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Existence and Uniqueness of Solution of Unilateral Problems with L^1 Data

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We prove an existence and uniqueness theorem for the solution of unilateral problems with L^1 data.

1. Introduction

In this paper we consider the obstacle problem with L^1 data associated to differential operators A of monotone type and we prove an existence and uniqueness theorem.

We recall that, when the data belong to $L^1(\Omega)$, the classical formulation of unilateral problems is inadequate. As a matter of fact, even in the case of the equations, the term $\int_\Omega fu$ as no meaning, if u is a weak solution. If A is the p-Laplacian operator this difficulty is overcome by using a new definition of solution, whose existence is proved in [5] by approximation. Other proofs of the existence result can be found in [3].

Recently,(in [2]) it was proved that the solutions of nonlinear equations with L^1 data, satisfying an additional entropy condition, are unique.

Here, following this idea, we introduce a formulation of unilateral problems with L^1 data, quite similar to the definition of entropy solution of the Dirichlet problem with L^1 data. In this way, we can prove the uniqueness of solution of the obstacle problem as the uniqueness of entropy solutions of the Dirichlet problem is proved in [2].

We shall prove the existence of solutions using a classical approximation and a penalization method. Moreover, we shall present another approach, introduced in [8] in the variational framework, which allows us to prove the Lewy-Stampacchia inequality.

2. Statement of the results

Let Ω be an open bounded subset of $\Re^N$, $N \geq 2$.

Let us consider the nonlinear operator

$$Av = -\operatorname{div} a(x, Dv) \tag{2.1}$$

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where $a : \Omega \times \Re^N \to \Re^N$ is a Caratheodory function satisfying the following assumptions, for a.e. $x \in \Omega$ and $\forall \xi, \eta \in \Re^N$, $(\xi \neq \eta)$, $1 < p < \infty$:

$$a(x,\xi)\xi \geq \alpha|\xi|^p \tag{2.2}$$

$$|a(x,\xi)| \leq \beta[h(x) + |\xi|^{p-1}] \tag{2.3}$$

$$[a(x,\xi) - a(x,\eta)][\xi - \eta] > 0 \tag{2.4}$$

with $\alpha, \beta > 0$ and $h(x) \in L^{p'}(\Omega)$ (here p' denotes the conjugate exponent of p).
Assume that

$$f \in L^1(\Omega) \tag{2.5}$$

and

$$\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \tag{2.6}$$

Let us define

$$\mathcal{K} = \left\{ v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : v(x) \geq \psi(x) \text{ in } \Omega \right\}.$$

In the following, we shall denote by $T_k(s)$ the truncation defined by

$$T_k(s) = \begin{cases} k & \text{if } s > k \\ s & \text{if } |s| \leq k \\ -k & \text{if } s < -k \end{cases}$$

The aim of this paper is to prove the following

Theorem 2.1. *Let $2 - \frac{1}{N} < p < N$. Under assumptions (2.2), ... , (2.6) there exists a unique solution u of the problem*

$$\begin{cases} u \in W_0^{1,q}(\Omega) \quad 1 < q < \frac{N(p-1)}{N-1} \\ u(x) \geq \psi(x) \quad in\ \Omega \\ T_k(u) \in W_0^{1,p}(\Omega) \quad \forall k > 0 \\ \langle Au, T_k(u-v) \rangle \leq \int_\Omega f T_k(u-v) \quad \forall v \in \mathcal{K} \end{cases} \tag{2.7}$$

As we have stated in the Introduction, in Section 4, under an additional assumption, we shall prove the following stronger result:

Theorem 2.2. *Assume that the hypotheses of* Theorem 2.1 *hold and that*

$$A\psi \in L^1(\Omega). \tag{2.8}$$

Let u be the solution of problem (2.7)*. Then the following inequality holds*

$$f \leq Au \leq f + (f - A\psi)^-. \tag{2.9}$$

Remark 2.3. We point out that the previous definition of solution of the obstacle problem is similar to the definition of "entropy solution" of the Dirichlet problem, introduced in [2].

Remark 2.4. The first inequality of the assumption on p ($p > 2 - \frac{1}{N}$) guarantees that the gradient of the solution belongs to $L^1(\Omega)$ $\left(\frac{N(p-1)}{N-1} > 1\right)$.

