).$$

(2) If $\partial^Q f(x_0) \neq \emptyset$ then f is l.s.c. at x_0.
(3) A point $x_0 \in X$ is a global minimizer of f if and only if $0 \in \partial^Q f(x_0)$.
(4) If $x^* \in \partial^Q f(x_0)$ and if x_0 is a local maximizer of f then $x^* = 0$.
(5) If $f(x_0) > \inf\{f(x) : x \in X\}$ and x_0 is a local maximizer of f then $\partial^Q f(x_0) = \emptyset$.
(6) The operator $\partial^Q f : X \rightrightarrows X^*$ is quasimonotone, i.e.,

$$x_i^* \in \partial^Q f(x_i) \quad (i = 1, 2) \Longrightarrow \max\{\langle x_1^*, x_1 - x_2\rangle, \langle x_2^*, x_2 - x_1\rangle\} \geq 0.$$

The following result gives a chain rule for composition with non-decreasing functions:

Proposition 2.5. *Let $f : X \to \bar{R}$ and assume that $g : \bar{R} \to \bar{R}$ is non-decreasing. If $|f(x_0)| < +\infty$ and g is differentiable at $f(x_0)$ with $g'(f(x_0)) > 0$ then*

$$g'(f(x_0))\partial^Q f(x_0) \subset \partial^Q (g \circ f)(x_0); \tag{2.11}$$

if, moreover, f is convex then equality holds.

Proof. Let $x^* \in \partial^Q f(x_0)$. Then, by Definition 2.1, one has (2.1) for a function $\varphi : R \to \bar{R}$ satisfying (2.2)–(2.4). By the non-decreasing property of g, (2.1) implies

$$g(f(x)) \geq g(f(x_0) + \varphi(\langle x^*, x - x_0\rangle)). \tag{2.12}$$

For $t \in \bar{R}$ let us define

$$\psi(t) = g(f(x_0) + \frac{t}{g'(f(x_0))}) - g(f(x_0)). \tag{2.13}$$

Obviously, $\psi_{|R}$ has properties (2.2)–(2.4). Setting $t = g'(f(x_0))\varphi(\langle x^*, x - x_0\rangle)$ in (2.12), we get

$$g(f(x_0) + \varphi(\langle x^*, x - x_0\rangle)) = g(f(x_0)) + \psi(g'(f(x_0))\varphi(\langle x^*, x - x_0\rangle)).$$

Combining this with (2.12) yields

$$g(f(x)) \geq g(f(x_0)) + \psi(g'(f(x_0))\varphi(\langle x^*, x - x_0\rangle).$$

Hence to show that

$$g'(f(x_0))x^* \in \partial^Q (g \circ f)(x_0)$$

it suffices to observe that the function

$$t \in R \to \psi(g'(f(x_0)))\varphi(\frac{t}{g'(f(x_0))}) \in \bar{R}$$

satisfies (2.2)–(2.4). This proves the inclusion in the statement.

It remains to prove that the opposite inclusion holds when f is convex. Let $x^* \in \partial^Q(g \circ f)(x_0)$. By Property (1) above, x^* is a Fréchet subgradient of $g \circ f$ at x_0, that is,

$$(g \circ f)(x) \geq (g \circ f)(x_0) + \langle x^*, x - x_0 \rangle + o(x - x_0) \ \forall x \in X,$$

for a suitable function $o : X \to \bar{R}$ satisfying (2.10). In particular, for $d \in X$ and $\lambda > 0$ we have

$$g(f(x_0 + \lambda d)) - g(f(x_0)) \geq \lambda \langle x^*, d \rangle + o(\lambda d).$$

Dividing by λ and setting $\lambda \to +0$, we obtain

$$g'(f(x_0)) f'(x_{0;} d) \geq \langle x^*, d \rangle,$$

where $f'(x_{0;} d)$ denotes the directional derivative of f at x_0 in direction d. Since d was arbitrarily chosen, we get

$$\frac{1}{g'(f(x_0))} x^* \in \partial f(x_0),$$

as desired. □

Remark 2.6. In view of Property (1), under the assumptions of Proposition 2.5, for a convex function f one has

$$\partial^Q(g \circ f)(x_0) = g'(f(x_0)) \partial f(x_0).$$

Corollary 2.7. Proposition 2.5 *remains true when the hypothesis* $g'(f(x_0)) > 0$ *is replaced by the assumption that* x_0 *is a global minimizer of* $g \circ f$.

