Journal of Convex Analysis
Volume 7 (2000), No. 1, 47–71

Epi-Distance Convergence of Parametrised Sums of Convex Functions in Non-Reflexive Spaces

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Received April 24, 1998

A weakened set of conditions is established for the epi-distance convergence of a sum $\{f_v + g_v\}_{v\in W}$ of parametrised closed convex functions $\{f_v\}_{v\in W}$ and $\{g_v\}_{v\in W}$ for $v \to w$, on an arbitrary Banach space. They are as follows: (1) $0 \in \operatorname{sqri}(\operatorname{dom} f_w - \operatorname{dom} g_w)$; and (2) $X_w := \operatorname{cone}(\operatorname{dom} f_w - \operatorname{dom} g_w)$ has closed algebraic complement Y_w; and (3) $X_v \cap Y_w = \{0\}$ for all v near w, (where $X_v := \overline{\operatorname{span}}(\operatorname{dom} f_v - \operatorname{dom} g_v)$). These are motivated by similar interiority conditions found in Fenchel duality theory. Our results are then used to investigate saddle-point convergence in Young-Fenchel duality in which both functions vary in a very general fashion.

1. Introduction

Over the last thirty years a number of authors have proposed topologies for the hyperspace of closed convex sets, the most famous being the Mosco topology. It has become evident that despite the success of this topology in characterising convergence of convex sets in reflexive Banach spaces, it fails to have many desirable properties in non-reflexive spaces (see [11]). On the other hand the Attouch-Wets topology (or epi-distance topology) has proved itself a valuable tool for obtaining quantitative estimates outside of the reflexive context (see [3],[4],[5]). Another candidate for a replacement of the Mosco topology is the slice topology of Beer [10]. A comprehensive treatment of all of these is given by Beer in [14]. Our paper concentrates on utilizing special features of Banach spaces to analyse epi-distance convergence. This has only previously been embarked upon in one instance [8]. Since the definition of the epi-distance topology only requires the existence of a metric it is usually the case that authors have only specialised to the context of a normed linear space (possibly not complete), [14]. Our considerations here are largely topological and so we do not attempt to obtain error bounds or estimates of metrics generating the Attouch-Wets topology. This endeavour has been undertaken in [8] under stronger assumptions.

Azé & Penot [8] and Beer [12] have considered the following problem: Given two families of closed proper convex functions $\{f_v\}_{v\in W}$ and $\{g_v\}_{v\in W}$, mapping a normed space to the extended reals, both converging in the epi-distance sense to f_w and g_w (as $v \to w$) respectively, when is the family $\{f_v + g_v\}_{v\in W}$ epi-distance convergent to $f_w + g_w$? Beer [12] gives a simple condition, namely $(\operatorname{int} \operatorname{dom} f_w) \cap \operatorname{dom} g_w \neq \emptyset$. This interiority condition transforms to the familiar Slater constraint-qualification on application to perturbed convex optimization problems. In [8] this is weakened to a more complex condition designed to extract convergence estimates. We observe here that this condition is equivalent to $0 \in \operatorname{int}(\operatorname{dom} f_v - \operatorname{dom} g_v)$ for all v in a neighbourhood of w. Although the condition

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$(\operatorname{int}\operatorname{dom} f_w) \cap \operatorname{dom} g_w \neq \emptyset$ is a natural and often-used assumption in convex analysis it has been noted in recent years that it is unduly restrictive in some infinite-dimensional spaces (such as l^1), due to the lack of interior in the cone of positive vectors. This has motivated the introduction of weaker interiority notions such as the concept of "quasi relative interior" [15] and "strong quasi relative interior" [18]. We refer the reader to Gowda and Teboulle [16] for an insightful comparison of these constraint qualifications.

In the context of conjugate duality Robinson [20] first showed that in non-reflexive spaces the weakened condition $0 \in \operatorname{core}(\operatorname{dom} f_w - \operatorname{dom} g_w)$ is sufficient to ensure strong duality. Jeyakumar [19] has shown in this context that such conditions may be used to derive other seemingly weaker constraint-qualifications $0 \in \operatorname{sqri}(\operatorname{dom} f_w - \operatorname{dom} g_w)$, involving the concept of strong quasi relative interior (i.e. $cone(\operatorname{dom} f_w - \operatorname{dom} g_w) = X_w$ is a closed subspace). In this paper we will show that for functions on a Banach space, the conditions: (1) $0 \in \operatorname{sqri}(\operatorname{dom} f_w - \operatorname{dom} g_w)$; and (2) the algebraic complement Y_w of X_w is closed and satisfies $X_v \cap Y_w = \{0\}$ for all v near w, (where $X_v := \overline{\operatorname{span}}(\operatorname{dom} f_v - \operatorname{dom} g_v)$), are sufficient for establishing epi-distance convergence of the sum $\{f_v + g_v\}_{v \in W}$ (as outlined above). The condition $0 \in \operatorname{core}(\operatorname{dom} f_w - \operatorname{dom} g_w)$ (and so also $0 \in \operatorname{sqri}(\operatorname{dom} f_w - \operatorname{dom} g_w)$) is implied by $0 \in \operatorname{int}(\operatorname{dom} f_w - \operatorname{dom} g_w)$. Surprisingly, for closed proper convex functions $0 \in \operatorname{core}(\operatorname{dom} f_w - \operatorname{dom} g_w)$ implies $0 \in \operatorname{int}(\operatorname{dom} f_w - \operatorname{dom} g_w)$, (and so both are equivalent). This equivalence does not seem to receive due acknowledgement in the literature (compare this with the well-known and well-documented equality of core and interior for the domains of single closed convex functions). Moreover $0 \in \operatorname{int}(\operatorname{dom} f_w - \operatorname{dom} g_w)$ holding at w implies it holds locally (i.e. $X_v = X$, locally). Thus our result is a non-trivial extension of the sufficient conditions given in [8] for epi-distance convergence of a sum of parametrised closed convex functions.

These observations lead one to consider a weak set of conditions ensuring strong duality for the Young-Fenchel duality scheme which also imply stability of the perturbed problem (in the epi-distance sense). We also investigate saddle-point convergence of the associated saddle function. From within the framework of variational analysis this allows one to investigate the convergence of approximate solutions of the perturbed primal and dual optimization problems to solutions of the limiting problem. It may be shown that one can quite generally deduce the existence of an accumulation point of the approximating dual solutions. These results augment those obtained by Attouch, Wets, Brézis, Azé and Penot as well as that by other authors.

2. Preliminaries

In this section we draw together a number of results and definitions. This is done to make the development self-contained.

In the following the upper-case roman letters A, B etc, are used to denote sets in a (real) Banach space. $\overline{\mathbb{R}}$ shall stand for the extended reals $[-\infty, +\infty]$. We will let $C(X)$ stand for the class of all nonempty closed convex subsets of a Banach space X and $CB(X)$ the closed bounded convex sets. Place $d(a, B) = \inf\{\, \|a-b\| \mid b \in B \,\}$, $B(0, \rho) = \{\, x \in X \mid \|x\| < \rho \}$ and $\bar{B}(0, \rho) = \{x \in X \mid \|x\| \leq \rho\}$. Corresponding balls in the dual space X^* will be denoted $B^*(0, \rho)$ and $\bar{B}^*(0, \rho)$ respectively. The indicator function of a set A will be denoted δ_A.