If $p \leq 2 - \frac{1}{N}$, we can adapt the method used in [2] for the study of the Dirichlet problem.

Moreover, if $p > N$, we are in the "variational framework", since $L^1(\Omega) \subset (W_0^{1,p}(\Omega))'$ and the existence of solutions is a consequence of classical results (see for example [11])

Remark 2.5. The existence result still holds if the data have the form $f(x) - \operatorname{div}(F)$, with $f \in L^1(\Omega)$ and $F \in [L^{p'}(\Omega)]^N$, that is , in particular, if they are measures absolutely continuous with respect to the $p-$capacity (see [7]).

Remark 2.6. If the differential operator A is the p-laplacian, an existence result for (2.7) can be found in [5], [3], where a slight different definition of solution is given.

3. Existence by approximation

Let $\{f_n\}$ be a sequence of smooth functions such that:

$$\begin{cases} f_n \to f & \text{in } \; L^1(\Omega) \\ \|f_n\|_1 \leq \|f\|_1 & \forall n \in \mathcal{N} \end{cases} \tag{3.1}$$

Let u_n be the solution of the problem:

$$\begin{cases} u_n \in W_0^{1,p}(\Omega) \;\; u_n(x) \geq \psi(x) \;\; \text{in } \;\Omega \\ \langle Au_n, u_n - v\rangle \leq \int_\Omega f_n(u_n - v) \\ \forall v \in W_0^{1,p}(\Omega), \;\; v(x) \geq \psi(x) \;\; \text{in } \;\Omega \end{cases} \tag{3.2}$$

Thanks to the hypotheses (2.2), (2.3), (2.4) A is a nonlinear operator of Leray-Lions type, so the existence of u_n follows from the classical results of [11].

In order to prove the existence and uniqueness theorem we need the following:

Lemma 3.1. *There exists a constant $c_0(q)$, independent on n, such that:*

$$\|u_n\|_{W_0^{1,q}(\Omega)} \leq c_0(q) \;\; \forall n \in \mathcal{N}, \quad 1 < q < \frac{N(p-1)}{N-1}. \tag{3.3}$$

Proof. Let $k > 0$ be an integer and $\varphi_k : \Re \to \Re$ be the function defined by:

$$\varphi_k(s) = \begin{cases} 1 & \text{if } s \geq k+1 \\ s-k & \text{if } k \leq s < k+1 \\ 0 & \text{if } 0 \leq s < k \\ -\varphi_k(-s) & \text{if } s < 0 \end{cases}$$

Let $k \geq \|\psi\|_\infty$; taking as test function in (3.2)

$$v = u_n - \varphi_k(u_n),$$

we obtain

$$\langle Au_n, \varphi_k(u_n)\rangle \leq \int_\Omega f_n \varphi_k(u_n) \quad \forall k \geq \|\psi\|_\infty.$$

Thanks to assumption (2.2) we have

$$\int_{B_k^n} |Du_n|^p \leq \frac{\|f\|_1}{\alpha} \quad \forall k \geq \|\psi\|_\infty, \tag{3.4}$$

where

$$B_k^n = \{x \in \Omega : \quad k \leq |u_n(x)| < k+1\}.$$

Let

$$v = u_n - T_k(u_n - \psi), \quad k > 0$$

Since $v \in W_0^{1,p}(\Omega)$ and $v(x) \geq \psi(x)$ in Ω, v is an admissible test function in (3.2), so we have

$$\int_{\{u_n - \psi < k\}} a(x, Du_n) D(u_n - \psi) \leq k\|f\|_1.$$

By virtue of hypotheses (2.2), (2.3) and using Young's inequality (with exponents p, p') we get

$$\alpha \int_{\{u_n-\psi<k\}} |Du_n|^p \leq$$

$$k\|f\|_1 + \frac{\alpha}{2} \int_{\{u_n-\psi<k\}} (|h(x)|^{p'} + |Du_n|^p) + c_1 \|\psi\|^p_{W_0^{1,p}(\Omega)}$$

and finally

$$\int_{\{|u_n|<k\}} |Du_n|^p \leq \int_{\{u_n-\psi<k+\|\psi\|_\infty\}} |Du_n|^p \leq c_2(k + \|\psi\|_\infty + \|\psi\|^p_{W_0^{1,p}} + \|h\|^p_{p'})), \quad \forall k > 0 \tag{3.5}$$