Proof. We only need to consider the case when $g'(f(x_0)) = 0$. Then, by properties (1) and (3) listed above we have

$$g'(f(x_0)) \partial f(x_0) = \{0\} \subset \partial^Q(g \circ f)(x_0) \subset \partial^F(g \circ f)(x_0).$$

Let us prove that these inclusions are nonstrict. If $x^* \in \partial^F(g \circ f)(x_0)$ then, using the same argument as in the second part of the proof of Proposition 2.5, we get $0 \geq \langle x^*, d \rangle$ for all $d \in X$, i.e., $x^* = 0$. Thus, $\partial^F(g \circ f)(x_0) = \{0\}$. □

To conclude this section, we give two results relating Q-subdifferentials with tangent cones and normal cones. We recall the following well known definitions of the normal cone $N(A, x_0)$ and the Bouligand tangent cone $T(A, x_0)$ of a set $A \subset X$ at $x_0 \in X$:

$$N(A, x_0) = \{x^* \in X^* : \langle x^*, x - x_0 \rangle \leq 0 \quad \forall x \in A\},$$

$$T(A, x_0) = \{d \in X : \text{there are sequences } t_n \searrow 0 \text{ and } d_n \to d \text{ such that } x_0 + t_n d_n \in A \quad \forall n\}.$$

Proposition 2.8. *Let $A \subset X$ and $x_0 \in A$, and denote by $\delta_A : X \to \bar{R}$ the indicator function of A, defined by*

$$\delta_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$\partial^Q \delta_A(x_0) = N(A, x_0).$$

Proof. The inclusion $\supset$ follows from the first inclusion in Property (1) above, since $N(A, x_0) = \partial \delta_A(x_0)$. Conversely, if (2.1) holds for some $\varphi : R \to \bar{R}$ satisfying (2.2)–(2.4) then $0 \geq \varphi(\langle x^*, x - x_0 \rangle)$ for all $x \in A$. Since (2.2)–(2.4) imply that $\varphi(t) > 0$ for $t > 0$, it follows that $\langle x^*, x - x_0 \rangle \leq 0$ for all $x \in A$. Hence $x^* \in N(A, x_0)$. □

The next result relates the Q-subdifferential of $f : X \to \bar{R}$ at $x_0 \in f^{-1}(R)$ to the normal cone to $\operatorname{epi} f = \{(x, \lambda) \in X \times R : f(x) \leq \lambda\}$, the epigraph of f, at $(x_0, f(x_0))$ and to the polar to the Bouligand tangent cone of $\operatorname{epi} f$ at the same point (recall that the polar K^0 to a cone K is the normal cone of K at 0).

Proposition 2.9. *Let $f : X \to \bar{R}$ and $x_0 \in f^{-1}(R)$. Then*

$$\begin{aligned} \{x^* &\in X^*/(x^*, -1) \in N(\operatorname{epi} f, (x_0, f(x_0)))\} \subset \partial^Q f(x_0) \\ &\subset \{x^* \in X^*/(x^*, -1) \in (T(\operatorname{epi} f, (x_0, f(x_0))))^o\}. \end{aligned}$$

Proof. The first inclusion follows from the first inclusion in Property (1), since $N(\operatorname{epi} f, (x_0, f(x_0))) = \partial f(x_0)$. To prove the second one, let $x^* \in \partial^Q f(x_0)$ and $(u, v) \in T(\operatorname{epi} f, (x_0, f(x_0)))$. There are sequences $t_n \searrow 0$ and $(u_n, v_n) \to (u, v)$ such that

$$f(x_0) + t_n v_n \geq f(x_0 + t_n u_n) \qquad \forall n.$$

Hence

$$v_n \geq \frac{f(x_0 + t_n u_n) - f(x_0)}{t_n} \geq \frac{\varphi(t_n \langle x^*, u_n \rangle)}{t_n},$$

with $\varphi : R \to \bar{R}$ being a function satisfying (2.2)–(2.4). We distinguish two cases, according to whether there exist infinitely many terms u_n such that $\langle x^*, u_n \rangle \neq 0$ or not. In the first case, for such terms we have

$$v_n \geq \langle x^*, u_n \rangle \frac{\varphi(t_n \langle x^*, u_n \rangle)}{t_n \langle x^*, u_n \rangle} \longrightarrow \langle x^*, u \rangle,$$