We will use u.s.c. to denote upper-semicontinuity and l.s.c. to denote lower-semicontinuity.

Recall that a function $f : X \to \overline{\mathbb{R}}$ is called closed, proper convex on X if and only if f is convex l.s.c. is never $-\infty$ and not is identically $+\infty$. The class of all closed proper convex functions on X is denoted by $\Gamma(X)$, and the class of all weak* closed proper convex functions on X^* will be denoted by $\Gamma^*(X^*)$. We shall use the notation $\operatorname{cl} A$ and $\overline{A}$ interchangeably for the closure of a set A in a topological space (Z, τ) and, to emphasise the topology, we may write $\operatorname{cl}_\tau A$ or $\overline{A}^\tau$. For $x \in Z$, $\mathcal{N}_\tau(x)$ denotes the collection of all τ-neighborhoods of x. For a function $f : Z \to \overline{\mathbb{R}}$, the epigraph of f, denoted $\operatorname{epi} f$, is the set $\{(x, \alpha) \in Z \times \mathbb{R} \mid f(x) \leq \alpha\}$, and the strict epigraph $\operatorname{epi}_s f$ is the set $\{(x, \alpha) \in Z \times \mathbb{R} \mid f(x) < \alpha\}$. The domain, denoted $\operatorname{dom} f$ is the set $\{x \in Z \mid f(x) < +\infty\}$. The (sub-)level set $\{x \in Z \mid f(x) \leq \alpha\}$ (where $\alpha > \inf_Z f$) will be given the abbreviation $\{f \leq \alpha\}$. Any product $X \times Y$ of normed spaces will always be understood to be endowed with the box norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$; any balls in such product spaces will always be with respect to the box norm.

If $f : (Z, \tau) \to \overline{\mathbb{R}}$, its τ-l.s.c. hull, denoted $\overline{f}^\tau$ or $\operatorname{cl}_\tau f$, is defined by $\overline{f}^\tau(x) = \liminf_{x' \xrightarrow{\tau} x} f(x')$. The (extended) lower closure $\underline{\operatorname{cl}}_\tau f$ is defined to coincide with $\operatorname{cl}_\tau f$ if the latter does not take the value $-\infty$ anywhere, and to be identically $-\infty$ otherwise.

Definition 2.1. Let $F{:}W \to 2^X$ be a multifunction from topological spaces W to X.

(1) $\limsup_{v \to w} F(v) = \bigcap_{V \in \mathcal{N}(w)} \overline{\bigcup_{v \in V} F(v)}$.

(2) $\liminf_{v \to w} F(v) = \bigcap_{\{B \subset W \mid w \in \overline{B}\}} \overline{\bigcup_{v \in B} F(v)}$.

(3) $F(\cdot)$ is lower-semicontinuous at w iff $F(w) \subseteq \liminf_{v \to w} F(v)$.

Remark 2.2. It is easily seen that this notion of lower-semicontinuity is equivalent to the classical formulation–namely: For any open set U intersecting $F(w)$ there is a neighborhood V of w for which $F(v) \cap U$ is nonempty for every v in V.

Remark 2.3. For metrizable X, the above definitions can be shown to have the equivalent forms:

(1)

$$\begin{aligned}\limsup_{v \to w} F(v) &= \{x \in X \mid \exists \text{ nets } v_\beta \to w,\ x_\beta \to x \text{ with } x_\beta \in F(v_\beta)\, \forall \beta\,\} \\ &= \{x \in X \mid \liminf_{v \to w} d(x, F(v)) = 0\}\end{aligned}$$

(2)

$$\begin{aligned}\liminf_{v \to w} F(v) &= \{\, x \in X \mid \forall \text{ nets } v_\beta \to w,\ \exists\, x_\beta \to x \text{ s.t. } x_\beta \in F(v_\beta) \text{ eventually } \} \\ &= \{x \in X \mid \limsup_{v \to w} d(x, F(v)) = 0\}\end{aligned}$$

Definition 2.4. Let A be a convex set in a topological vector space and $x \in A$. Then

(1) $\operatorname{cone} A = \cup_{\lambda > 0} \lambda A$ (the smallest convex cone containing A);

(2) $x \in \operatorname{qri} A$ (quasi relative interior) iff $\overline{\operatorname{cone}(A - x)}$ is a subspace of X;

(3) $x \in \operatorname{sqri} A$ (strong quasi relative interior) iff $\operatorname{cone}(A - x)$ is a closed subspace of X;

(4) $x \in \operatorname{core} A$ (core) iff $\forall y \in X\ \exists \epsilon > 0$ such that $\forall \lambda \in [-\epsilon, \epsilon]$ we have $x + \lambda y \in A$;

(5) For $a \in A$ place $\operatorname{aff} A = a + \operatorname{span}(A - a)$, the affine hull; (this is independent of the choice of $a \in A$)

(6) $\operatorname{icr} A$ (intrinsic core) is the core of A relative to $\operatorname{aff} A$;
(7) $y \in \operatorname{ri} A$ (relative interior) iff 0 is an interior point of $A - y$ relative to $\operatorname{aff}(A - y)$.

From [17] and [22] we have the following. Recall that a set A in a topological linear space X is ideally convex if for any bounded sequence $\{x_n\} \subseteq A$ and $\{\lambda_n\}$ of nonnegative numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$, the series $\sum_{n=1}^{\infty} \lambda_n x_n$ either converges to an element of A, or else does not converge at all. Open or closed convex sets are ideally convex, as is any finite-dimensional convex set. In particular, if X is Banach, then such series always converge, and the definition of ideal convexity only requires that $\sum_{n=1}^{\infty} \lambda_n x_n$ be in A. From [17, Section 17E] we have the following

Proposition 2.5. *For a Banach space X,*

(i) *If $C \subset X$ is closed convex, it is ideally convex.*
(ii) *For ideally convex C, $\operatorname{core} C = \operatorname{core} \overline{C} = \operatorname{int} \overline{C} = \operatorname{int} C$.*
(iii) *If A and B are ideally convex subsets of X, one of which is bounded, then $A - B$ is ideally convex.*

Proof. We prove the last assertion only; the rest can be found in the cited reference. Let $\{a_n - b_n\} \subset A - B$ be a bounded sequence, let $\lambda_n \geq 0$ be such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Then $\{a_n\} \subset A$ and $\{b_n\} \subset B$ are both bounded, so $\sum_{n=1}^{\infty} \lambda_n a_n \in A$ and $\sum_{n=1}^{\infty} \lambda_n b_n \in B$ (both convergent). Thus $\sum_{n=1}^{\infty} \lambda_n (a_n - b_n) = \sum_{n=1}^{\infty} \lambda_n a_n - \sum_{n=1}^{\infty} \lambda_n b_n \in A - B$. □