Let $1 < q < \frac{N(p-1)}{N-1}$ and $\overline{k} \geq \|\psi\|_\infty$. It results:

$$\int_\Omega |Du_n|^q \leq$$

$$\Big(\int_{\{|u_n|<\overline{k}\}} |Du_n|^p\Big)^{\frac{q}{p}} |\Omega|^{1-\frac{q}{p}} + \sum_{j=\overline{k}}^{\infty}\Big(\frac{1}{(1+j)^\lambda}\int_{B_j^n} |Du_n|^p\Big)^{\frac{q}{p}}\Big(\int_{B_j^n}(1+|u_n|)^{\frac{\lambda q}{p-q}}\Big)^{1-\frac{q}{p}}$$

where $\lambda = \frac{N(p-q)}{N-q}$; observe that $\lambda > 1$, since $1 < q < \frac{N(p-1)}{N-1}$.

From this inequality, using (3.4) and (3.5) we obtain

$$\int_\Omega |Du_n|^q \leq c_3 + c_4\Big(\int_\Omega |u_n|^{q^\star}\Big)^{1-\frac{q}{p}}. \tag{3.6}$$

From this estimate, by Sobolev's inequality, we get

$$\int_\Omega |u_n|^{q^\star} \leq c_5.$$

Finally, from (3.6) we obtain (3.3). $\square$

First Proof of Theorem 2.1. The estimate (3.3) guarantees that there exists a subsequence, still denoted by $\{u_n\}$, such that $\forall q < N(p-1)/(N-1)$:

$$\begin{cases} u_n \to u & \text{weakly- } W_0^{1,q}(\Omega) \\ u_n \to u & \text{strongly-}L^q(\Omega) \\ u_n \to u & \text{almost everywhere in } \Omega \end{cases} \tag{3.7}$$

Since $u_n(x) \geq \psi(x)$ in Ω, $\forall n \in \mathcal{N}$,

$$u(x) \geq \psi(x) \ \text{ in } \Omega.$$

Moreover, we shall prove that

$$Du_n \to Du \qquad \text{almost everywhere in } \Omega \tag{3.8}$$

As a matter of fact, if we take as test function in (3.2) $u_n - T_k(u_n - u_m)$ and then $u_m + T_k(u_n - u_m)$, we obtain

$$\langle Au_n, T_k(u_n - u_m)\rangle \leq \int_\Omega f_n T_k(u_n - u_m)$$

and

$$-\langle Au_m, T_k(u_n - u_m)\rangle \leq -\int_\Omega f_m T_k(u_n - u_m).$$

Adding up these inequalities, we get

$$\langle Au_n - Au_m, T_k(u_n - u_m)\rangle \le \int_\Omega (f_n - f_m)T_k(u_n - u_m) \tag{3.9}$$

The right hand side of (3.9) tends to zero, if $n,\ m \to \infty$ and so, thanks to the monotonicity of the operator A, also $\langle Au_n - Au_m, T_k(u_n - u_m)\rangle \to 0$. Thus Lemma 1 of [6] implies that

$$Du_n \to Du \quad \text{almost everywhere in } \Omega. \tag{3.10}$$

Let $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We observe that, $\forall k > 0$ the sequence $\{T_k(u_n - w)\}$ is bounded in $W_0^{1,p}(\Omega)$. Indeed:

$$\int_\Omega |DT_k(u_n - w)|^p \le c_6 \int_{\{|u_n|<k+\|w\|_\infty\}} |Du_n|^p + c_7 \int_\Omega |Dw|^p$$

and the right hand side of the previous inequality is bounded thanks to (3.5).

Let $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $w(x) \ge \psi(x)$ in Ω; the function

$$u_n - T_k(u_n - w)$$

is an admissible test function in (3.2).