whence, as $v_n \to v$, we get $v \geq \langle x^*, u \rangle$. In the second case we have $\langle x^*, u_n \rangle = 0$ for sufficiently large n, and hence, by $v_n \geq \frac{\varphi(0)}{t_n} = 0$, we get $v \geq 0 = \langle x^*, u \rangle$. Thus in both cases we have

$$\langle (u^*, -1), (u, v) \rangle = \langle x^*, u \rangle - v \leq 0.$$

This shows that

$$(x^*, -1) \in (T(\operatorname{epi} f, (x_0, f(x_0))))^o.$$

□

3. Quasiconvexity and Q-subdifferentials

We first recall the well-known notions of quasiconvexity and semistrict quasiconvexity of a function $f : X \to \bar{R}$ (see [2]).

Definition 3.1. A function $f : X \to \bar{R}$ is called quasiconvex if, for all $x_i \in X$ $(i = 1, 2)$ and $\lambda \in (0, 1)$,

$$f(x_2) \le f(x_1) \Rightarrow f(x_\lambda) \le f(x_1), \tag{3.1}$$

where $x_\lambda = (1 - \lambda)x_1 + \lambda x_2$. If the implication (3.1) is replaced by $f(x_2) < f(x_1) \Rightarrow f(x_\lambda) < f(x_1)$ (resp. $f(x_2) \le f(x_1) \Rightarrow f(x_\lambda) < f(x_1)$) for all $x_i \in X$ $(i = 1, 2)$ (resp. for all $x_i \in X$ $(i = 1, 2)$, $x_1 \neq x_2$) and $\lambda \in (0, 1)$, then f is called semistrictly quasiconvex (resp. strictly quasiconvex).

We also recall the definition of the upper Dini derivative of f at x_0, denoted $f'_+(x_0, d)$:

$$f'_+(x_0, d) = \limsup_{\lambda \to +0} \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda}.$$

Theorem 3.2.

(i) *If $0 \neq x^* \in \partial^F f(x_0)$ and $f : X \to \bar{R}$ is semistrictly quasiconvex and u.s.c. on the set of all x such that $f(x) < f(x_0)$, then $x^* \in \partial^{GP} f(x_0)$.*

(ii) *If $x^* \in \partial^F f(x_0)$, $f : R^n \to \bar{R}$ is strictly quasiconvex, u.s.c. on the set of all x such that $f(x) < f(x_0)$, continuous on a neighbourhood of x_0, and there is $d \in R^n$ such that $f'_+(x_0; d) = \langle x^*, d\rangle > 0$ and $f'_+(.; d)$ is continuous at x_0, then $x^* \in \partial^Q f(x_0)$.*

Proof. (i) Let x be such that $f(x) < f(x_0)$. For $\lambda > 0$ sufficiently small we have

$$f(x_0) > f(x_0 + \lambda(x - x_0)) \qquad (\text{ by semistrict quasiconvexity})$$

$$\ge f(x_0) + \lambda \langle x^*, x - x_0\rangle + o(\lambda \|x - x_0\|) \qquad (\text{by } (2.9)).$$

Therefore

$$\langle x^*, x - x_0\rangle \le \lim_{\lambda \to +0} -\frac{o(\lambda \|x - x0\|)}{\lambda} = 0.$$

We have thus proved that

$$f(x) < f(x_0) \Rightarrow \langle x^*, x - x_0\rangle \le 0. \tag{3.2}$$

We show that, in fact,

$$\langle x^*, x - x_0\rangle < 0.$$

Indeed, if $\langle x^*, x - x_0\rangle = 0$, then by the upper semicontinuity of f and by the property that $x^* \neq 0$, we can find a point $\bar{x}$ near x such that $f(\bar{x}) < f(x_0)$ and $\langle x^*, \bar{x} - x_0\rangle > 0$. This contradicts (3.2). So

$$f(x) < f(x_0) \Rightarrow \langle x^*, x - x_0\rangle < 0.$$

This implication shows that $x^* \in \partial^{GP} f(x_0)$, as desired.