Proposition 2.6. *Let f, $g \in \Gamma(X)$ (X a Banach space). Then*

$$\operatorname{core}(\operatorname{dom} f - \operatorname{dom} g) = \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g).$$

Proof. It suffices to show that $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$ implies $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$ since for any $x \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$ we may define $\tilde{g}(y) := g(y - x)$, a closed proper convex function, for which $\operatorname{dom} \tilde{g} = \operatorname{dom} g + x$ and so $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} \tilde{g})$. If $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$ then by a simple argument $0 \in \operatorname{core}(\operatorname{epi} f - \operatorname{epi} g)$. From this it follows that whenever $\rho > 0$ satisfies $\rho > \|(\bar{x}, \bar{\alpha})\|$ for some $(\bar{x}, \bar{\alpha}) \in \operatorname{epi} f \cap \operatorname{epi} g$, we have

$$0 \in \operatorname{core}(\operatorname{epi} f \cap \overline{B}(0, \rho) - \operatorname{epi} g \cap \overline{B}(0, \rho)). \tag{2.1}$$

Indeed, if $(x, \alpha) \in X \times \mathbb{R}$, then since $X \times \mathbb{R} = \operatorname{cone}(\operatorname{epi} f - \operatorname{epi} g)$ we have for some $\lambda > 0$, $(x_1, \alpha_1) \in \operatorname{epi} f$ and $(x_2, \alpha_2) \in \operatorname{epi} g$ that

$$\begin{aligned} (x, \alpha) &= \lambda \left((x_1, \alpha_1) - (x_2, \alpha_2)\right) \\ &= t\lambda \left([\frac{1}{t}(x_1, \alpha_1) + (1 - \frac{1}{t})(\bar{x}, \bar{\alpha})] - [\frac{1}{t}(x_2, \alpha_2) + (1 - \frac{1}{t})(\bar{x}, \bar{\alpha})] \right) \\ &\in t\lambda \left(\operatorname{epi} f \cap \overline{B}(0, \rho) - \operatorname{epi} g \cap \overline{B}(0, \rho)\right) \end{aligned}$$

if $t > 1$ is selected so that $\frac{1}{t}\|(x_i, \alpha_i)\| + \|(\bar{x}, \bar{\alpha})\| < \rho$ for $i = 1, 2$. As $(x, \alpha) \in X \times \mathbb{R}$ was arbitrary we have

$$X \times \mathbb{R} = \operatorname{cone}(\operatorname{epi} f \cap \overline{B}(0, \rho) - \operatorname{epi}\ g \cap \overline{B}(0, \rho))$$

implying (2.1). As $\operatorname{epi} f \cap \overline{B}(0, \rho)$ and $\operatorname{epi} g \cap \overline{B}(0, \rho)$ are closed and convex they are ideally convex. By Proposition 2.5, $\operatorname{epi} f \cap \overline{B}(0, \rho) - \operatorname{epi} g \cap \overline{B}(0, \rho)$ is ideally convex and so

$$0 \in \operatorname{int}\left(\operatorname{epi} f \cap \overline{B}(0, \rho) - \operatorname{epi} g \cap \overline{B}(0, \rho)\right). \tag{2.2}$$

Using the fact we are using box norms in $X \times \mathbb{R}$ we may project onto X to obtain

$$0 \in \operatorname{int}\left(\{f \le \rho\} \cap \overline{B}(0,\rho) - \{g \le \rho\} \cap \overline{B}(0,\rho)\right) \subseteq \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g).$$

□

The following is a consequence of the previous proof.

Corollary 2.7. *If $f,\ g \in \Gamma(X)$, where X is a Banach space and $\rho > \inf\{\|(x,\alpha)\| \mid (x,\alpha) \in \operatorname{epi} f \cap \operatorname{epi} g\}$ then*

$$0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g) \quad \textit{implies} \quad 0 \in \operatorname{int}\left(\{f \le \rho\} \cap \overline{B}(0,\rho) - \{g \le \rho\} \cap \overline{B}(0,\rho)\right).$$

From [16] we have a characterization of $\operatorname{sqri} C$.

Lemma 2.8. *If $C \subseteq X$ a linear topological space then*

$$\left\{ \begin{array}{l} x \in \operatorname{icr}(C) \\ \quad \textit{and } \operatorname{aff}(C - x) \textit{ is a closed subspace} \end{array} \right\} \textit{ if and only if } x \in \operatorname{sqri} C$$

further, if $x \in \operatorname{sqri} C$ then $\operatorname{aff}(C - x) = \operatorname{cone}(C - x)$ also.

We immediately have the following confirming the conjecture in [16] that strong quasi relative interior is close to being a relative interior (see [16, page 932]).

Lemma 2.9. *Let $f,\ g \in \Gamma(X)$ (X a Banach space). Then $0 \in \operatorname{sqri}(\operatorname{dom} f - \operatorname{dom} g)$ if and only if $0 \in \operatorname{ri}(\operatorname{dom} f - \operatorname{dom} g)$ and $\operatorname{aff}(\operatorname{dom} f - \operatorname{dom} g)$ is a closed subspace of X.*

The infimal convolution plays a central role in our development.

Definition 2.10. Let f and g be closed convex functions on a linear topological space X into the extended reals. Then

$$(f \Box g)(x) := \inf_{y \in X} (f(y) + g(x - y))$$

is called the inf-convolution.

It is well known that the *strict* epigraph of the inf-convolution is equal to the set-addition of the strict epigraphs of the individual functions:

$$\operatorname{epi}_s(f \Box g) = \operatorname{epi}_s f + \operatorname{epi}_s g.$$

Also $\operatorname{dom}(f \Box g) = \operatorname{dom} f + \operatorname{dom} g$ and

$$(f \Box g)^* = f^* + g^*$$

where $f^*(x^*) = \sup_{x \in X}(\langle x, x^* \rangle - f(x))$ is the Young-Fenchel conjugate of f. From [1] we have the following (which establishes a condition for $f^* \Box g^*$ to be lower-semicontinuous and so for $\operatorname{epi} f^* + \operatorname{epi} g^*$ to be weak* closed) giving $\operatorname{epi}(f^* \Box g^*) = \operatorname{epi} f^* + \operatorname{epi} g^*$.

Proposition 2.11. *Assume that f, g are proper closed convex functions from a Banach space X into the extended reals. Suppose also that $0 \in \operatorname{sqri}(\operatorname{dom} f - \operatorname{dom} g)$. Then*

$$(f+g)^* = f^* \Box\, g^*$$

and, moreover the inf-convolution is exact, that is: for each $x^* \in X^*$ we have $(f^* \Box\, g^*)(x^*) = f^*(x^* - y^*) + g^*(y^*)$ for some $y^* \in X^*$.

The proof of this theorem involved the following statement, which will be of importance in the development of our results. We state this as a lemma and for completeness we include the proof, excerpted from the indicated reference.