This choice yields

$$\langle Au_n - Aw, T_k(u_n - w)\rangle + \langle Aw, T_k(u_n - w\rangle \le \int_\Omega f_n T_k(u_n - w) \tag{3.11}$$

Since $\{T_k(u_n - w)\}$ is bounded in $W_0^{1,p}(\Omega)$and in $L^\infty(\Omega)$, we deduce:

$$\liminf \langle Au_n - Aw, T_k(u_n - w)\rangle \ge \langle Au - Aw, T_k(u - w)\rangle$$

$$\lim \langle Aw, T_k(u_n - w)\rangle = \langle Aw, T_k(u - w)\rangle$$

$$\lim \int_\Omega f_n T_k(u_n - w) = \int_\Omega f T_k(u - w)$$

Thus, taking the limit as $n \to \infty$ in (3.11), we have proved that u is a solution of (2.7).

Moreover, its uniqueness can be proved exactly as the uniqueness of the entropy solutions of the Dirichlet problem for equations is proved in [2]. $\square$

4. The penalization method

For sake of simplicity, in this section, we assume that

$$\psi = 0. \tag{4.1}$$

Let $\{f_\epsilon\}$ be a sequence of smooth functions such that:

$$\begin{cases} f_\epsilon \to f & \text{in } L^1(\Omega) \\ \|f_\epsilon\|_1 \leq \|f\|_1 & \forall \epsilon > 0 \end{cases} \tag{4.2}$$

and

$$j(s) = |s|^{p-2}s. \tag{4.3}$$

Let us consider the following problem:

$$\begin{cases} u_\epsilon \in W_0^{1,p}(\Omega) \\ Au_\epsilon - j(\frac{(u_\epsilon)^-}{\epsilon}) = f_\epsilon \end{cases} \tag{4.4}$$

As usual, the existence of u_ϵ follows from well-known results (see [11]).

We shall prove the following:

Lemma 4.1. *There exists a constant $c(q) > 0$, independent on ϵ, such that:*

$$\|u_\epsilon\|_{W_0^{1,q}(\Omega)} \leq c(q), \qquad \forall \epsilon > 0 \tag{4.5}$$

with $1 < q < \frac{N(p-1)}{N-1}$.

Proof. Let $k > 0$. Let us take as test function in the weak formulation of (4.4)

$$v = \varphi_k(u_\epsilon),$$

where $\varphi_k(s)$ is the function defined in the proof of Lemma 3.1.

This choice yields

$$\langle Au_\epsilon, \varphi_k(u_\epsilon)\rangle - \int_\Omega \left(\frac{(u_\epsilon)^-}{\epsilon}\right)^{p-1} \varphi_k(u_\epsilon) \leq \|f\|_1. \tag{4.6}$$

Since $k > 0$, we have:

$$\int_\Omega \left(\frac{(u_\epsilon)^-}{\epsilon}\right)^{p-1} \varphi_k(u_\epsilon) \leq 0$$

and from (4.6) we deduce

$$\int_{B_k^\epsilon} |Du_\epsilon|^p \leq c_1 \qquad \forall k > 0, \forall \epsilon > 0 \ , \tag{4.7}$$

where B_k^ϵ is the set defined in the previous section and c_1 is a positive constant independent on ϵ.

Taking as test function in (4.4) $v = T_k(u_\epsilon)$, we obtain

$$\int_{\{|u_\epsilon|<k\}} a(x, Du_\epsilon)Du_\epsilon \leq k\|f\|_1.$$

From this estimate, by the ellipticity condition (2.2), we obtain:

$$\int_\Omega |DT_k(u_\epsilon)|^p \leq c_4 k \qquad \forall k > 0. \tag{4.8}$$

Working as in the previous section, from (4.7) and (4.8) we obtain the estimate (4.5). □

Second Proof of Theorem 2.1. Since $\{u_\epsilon\}$ is bounded in $W_0^{1,q}(\Omega)$, there exists a subsequence, still denoted by $\{u_\epsilon\}$, such that $\forall q < N(p-1)/(N-1)$:

$$\begin{cases} u_\epsilon \to u & \text{weakly- } W_0^{1,q}(\Omega) \\ u_\epsilon \to u & \text{strongly-}L^q(\Omega) \\ u_\epsilon \to u & \text{almost everywhere in } \Omega \end{cases} \tag{4.9}$$