(ii) By the first part of the theorem and Proposition 2.3, it is enough to show that $1 \in \partial^Q h(t_0)$. To this purpose, set

$$\varphi(t) = h(t + t_0) - h(t_0) \quad (i.e. \quad \varphi(t - t_0) = h(t) - h(t_0)),$$

where t_0 and h are given by (2.5) and (2.6), respectively. Obviously, φ satisfies (2.2) and (2.3). Condition (2.4) follows from a result of Crouzeix [6, Prop 5.1, p.161] which says that h is differentiable at t_0 and

$$h'(t_0) = \frac{f'_+(x0; d)}{\langle x^*, d\rangle} = 1,$$

where d is the point appearing in the formulation of the theorem. □

An example of a nonsmooth function which satisfies the hypotheses of Theorem 3.2 part (ii) is provided by any nonsmooth strictly convex function $f : R^n \to R$ (in particular, one can take f nondifferentiable on a dense subset of R^n). Indeed, if f is differentiable at $x_0 \in R^n$ and this point is not the global minimum of f. then there is some $d \in R^n$ such that $f'(x_0; d) > 0$; moreover, one has $f'(x_0; d) = \langle x^*, d\rangle$ for $x^* = f'(x_0)$, the gradient of f at x_0, and $f'(., d)$ is continuous at x_0 (this easily follows from the fact that the subdifferential operator is compact-valued, upper semicontinuous, and reduces to the gradient at differentiability points). Using Proposition 2.8, one can modify this example so as to exhibit a nonconvex strictly quasiconvex function satisfying all the assumptions of Theorem 3.2 part (ii).

Corollary 3.3. *Assume that $f : R^n \to \bar{R}$ is differentiable on a neighbourhood of x_0, strictly quasiconvex, and u.s.c. on the set of all x such that $f(x) < f(x_0)$. If the Fréchet derivative $f'(\cdot)$ is continuous at x_0 and $f'(x_0) \neq 0$, then $\partial^Q f(x_0) = \{f'(x_0)\}$.*

Proof. This follows from the second statement of Theorem 3.2 and property (1) of Q-subdifferentials. □

Remark 3.4. Corollary 3.3 fails to hold if f is not quasiconvex. As an example, take $f(t) = t - t^2$ $(t \in R)$. Then $f'(t_0) \neq 0$ for all $t_0 \neq \frac{1}{2}$, and $\partial^Q f(t) = \emptyset$ for all t. Observe also that the assumption $f'(x_0) \neq 0$ cannot be removed. Indeed, for the strictly quasiconvex function $f(t) = t^3$ $(t \in R)$, we have $\partial^Q f(0) = \emptyset$.

From now on we asume that C is a subset of X and $f : C \to R$ is a given function. We shall set $f(x) = +\infty$ for all $x \notin C$. In this context, instead of saying that $f : X \to \bar{R}$ is quasiconvex, we say that $f : C \to R$ is quasiconvex on C.

Proposition 3.5. *Let C be a convex set of X and $f : C \to R$.*

(i) *Assume that f is l.s.c. on C and, for all $\lambda > \inf\{f(x) : x \in C\}$, the level set $S_\lambda(f) = \{x \in C : f(x) \leq \lambda\}$ is solid (i.e. $\operatorname{int} S_\lambda(f) \neq \emptyset$). If $\partial^{GP} f(x) \neq \emptyset$ for all x from a subset C' which is dense in C, then f is quasiconvex on C.*

(ii) *Assume that f is l.s.c. on C and $\partial^{GP} f(x) \neq \emptyset$ for all x from a subset C' which is strongly (or radially) dense in C in the sense that the intersection of C' and any interval $(x_1, x_2) \subset C$ is dense in (x_1, x_2). Then f is quasiconvex on C.*

Proof. (i) Let $\lambda > \inf\{f(x) : x \in C\}$ and $x_1 \in C \backslash S_\lambda(f)$. Since $S_\lambda(f)$ is closed in C (by the lower semicontinuity of f), there is an open neighbourhood V of x_1 in C such that $V \cap S_\lambda(f) = \emptyset$. Using the fact that $\operatorname{int} S_\lambda(f) \neq \emptyset$, we can easily check that

$$V \cap \operatorname{int} Q \neq \emptyset,$$

where $Q = co(S_\lambda(f)_\cup\{x_1\})$ denotes the convex hull of $S_\lambda(f) \cup \{x_1\}$. Indeed, let $\hat{x} \in \operatorname{int} S_\lambda(f)$. Then, for some $\epsilon > 0$,