Lemma 2.12. *Assume that f, g are proper closed convex functions from a Banach space X into the extended reals. Suppose also that $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$. Then*

$$H(K,r) := \{(x_1^*, x_2^*) \in X^* \times X^* \mid f^*(x_1^*) + g^*(x_2^*) \leq K \text{ and } \|x_1^* + x_2^*\| \leq r\}$$

is norm-bounded in $X^* \times X^*$ for any $r > 0$ and $K > \inf_{x^* \in X^*} (f^* \Box\, g^*)(x^*)$.

Proof. In view of the Uniform Boundedness Principle, it suffices to check that for each element (x, y) of $X \times X$ there is a constant $C(x, y)$ such that for all $(x^*, y^*) \in H(K, r)$, we have $\langle x^*, x\rangle + \langle y^*, y\rangle \leq C(x, y)$. Indeed, $x - y = \lambda(u - v)$ for some $\lambda > 0$, $u \in \operatorname{dom} f$ and $v \in \operatorname{dom} g$, and then

$$\begin{aligned}
\langle x^*, x\rangle + \langle y^*, y\rangle &= \lambda\langle x^*, u\rangle + \lambda\langle y^*, v\rangle + \langle x^* + y^*, y - \lambda v\rangle \\
&\leq \lambda(f^*(x^*) + f(u) + g^*(y^*) + g(v)) + \|x^* + y^*\|\|y - \lambda v\| \\
&\leq \lambda(K + f(u) + g(v)) + r\|y - \lambda v\| := C(x, y).
\end{aligned}$$

□

Lower semi-continuity of the epi-graphical multi-function $v \mapsto \operatorname{epi}_s(f_v \Box\, g_v)$ may be deduced from that of its components using the following lemma. We include its derivation due to a lack of a solid reference.

Lemma 2.13. *If $F_1(\cdot)$ and $F_2(\cdot)$ are multi-functions l.s.c. at x_0 then $F(x) = F_1(x) + F_2(x)$ is l.s.c. at x_0.*

Proof. We use the classical formulation of l.s.c. (Remark 2.2). Let N be a neighborhood of a point $y_0 \in F(x_0)$. There exist $y_i \in F_i(x_0)$ such that $y_0 = y_1 + y_2$. Taking two neighborhoods N_i of y_i such that $N_1 + N_2 \subseteq N$, there then exist neighborhoods V_i of x_0 for which $F_i(x) \cap N_i \neq \emptyset$ for all $x \in V_i$. Hence for all $x \in V_1 \cap V_2$,

$$\emptyset \neq F_1(x) \cap N_1 + F_2(x) \cap N_2 \subseteq (F_1(x) + F_2(x)) \cap (N_1 + N_2) \subseteq F(x) \cap N$$

giving the result. □

In [24, Lemma 4.1], it is shown that the epigraphs of the closures $\overline{f_v \Box\, g_v}$ (in the strong topology on X) satisfy

$$\liminf_{v \to w} \operatorname{epi}\left(\overline{f_v \Box\, g_v}\right) \supseteq \overline{\operatorname{epi}}\,(f_w \Box\, g_w)$$

under the condition that $\operatorname{dom} f_v^* \cap \operatorname{dom} g_v^* \neq \emptyset$ for v in a neighborhood of w. This result is of limited use here as we are interested in this occurring in the dual space.

3. Convergences Used

Here we summarise the variational limit notions used in this paper. A reader conversant with variational convergences need only consult this section for definitions and notation. Let X and W be topological spaces, then for $x \in X$, $w \in W$, and $\{f_v\}_{v\in W}$ a collection of $\overline{\mathbb{R}}$-valued functions, define the lower and upper epi-limits by:

$$\begin{aligned}(\text{e-li}_{v\to w} f_v)(x) &:= \sup_{U\in\mathcal{N}(x)} \sup_{V\in\mathcal{N}(w)} \inf_{v\in V} \inf_{y\in U} f_v(y) \ , \\ (\text{e-ls}_{v\to w} f_v)(x) &:= \sup_{U\in\mathcal{N}(x)} \inf_{V\in\mathcal{N}(w)} \sup_{v\in V} \inf_{y\in U} f_v(y) \ .\end{aligned}$$

It is well known [21] that these limits correspond to the Kuratowski limit of the epi-graph multifunction in the sense that

$$\begin{aligned}\text{epi}(\text{e-ls}_{v\to w} f_v) &= \liminf_{v\to w} \text{epi}\, f_v \text{ and} \\ \text{epi}(\text{e-li}_{v\to w} f_v) &= \limsup_{v\to w} \text{epi}\, f_v .\end{aligned} \tag{3.1}$$

Also, for a sequence $\{f_n\}_{n=1}^{\infty}$ of functions, on a first-countable space X, we have [2] for each $x \in X$

$$\begin{aligned}(\text{e-li}_{n\to\infty} f_n)(x) &= \min_{\{x_n\}_{n=1}^{\infty}\to x} \liminf_{n\to\infty} f_n(x_n) \quad \text{and} \\ (\text{e-ls}_{n\to\infty} f_n)(x) &= \min_{\{x_n\}_{n=1}^{\infty}\to x} \limsup_{n\to\infty} f_n(x_n) \ .\end{aligned} \tag{3.2}$$

Definition 3.1. Let $\{f_v\}_{v\in W}$ be a family of functions. We say that $\{f_v\}_{v\in W}$ is epi-u.s.c. (resp. l.s.c.) at $w \in W$ if for all $x \in X$ we have

$$(\text{e-ls}_{v\to w} f_v)(x) \le f_w(x), \quad (\text{resp. } f_w(x) \le (\text{e-li}_{v\to w} f_v)(x)).$$

Equivalently for an epi-u.s.c. family the epi-graphs of f_v are lower Kuratowski-convergent to epi f_w.

Definition 3.2. A family of functions $\{f_v\}_{v\in W}$ in $\overline{\mathbb{R}}^X$ is epi-convergent to a function f_w (as $v \to w$) if it is both epi-u.s.c. and epi-l.s.c. at w.

Since e-li$\,_{v\to w} f_v \le$ e-ls$\,_{v\to w} f_v$ on X, the relation defining epi-convergence is in fact an equality. Now consider the dual space and what would correspond to one half of dual slice convergence.

Definition 3.3. Let $\{f_v\}_{v\in W}$ be a family of functions on a normed space X, and $\{f_v^*\}_{v\in W}$ the family of conjugate functions on X^*. We define a bounded-weak* upper epi-limit (as $v \to w$) of $\{f_v^*\}_{v\in W}$ by

$$\begin{aligned} bw^*\text{-}\limsup_{v\to w} \text{epi}\, f_v^* := \ & \{(x^*,\alpha) \ \in X^* \times \mathbb{R} \mid \exists \text{ nets } v_\beta \to w\,;\ (y_\beta^*, \alpha_\beta) \in \text{epi}\, f_{v_\beta}^* \\ & \text{such that } \alpha_\beta \to \alpha\,;\ y_\beta^* \text{ norm bounded; } y_\beta^* \xrightarrow{w^*} x^*\}.\end{aligned}$$

It is an elementary exercise to see that this set recedes to $+\infty$ in the vertical direction and hence resembles the epigraph of some function. This prompts us to define

Definition 3.4. For $x^* \in X^*$,

$$(bw^*\text{-e-li}_{v\to w} f_v^*)(x^*) := \inf\{\alpha \in \mathbb{R} \mid (x^*, \alpha) \in bw^*\text{-}\limsup_{v\to w} \text{epi } f_v^*\}. \tag{3.3}$$

It then follows that

$$\text{epi}_s\,(bw^*\text{-e-li}_{v\to w} f_v^*) \subseteq bw^*\text{-}\limsup_{v\to w} \text{epi } f_v^* \subseteq \text{epi}\,(bw^*\text{-e-li}_{v\to w} f_v^*)\,. \tag{3.4}$$

Thus $bw^*\text{-e-li}_{v\to w} f_v^*$ is essentially a variational limit in the sense of [6] or [21]. The next lemma is a modification for the bounded-weak* topology of a result in [6, page 290].