Now, we shall prove that

$$u \geq \psi \qquad \text{a.e. in } \Omega \tag{4.10}$$

Let $\{\theta_n(s)\}$ be a sequence of increasing functions converging to

$$\theta(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}$$

Let us choose $v = \theta_n(u_\epsilon - \psi)$ as test function in the weak formulation of problem (4.4); then, there exists $\nu \in \mathcal{N}$ such that:

$$-\left(\frac{1}{\epsilon}\right)^{p-1} \int_\Omega \left((u_\epsilon - \psi)^-\right)^{p-1} \theta_n(u_\epsilon - \psi) \leq \|f\|_1 \qquad \forall n > \nu$$

Passing to the limit as $n \to \infty$ we get:

$$\int_\Omega \left((u_\epsilon - \psi)^-\right)^{p-1} \leq \epsilon^{p-1} \|f\|_1 \tag{4.11}$$

From this inequality we deduce

$$\left((u_\epsilon - \psi)^-\right)^{p-1} \to 0 \ \text{ strongly } \ \text{ in } \ L^1(\Omega).$$

Thanks to (4.9) we obtain:

$$(u - \psi)^- = 0 \ \text{ a.e. in } \Omega$$

which proves (4.10).

Moreover, by (4.11), $j(\frac{(u_\epsilon - \psi)^-}{\epsilon}) + f_\epsilon$ is bounded in $L^1(\Omega)$, and from the results of [6] it thus follows

$$Du_\epsilon \to Du \quad \text{almost everywhere in } \Omega \tag{4.12}$$

Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v \geq \psi$ a.e. in Ω, and $k > 0$.

Using the estimate (4.8) we can prove that $\{T_k(u_\epsilon - v)\}$ is bounded in $W_0^{1,p}(\Omega)$; moreover, taking $T_k(u_\epsilon - v)$ as test function in the weak formulation of problem (4.4) we obtain:

$$\langle Au_\epsilon - Av, T_k(u_\epsilon - v)\rangle + \langle Av, T_k(u_\epsilon - v)\rangle \leq \int_\Omega f_\epsilon T_k(u_\epsilon - v).$$

Taking the limit as $\epsilon \to 0$ in the last inequality we conclude the proof of Theorem 2.1. $\square$

5. The omographic approximation

In this section we shall prove Theorem 2.1 using another approximation, introduced by [8] in the variational framework and already used in [3] for unilateral problem with L^1 data.

We recall that, differently from the previous methods, here we need hypothesis (2.8) even in the proof of Theorem 2.1. However this approach allows us to prove the Lewy-Stampacchia inequality.

Let $\lambda > 0$ and $\{f_\lambda\}$ be a sequence of smooth function such that $\forall q < N(p-1)/(N-1)$:

$$\begin{cases} f_\lambda \to f & \text{in } L^1(\Omega) \\ \|f_\lambda\|_1 \leq \|f\|_1 & \forall \lambda > 0 \end{cases}$$

Let us consider the following problem:

$$\begin{cases} u_\lambda \in W_0^{1,p}(\Omega) \\ Au_\lambda + g\frac{u_\lambda - \psi}{\lambda + |u_\lambda - \psi|} = f_\lambda + g & \text{in } \Omega \end{cases} \tag{5.1}$$

where:

$$g = (f_\lambda - A\psi)^-.$$

We observe that g is non negative and $\|g\|_1 \leq \|f\|_1 + \|(A\psi)^+\|_1$.

The existence of u_λ follows from the results of [12].

In order to prove Theorem 2.1 we need the following:

Lemma 5.1. *Assume that hypotheses* (2.2), (2.3), (2.4) *and* (2.8) *are satisfied. Then:*

$$u_\lambda \geq \psi \quad \forall \lambda > 0 \qquad \textit{almost everywhere in } \Omega \tag{5.2}$$

Moreover, there exists a constant $c(q) > 0$, independent on λ, such that:

$$\|u_\lambda\|_{W_0^{1,q}(\Omega)} \leq c(q) \qquad \forall 1 < q < \frac{N(p-1)}{N-1}. \tag{5.3}$$