$$\|x - \hat{x}\| < \epsilon \Rightarrow x \in S_\lambda(f).$$

Since V is open in C, for some $\gamma > 0$ we have $y = (1-\gamma)x_1 + \gamma\hat{x} \in V$. We claim that $y \in \operatorname{int} Q$. Indeed, let x be such that $\|x - y\| < \gamma\epsilon$. Obviously, $x = (1-\gamma)x_1 + \gamma(\hat{x} + \frac{1}{\gamma}z)$, where $z = x - y$. Since $\left\|\frac{1}{\gamma}z\right\| < \epsilon$, $\hat{x} + \frac{1}{\gamma}z \in S_\lambda(f)$, and hence $x \in Q$. This proves that $y \in \operatorname{int} Q$ and thus $y \in V \cap \operatorname{int} Q$. Therefore, $V \cap \operatorname{int} Q \neq \emptyset$. Since $V \cap \operatorname{int} Q$ is open in C and C' is dense in C, we also have

$$C' \cap V \cap \operatorname{int} Q \neq \emptyset.$$

Take a point $\bar{x}$ from this latter intersection. As $\bar{x} \in C'$ we have by assumption $\partial^{GP} f(\bar{x}) \neq \emptyset$. Pick $x^* \in \partial^{GP} f(\bar{x})$. Then the following implication holds:

$$[x \in C, \quad \langle x^*, x - \bar{x}\rangle \geq 0] \Rightarrow f(x) \geq f(\bar{x}).$$

Since $\bar{x} \in V$, we get $\bar{x} \notin S_\lambda(f)$, i.e. $f(\bar{x}) > \lambda$. Therefore, the following implication is also satisfied:

$$[x \in C, \quad \langle x^*, x - \bar{x}\rangle \geq 0] \Rightarrow f(x) > \lambda.$$

In other words, $S_\lambda(f) \subset H$, where

$$H = \{x \in C : \langle x^*, x - \bar{x}\rangle < 0\}.$$

Assume now that $x_1 \in H$. Then $S_\lambda(f) \cup \{x_1\} \subset H$, whence, by the convexity of H, $Q \subset H$. But this implies that $\bar{x} \in H$, which is false. Therefore $x_1 \notin H$. Thus, for any $x_1 \in C \backslash S_\lambda(f)$ we can find a convex set H such that $S_\lambda(f) \subset H$ and $x_1 \notin H$. This means that $S_\lambda(f)$ is an intersection of convex sets, and therefore it is convex. This proves the quasiconvexity of f.

(ii) Assume to the contrary that, for some $\lambda \in R$, the level set $S_\lambda(f)$ is not convex. Then there are three points x_i $(i = 1, 2, 3)$, such that $x_i \in S_\lambda(f)$ $(i = 1, 2)$ and $x_3 \in (x_1, x_2) \backslash S_\lambda(f)$. Since $S_\lambda(f)$ is closed in C and C' is strongly dense in C, we can find a point $\bar{x} \in (x_1, x_2) \backslash S_\lambda(f)$ such that $\partial^{GP} f(\bar{x}) \neq \emptyset$. Pick $x^* \in \partial^{GP} f(\bar{x})$. Then, as in the proof of the first statement, we conclude that $x \in S_\lambda(f) \Rightarrow \langle x^*, x - \bar{x}\rangle > 0$. In particular, since $x \in S_\lambda(f)$ for $i = 1, 2$, we have $\langle x^*, x_i - \bar{x}\rangle < 0 \quad (i = 1, 2)$. This contradicts that $\bar{x} \in (x_1, x_2)$. □

Remark 3.6. Proposition 3.5 fails to hold if the lower semicontinuity property of f is replaced by upper semicontinuity. Also, the lower semicontinuity on the whole set C

cannot be weakened to lower semicontinuity on a proper dense (or strongly dense) subset in C. As a counterexample we can take

$$f(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, f is not quasiconvex, although it is u.s.c. (and Q-subdifferentiable and Greenberg-Pierskalla subdifferentiable everywhere on the real line except for $t = 0$).

Remark 3.7. It is known since a long time (see, e.g. [13]) that f is quasiconvex on C if $\partial^{GP} f(x) \neq \emptyset$ at every point $x \in C$.

Using the fact that the Q-subdifferential is a subset of the Greenberg-Pierskalla subdifferential, we obtain

Theorem 3.8. *The conditions stated in* Proposition 3.5 *and* Remark 3.7 *remain being sufficient for quasiconvexity when $\partial^{GP} f$ is replaced by $\partial^{Q} f$ everywhere in their statements.*

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