Lemma 3.5. *Let X be normed, W topological, and $\{f_v\}_{v\in W}$ be a family of closed proper convex extended-real-valued functions on X. Then*

$$\text{e-ls}_{v\to w} f_v \geq (bw^*\text{-e-li}_{v\to w} f_v^*)^*.$$

Proof. Write $U := bw^*\text{-e-li}_{v\to w} f_v^*$; $V := \text{e-ls}_{v\to w} f_v$. We show that $U^* \leq V$ on X. Let $x \in X$. If $V(x) = +\infty$ or $U \equiv +\infty$, (so $U^* \equiv -\infty$) there is nothing to prove. We then give a proof in the case where $V(x) < +\infty$ and U not identically $+\infty$ on X^*. Let $x^* \in X^*$ with $U(x^*) < +\infty$. Let $\alpha,\ \beta \in \mathbb{R}$ with $\alpha > V(x)$, $\beta > U(x^*)$. Then $(x, \alpha) \in \text{epi}\, V$, $(x^*, \beta) \in \text{epi}_s\, U \subseteq bw^*\text{–}\limsup_{v\to w} \text{epi}\, f_v^*$, so there are nets $v_\gamma \to w$, $x_\gamma^* \to x^*$ (weak*), $\beta_\gamma \to \beta$, with $(x_\gamma^*, \beta_\gamma) \in \text{epi}\, f_{v_\gamma}^*$ for each γ and the x_γ^* uniformly norm-bounded. Also, $(x, \alpha) \in \liminf_{v\to w} \text{epi}\ f_v$, and so there exists $(x_\gamma, \alpha_\gamma)(\in \text{epi}\, f_{v_\gamma}) \to (x, \alpha)$. For each γ, $\alpha_\gamma + \beta_\gamma \geq f_{v_\gamma}^*(x_\gamma^*) + f_{v_\gamma}(x_\gamma) \geq \langle x_\gamma, x_\gamma^* \rangle$ by the Fenchel inequality. As a result, passing to the limit, $\alpha + \beta \geq \langle x, x^* \rangle$. (This is permissible as the x_γ^* are norm-bounded and $x_\gamma \to x$ strongly). Rearrange to obtain $\alpha \geq \langle x, x^* \rangle - \beta$. Since the $\alpha > V(x)$ and $\beta > U(x^*)$ are arbitrary, we conclude that $V(x) \geq \langle x, x^* \rangle - U(x^*)$, and since $x^* \in \text{dom}\, U$ is also arbitrary, it follows that $V(x) \geq U^*(x)$ as claimed. □

The next result provides bounds which will be used repeatedly in subsequent proofs.

Lemma 3.6. *Let $\{f_v\}_{v\in W}$ be a family of proper closed convex extended-real-valued functions on a normed space X. Suppose that $\{f_v\}_{v\in W}$ is strongly epi-u.s.c. with respect to f_w. Then for each $M > 0$,*

$$(\exists V' \in \mathcal{N}(w))(\exists \mu \in \mathbb{R})(\forall v \in V')(\forall \|x^*\| \leq M)(f_v^*(x^*) \geq \mu). \tag{3.5}$$

Proof. Let $M > 0$ and suppose the assertion false. Then there are nets $v_\beta \to w$, $\|x_\beta^*\| \leq M$ such that $\lim_\beta f_{v_\beta}^*(x_\beta^*) = -\infty$. Taking a weakly* convergent subnet $x_{\beta_\gamma}^* \to x^*$, it follows for any real λ, that $(x^*, \lambda) \in bw^*\text{-}\limsup_{v\to w} \text{epi}\, f_v^* \subseteq \text{epi}(bw^*\text{-e-li}_{v\to w} f_v^*)$, so $(bw^*\text{-e-li}_{v\to w} f_v^*)(x^*) = -\infty$, which in turn implies via Lemma 3.5, that $f_w \geq \text{e-ls}_{v\to w} f_v \geq (bw^*\text{-e-li}_{v\to w} f_v^*)^* \equiv +\infty$, contradicting the properness of f_w. □

Definition 3.7. A family $\{f_v\}_{v\in W}$ of $\overline{\mathbb{R}}$-valued functions on a normed space X converges as $v \to w$ in the epi-distance (or Attouch-Wets) topology if for each $K \geq 0$ and $\epsilon > 0$ we have a neighborhood V of w such that for all $v \in V$

$$\text{epi}\, f_w \cap B(0, K) \subseteq \text{epi}\, f_v + B(0, \epsilon) \tag{3.6}$$

and

$$\operatorname{epi} f_v \cap B(0, K) \subseteq \operatorname{epi} f_w + B(0, \epsilon) . \tag{3.7}$$

where the balls are in the box-norm on $X \times \mathbb{R}$.

We can define Attouch-Wets convergence of sets in $C(X)$ in an analogous manner. It is known to be equivalent to the convergence of the corresponding indicator functions. It is an elementary exercise to show that the epi-distance convergence $f_v \to f_w$ implies the (norm-)epi-convergence of f_v to $\overline{f_w}$, where the latter denotes the norm-l.s.c. hull of f_w.

Let X be a normed linear space with continuous dual X^* and let $\Gamma(X)$ (resp. $\Gamma^*(X^*)$) denote the set of proper, lower-semicontinuous (resp. weak* lower semicontinuous) convex functions on X (resp. X^*). Then it is well known [22] that the Young-Fenchel transform $f \mapsto f^*$ is an order-reversing bijection of $\Gamma(X)$ onto $\Gamma^*(X^*)$. In [9] it was shown that the Young-Fenchel transform is a homeomorphism from $(\Gamma(X), \tau_d)$ to $(\Gamma^*(X^*), \tau_d)$ where τ_d is the epi-distance topology.

4. Epi-distance Convergence of Sums

Here we prove sufficient conditions for epi-distance convergence of a sum of parametrized convex functions in a general Banach space. Beer [12] has recently deduced epi-distance convergence for a sum of functions in a general normed space from the epi-distance convergence of the constituent functions, under the condition $(\operatorname{int} \operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$. In [8] a condition which is equivalent to $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$ is used to deduce a similar result, in Banach spaces (although metric estimates are obtained as well, under a more technical assumption). The result proved here in Banach spaces use a strong quasi relative interiority condition more in the spirit of that found in duality theory.