Proof. Let us take $(u_\lambda - \psi)^-$ as test function in the weak formulation of (5.1) ; we have

$$\langle Au_\lambda, (u_\lambda - \psi)^- \rangle + \int_\Omega g(u_\lambda - \psi)\frac{(u_\lambda - \psi)^-}{\lambda + |u_\lambda - \psi|} = \langle f_\lambda + g, (u_\lambda - \psi)^- \rangle$$

Since

$$\int_\Omega g(u_\lambda - \psi)\frac{(u_\lambda - \psi)^-}{\lambda + |u_\lambda - \psi|} \le 0,$$

and

$$\langle f_\lambda + g - A\psi, (u_\lambda - \psi)^- \rangle \ge 0,$$

we obtain

$$\int_{\{u_\lambda - \psi < 0\}} (a(x, Du_\lambda) - a(x, D\psi))D(u_\lambda - \psi) \le 0.$$

From this estimate, using also assumption (2.4) we obtain

$$D(u_\lambda(x) - \psi(x)) = 0 \ \text{ almost everywhere in } \{x \in \Omega : u_\lambda(x) < \psi(x)\}.$$

Then we get:

$$\|(u_\lambda - \psi)^-\|_{W_0^{1,p}(\Omega)} = 0$$

from which easily follows (5.2).

Thanks to (5.2), u_λ is solution of the following equation

$$Au_\lambda - g\frac{\lambda}{\lambda + (u_\lambda - \psi)} = f_\lambda \qquad \text{in } \Omega. \tag{5.4}$$

Let $k > 0$; choosing $\varphi_k(u_\lambda)$ ($\varphi_k(s)$ is the function defined in Section 3) as test function in the weak formulation of the last equation, we get ($f_\lambda + \lambda\frac{g}{\lambda + (u_\lambda - \psi)}$ is bounded in $L^1(\Omega)$)

$$\int_{B_k^\lambda} |Du_\lambda|^p \le c_1,$$

where B_k^λ is the set defined in Section 3.

From this estimate, working as in Section 3 we get the proof of (5.3). □

Proof of Theorem 2.1. We point out that we shall use $A\psi \in L^1$. Using Lemma 5.1 there exists a subsequence, still denoted by $\{u_\lambda\}$, such that:

$$\begin{cases} u_\lambda \to u & \text{weakly- } W_0^{1,q}(\Omega) \\ u_\lambda \to u & \text{strongly-}L^q(\Omega) \\ u_\lambda \to u & \text{almost everywhere in } \Omega \end{cases} \tag{5.5}$$

Moreover, since u_λ is solution of (5.4) and $f_\lambda + \lambda\frac{g}{\lambda+(u_\lambda-\psi)}$ is bounded in $L^1(\Omega)$, reasoning as before we have

$$Du_\lambda \to \quad Du \quad \text{almost everywhere} \quad \text{in} \quad \Omega \tag{5.6}$$

Let $k > 0$. Choosing $T_k(u_\lambda)$ as test function in the weak formulation of (5.4), we have

$$\|T_k(u_\lambda)\|_{W_0^{1,p}(\Omega)} \leq ck \qquad \forall k > 0 \tag{5.7}$$

From this estimate we can prove that

$$\{T_k(u_\lambda - v)\} \text{ is bounded in } W_0^{1,p}(\Omega), \tag{5.8}$$

for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.
Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v \geq \psi$ a.e. in Ω; then

$$\langle Au_\lambda - Av, T_k(u_\lambda - v)\rangle + \langle Av, T_k(u_\lambda - v)\rangle =$$

$$\int_\Omega f_\lambda T_k(u_\lambda - v) + \int_\Omega \lambda\frac{g}{\lambda + (u_\lambda - \psi)}T_k(u_\lambda - v).$$

Letting $\lambda \to 0$, and taking into account (5.5), (5.6) and (5.8) we can conclude the proof of Theorem 2.1. □

Proof of Theorem 2.2. Since u_λ satisfies (5.1) we have

$$Au_\lambda \leq f_\lambda + (f_\lambda - A\psi)^-. \tag{5.9}$$

Moreover, thanks to (5.4) we obtain

$$Au_\lambda \geq f_\lambda. \tag{5.10}$$

Taking the limit as $\lambda \to 0$ in (5.9) and (5.10) we get (2.9). □

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