Lemma 4.1. *Suppose that W is topological and that $\{f_v\}_{v \in W}$ and $\{g_v\}_{v \in W}$ are families of proper closed convex extended-real-valued functions on a Banach space X, which are epi-distance convergent (as $v \to w$) to f_w and g_w respectively.*

(i) *If $0 \in \operatorname{core}(\operatorname{dom} f_w - \operatorname{dom} g_w)$, then there exists $V \in \mathcal{N}(w)$ such that for $v \in V$ we have $0 \in \operatorname{core}(\operatorname{dom} f_v - \operatorname{dom} g_v)$. In particular if $\rho > \inf\{\|(x, \alpha)\| \mid (x, \alpha) \in \operatorname{epi} f_w \cap \operatorname{epi} g_w\}$ we have a fixed $\delta > 0$ such that for all $v \in V$,*

$$\overline{B}(0, \delta) \subseteq \{f_v \leq \rho\} \cap \overline{B}(0, \rho) - \{g_v \leq \rho\} \cap \overline{B}(0, \rho) .$$

(ii) *Suppose $0 \in \operatorname{sqri}(\operatorname{dom} f_w - \operatorname{dom} g_w)$. Place $X_v = \overline{\operatorname{span}}(\operatorname{dom} f_v - \operatorname{dom} g_v)$ for $v \in W$. Then for each $\epsilon > 0$, $\rho > 0$ there is some $V' \in \mathcal{N}(w)$ for which*

$$\overline{B}(0, \rho) \cap X_w \subseteq X_v + \overline{B}(0, \epsilon) \quad \text{for } v \in V'.$$

If also for each $\rho > 0$ (sufficiently large) there exist $V'' \in \mathcal{N}(w)$, $\eta > 0$ such that for all $v \in V''$, we have

$$\overline{B}(0, \eta) \cap X_v \subseteq \overline{\{f_v \leq \rho\} \cap \overline{B}(0, \rho) - \{g_v \leq \rho\} \cap \overline{B}(0, \rho)} \tag{4.1}$$

then $\{X_v\}_{v \in W}$ epi-distance converges to X_w.

Proof. (i) By Proposition 2.6 we have $0 \in \operatorname{core}(\operatorname{dom} f_w - \operatorname{dom} g_w)$ if and only if $0 \in \operatorname{int}(\operatorname{dom} f_w - \operatorname{dom} g_w)$. By Corollary 2.7 it follows that

$$0 \in \operatorname{int}\left(\{f_w \leq \rho\} \cap \overline{B}(0, \rho - \delta) - \{g_w \leq \rho\} \cap \overline{B}(0, \rho - \delta)\right)$$

for $\delta > 0$ sufficiently small that $\rho - \delta > \inf\{\|(x, \alpha)\| \mid (x, \alpha) \in \operatorname{epi} f_w \cap \operatorname{epi} g_w\}$. Take $\delta > 0$ sufficiently small so that

$$\overline{B}(0, 3\delta) \subseteq \{f_w \leq \rho\} \cap \overline{B}(0, \rho - \delta) - \{g_w \leq \rho\} \cap \overline{B}(0, \rho - \delta). \tag{4.2}$$

By proposition 7.1.7 of Beer [14] we have the epi-distance convergence of the level-sets of both $\{f_v\}_{v \in W}$ and $\{g_v\}_{v \in W}$ and so for $v \in V \in \mathcal{N}(w)$ we have both

$$\begin{aligned} &\{f_w \leq \rho\} \cap \overline{B}(0, \rho - \delta) \subseteq \{f_v \leq \rho\} \cap \overline{B}(0, \rho) + \bar{B}(0, \delta) \quad \text{and} \\ &\{g_w \leq \rho\} \cap \overline{B}(0, \rho - \delta) \subseteq \{g_v \leq \rho\} \cap \overline{B}(0, \rho) + \bar{B}(0, \delta). \end{aligned} \tag{4.3}$$

Combining this with (4.2) we obtain

$$\begin{array}{rcl} \overline{B}(0, \delta) + \overline{B}(0, 2\delta) & \subseteq & \{f_v \leq \rho\} \cap \overline{B}(0, \rho) - \{g_v \leq \rho\} \cap \overline{B}(0, \rho) + \overline{B}(0, 2\delta), \\ \text{which implies} \quad \overline{B}(0, \delta) & \subseteq & \overline{\{f_v \leq \rho\} \cap \overline{B}(0, \rho) - \{g_v \leq \rho\} \cap \overline{B}(0, \rho)} \end{array}$$

by the Rådström cancellation lemma [14]. The argument is completed by noting that the interior of $\{f_v \leq \rho\} \cap \overline{B}(0, \rho) - \{g_v \leq \rho\} \cap \overline{B}(0, \rho)$ coincides with that of its closure by ideal convexity (Proposition 2.5).

(ii) Since $0 \in \operatorname{core}_{X_w}(\operatorname{dom} f_w - \operatorname{dom} g_w)$, then for $\rho > 0$ as defined above, an argument analogous to that in the proof of Proposition 2.6 gives that $X_w = \operatorname{cone}(\{f_w \leq \rho\} \cap B(0, \rho) - \{g_w \leq \rho\} \cap B(0, \rho))$, which implies that for some $\eta > 0$,

$$\overline{B}(0, \eta) \cap X_w \subseteq \overline{\{f_w \leq \rho\} \cap \overline{B}(0, \rho) - \{g_w \leq \rho\} \cap \overline{B}(0, \rho)}. \tag{4.4}$$

Put $K = \frac{\rho}{\eta}$ and let $\epsilon > 0$ be arbitrary. In general we have $\overline{A} \subseteq A + B(0, \delta)$ so apply (4.3) with $\delta = \frac{\epsilon}{3K}$ to get for $v \in V'$

$$\begin{array}{rcl} \overline{B}(0, \eta) \cap X_w & \subseteq & \overline{\{f_v \leq \rho\} - \{g_v \leq \rho\} + \overline{B}(0, 2\delta)} \\ & \subseteq & \overline{\{f_v \leq \rho\} - \{g_v \leq \rho\}} + \overline{B}(0, 3\delta) \\ & \subseteq & \overline{\operatorname{span}\left(\{f_v \leq \rho\} - \{g_v \leq \rho\}\right)} + \overline{B}(0, 3\delta) \\ \text{implying} \quad \overline{B}(0, K\eta) \cap X_w & = & K\left(\overline{B}(0, \eta) \cap X_w\right) \subseteq K\left(X_v + \overline{B}(0, 3\delta)\right) \\ & = & X_v + \overline{B}(0, 3K\delta) \\ \text{or} \quad \overline{B}(0, \rho) \cap X_w & \subseteq & X_v + \overline{B}(0, \varepsilon) \quad \text{for } v \in V' \in \mathcal{N}(w). \end{array}$$

Using the symmetry in the definition of epi-distance convergence we have for $v \in V''$ that $\overline{B}(0, \rho) \cap X_v \subseteq X_w + \overline{B}(0, \epsilon)$ by a parallel argument, which utilized the uniform value of η along V'' in (4.1) to replace (4.4). Combining both for $V = V'' \cap V'$ we get the result. $\square$

Note that in the case of $0 \in \operatorname{core}(\operatorname{dom} f_w - \operatorname{dom} g_w)$ then it follows that $X_v = X$ for all $v \in V$. Thus we have a trivial kind of convergence of the spaces generated in this case.

Lemma 4.2. *Suppose that $\{f_v\}_{v\in W}$ and $\{g_v\}_{v\in W}$ are families of closed proper convex functions which epi-distance converge to f_w and g_w, respectively (as $v \to w$). Assume that*

$$0 \in \operatorname{sqri}(\operatorname{dom} f_w - \operatorname{dom} g_w).$$

Place

$$X_v = \overline{\operatorname{span}}\,(\operatorname{dom} f_v - \operatorname{dom} g_v)$$

for $v \in W$, and suppose additionally that the algebraic complement Y_w of X_w is closed, with $X_v \cap Y_w = \{0\}$ for all v near w. Then X_v epi-distance converges to X_w as $v \to w$. Further, there is $V \in \mathcal{N}(w)$ and $\delta > 0$ such that for all $v \in V$,

$$B(0,\delta) \subseteq X_v \cap \overline{B}(0,1) + Y_w \text{ and } 0 \in \operatorname{sqri}(\operatorname{dom} f_v - \operatorname{dom} g_v).$$

Proof. We show that for each $\rho > 0$ (sufficiently large) there exist $V \in \mathcal{N}(w)$ and $\eta > 0$ such that (4.1) holds for all $v \in V$. The convergence $X_v \to X_w$ will then follow from Lemma 4.1(ii). Place

$$A_v = \overline{\{f_v \le \rho\} \cap \overline{B}(0,\rho) - \{g_v \le \rho\} \cap \overline{B}(0,\rho)}.$$

where $\rho > \inf\{\|(x,\alpha)\| = \max\{\|x\|, |\alpha|\} \mid (x,\alpha) \in \operatorname{epi} f_w \cap \operatorname{epi} g_w\}$. By an argument analogous to that in the proof of Corollary 2.7, we have A_w generating the subspace X_w (i.e. $X_w = \operatorname{cone} A_w$) and hence $0 \in \operatorname{core}_{X_w} A_w = \operatorname{int}_{X_w} A_w$, since A_w is closed in the Banach space X_w.

By the epi-distance convergence of level-sets [14, Proposition 7.1.7], and by [14, Theorem 7.4.5], both $B_v := \{f_v \le \rho\} \cap \overline{B}(0,\rho)$ and $C_v := -\{g_v \le \rho\} \cap \overline{B}(0,\rho)$ epi-distance converge to B_w and C_w as $v \to w$ and therefore, by two applications of [14, exercise 7.4.7], follow in turn the convergence of $A_v = \overline{B_v + C_v}$ to $A_w = \overline{B_w + C_w}$ and hence that of $\overline{A_v + Y_w}$ to $\overline{A_w + Y_w}$.

Additionally, $0 \in \operatorname{int}(A_w + Y_w)$. Indeed, there is $\mu > 0$ such that $\overline{B}(0,\mu) \cap X_w \subseteq A_w$. Since the map from $X_w \times Y_w$ to X taking (x,y) to $x+y$ is a continuous linear surjection, the Open Mapping Theorem provides $\delta' > 0$ such that $\overline{B}(0,\delta') \subseteq X_w \cap B(0,\mu) + Y_w \subset A_w + Y_w$.

Combining the latter interiority with the convergence $\overline{A_v + Y_w} \to \overline{A_w + Y_w}$, the Rådström cancellation lemma [14, Chapter 7] yields, for any λ satisfying $0 < \lambda < \delta'$, a neighborhood V of w with $\overline{B}(0,\lambda) \subseteq \overline{A_v + Y_w}$ for all $v \in V$, and so, from the ideal convexity of $A_v + Y_w$ and Proposition 2.5, we have $B(0,\lambda) \subseteq A_v + Y_w$. (Note this implies that $X_v + Y_w = X$). Hence, choosing V sufficiently small that also $X_v \cap Y_w = \{0\}$, we obtain for such v,

$$B(0,\lambda) \cap X_v \subseteq (A_v + Y_w) \cap X_v \subseteq A_v + X_v \cap Y_w = A_v.$$

Taking $0 < \eta < \lambda$ now yields (4.1), as required. Thus follows the epi-distance convergence $X_v \to X_w$. Also, from $B(0,\lambda) \subseteq A_v + Y_w \subseteq X_v \cap \overline{B}(0,2\rho) + Y_w$ we obtain, on taking $\delta := \lambda/2\rho$, that $B(0,\delta) \subseteq X_v \cap \overline{B}(0,1) + Y_w$, which is the second assertion of the lemma. Furthermore, since $\{f_v \le \rho\} \cap \overline{B}(0,\rho) - \{g_v \le \rho\} \cap \overline{B}(0,\rho)$ is by Proposition 2.5 an ideally convex subset of X, it is also likewise as a subset of the Banach space X_v, so its interior in X_v is the same as that of its (X_v-)closure A_v and we obtain that $X_v = \operatorname{cone}(\{f_v \le \rho\} \cap \overline{B}(0,\rho) - \{g_v \le \rho\} \cap \overline{B}(0,\rho))$ so $0 \in \operatorname{sqri}(\operatorname{dom} f_v - \operatorname{dom} g_v)$. $\square$

Recall from Lemma 4.2 that for some neighborhood V of w we have for all $v \in V$ that $X_v \cap \overline{B}(0,1) + Y_w$ contains a δ-ball at 0. If for such v we also have $X_v \cap Y_w = \{0\}$, then at once $X = X_v \oplus Y_w$. Thus we can form natural extensions of functionals on X_v to X by the following scheme:

Definition 4.3. For $v \in W$, $x^* \in X_v^*$, define an extension $\widetilde{x^*} \in X^*$ by demanding that it vanish identically on Y_w, and extending by linearity.

Remark 4.4. For all $v \in V$, where V is small enough that the assumptions and conclusions of Lemma 4.2 are satisfied thereon,

$$\|\widetilde{x^*}\| = \frac{1}{\delta} \sup_{x \in B(0,\delta)} \langle \widetilde{x^*}, x\rangle \leq \frac{1}{\delta} \sup_{x \in \overline{B}(0,1) \cap X_v + Y_w} \langle \widetilde{x^*}, x\rangle = \frac{1}{\delta} \sup_{x \in \overline{B}(0,1) \cap X_v} \langle x^*, x\rangle = \frac{1}{\delta}\|x^*\|_{X_v^*}.$$

Note that the bound $\frac{1}{\delta}$ is uniform in $v \in V$. We may assume $\delta \leq 1$.

Lemma 4.5. *Suppose the family $\{X_v\}_{v \in W}$ of closed subspaces of X satisfy the following:*

(i) $\limsup_{v \to w} X_v \subseteq X_w$*;*

(ii) X_w *has finite codimension.*

Then $X_v \cap Y_w = \{0\}$ for all v in a neighborhood of w, where Y_w denotes the algebraic complement of X_w.

Proof. Suppose the result false. Then there exist nets $v_\alpha \to w$ and $x_\alpha \neq 0$ in $X_{v_\alpha} \cap Y_w$. Normalize so that x_α has unit norm for all α. By the compactness of the unit ball in Y_w, extracting a convergent subnet if necessary, we may assume that x_α has a limit $x \in Y_w \cap \overline{B}(0,1)$. Then $x \neq 0$, for if otherwise, $x_\alpha \to 0$, contradicting the normalization. Also, $x \in \limsup_{v \to w} X_v \subseteq X_w$ and we have $0 \neq x \in X_w \cap Y_w$, a contradiction. □

Remark 4.6. In the context of Lemma 4.2, it should be noted that the condition that $X_v \cap Y_w = \{0\}$ locally in v, is in fact equivalent to the epi-distance convergence $X_v \to X_w$, as is seen from the following lemma.

Lemma 4.7. *Let X_v be closed subspaces of a Banach space X, with X_v epi-distance converging to X_w. Let X_w have closed algebraic complement Y. Then $X_v \cap Y = \{0\}$ for all v near w.*

Proof. Since $X_w - Y = X$, [8, Corollary 2.9] implies the epi-distance convergence $X_v \cap Y \to X_w \cap Y = \{0\}$. Hence, if $0 < \epsilon < 1$, then for all v in a neighborhood of w, the unit ball in the space $X_v \cap Y$ is contained in the ϵ-ball at the origin in X. This forces $X_v \cap Y = \{0\}$. □

We now introduce the following notation:

$$\begin{aligned} H_K(X^*, v) &:= \{(x_1^*, x_2^*) \in X^* \times X^* \mid f_v^*(x_1^*) + g_v^*(x_2^*) < K\,,\ \|x_1^* + x_2^*\| < K\}, \\ H_K(X_v^*, v) &:= \{(x_1^*, x_2^*) \in X_v^* \times X_v^* \mid (f_v|_{X_v})^*(x_1^*) + (g_v|_{X_v})^*(x_2^*) < K\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \|x_1^* + x_2^*\|_{X_v^*} < K\}\,, \end{aligned}$$

where in the latter expression, the conjugation operation is with respect to the subspace X_v.

Also, place

$$\widetilde{H_K}(X_v^*, v) := \{(\widetilde{x_1^*}, \widetilde{x_2^*}) \mid (x_1^*, x_2^*) \in H_K(X_v^*, v)\} \subseteq X^* \times X^*,$$

where again the tilde denotes the extension operation of Definition 4.3.

Lemma 4.8. *Assume that W is topological and that $\{f_v\}_{v\in W}$ and $\{g_v\}_{v\in W}$ are families of proper closed convex extended-real-valued functions on a Banach space X, which are epi-distance convergent (as $v \to w$) to f_w and g_w respectively. Assume that $0 \in \operatorname{sqri}(\operatorname{dom} f_w - \operatorname{dom} g_w)$ and that for all v in a neighborhood of w, we have the following:*

(i) *at least one of $\operatorname{dom} f_v$, $\operatorname{dom} g_v$ intersects the subspace $X_v := \overline{\operatorname{span}}(\operatorname{dom} f_v - \operatorname{dom} g_v)$;*
(ii) *The algebraic complement Y_w of X_w is closed, and satisfies $X_v \cap Y_w = \{0\}$.*

Then for each sufficiently large $K > 0$, there exist $V \in \mathcal{N}(w)$ and $\gamma > 0$ such that for all $v \in V$,

$$\widetilde{H_K}(X_v^*, v) \subseteq B^*(0, \gamma)\,.$$

Proof. Since X_v is a subspace for each v, it follows from (i) that both $\operatorname{dom} f_v$ and $\operatorname{dom} g_v$ are contained in X_v. As $f_w^*\Box g_w^* = (f_w + g_w)^* \in \Gamma^*(X^*)$ by Proposition 2.11, and is therefore proper, there is K_0 such that $B^*(0, K_0) \cap \operatorname{epi}(f_w^*\Box g_w^*) \neq \emptyset$. From the epi-distance convergence (see [9]) $f_v^* \to f_w^*$ and $g_v^* \to g_w^*$, follows the lower-semicontinuity (at w) of $v \mapsto \operatorname{epi} f_v^*$ and $v \mapsto \operatorname{epi} g_v^*$ (with respect to the norm on X^*), so by Lemma 2.13, and the exactness of the inf-convolution $f_w^*\Box g_w^*$ (Proposition 2.11),

$$\operatorname{epi}(f_w^*\Box g_w^*) = \operatorname{epi} f_w^* + \operatorname{epi} g_w^* \subseteq \liminf_{v\to w}\,(\operatorname{epi} f_v^* + \operatorname{epi} g_v^*) \subseteq \liminf_{v\to w}\, \operatorname{epi}(f_v^*\Box g_v^*)\,.$$

Hence the map $v \mapsto \operatorname{epi} f_v^*\Box g_v^*$ is (norm-)lower-semicontinuous at w, so $B^*(0, K_0) \cap \operatorname{epi} f_v^*\Box g_v^* \neq \emptyset$ for all v in some neighborhood V_0 of w. Let $K \geq K_0$. Since $0 \in \operatorname{core}_{X_w}(\operatorname{dom} f_w|_{X_w} - \operatorname{dom} g_w|_{X_w})$ from the strong quasi relative interiority, we have by Lemma 2.12 that $H_K(X_w^*, w)$ is norm-bounded in $X_w^* \times X_w^*$. (Note that $H_K(X_v^*, v)$ is nonempty for each $v \in V_0$, for if $(x^*, \alpha) \in \operatorname{epi}(f_v^*\Box g_v^*) \cap B^*(0, K) \neq \emptyset$, then from the definition of the inf-convolution there is $y^* \in X^*$ such that $f_v^*(y^*) + g_v^*(x^* - y^*) < K$ and so $(y^*, x^* - y^*) \in H_K(X^*, v)$. Taking restrictions to X_v then yields elements of $H_K(X_v^*, v)$, since X_v contains both $\operatorname{dom} f_v$ and $\operatorname{dom} g_v$). For $K \geq K_0$, define $\hat{\gamma}(K) \in \mathbb{R}$ by

$$\hat{\gamma}(K) = \sup\{\|x_i^*\|_{X_w^*} \mid i = 1, 2;\ (x_1^*, x_2^*) \in H_K(X_w^*, w)\}. \tag{4.5}$$

Place

$$\gamma := \frac{3}{\delta}(\hat{\gamma}(\frac{K}{\delta} + 1) + 1), \tag{4.6}$$

where $0 < \delta \leq 1$ is provided by Lemma 4.2 (see also Definition 4.3 and Remark 4.4).

Let $M > \max\{\gamma, 1\}$ be large enough that $H_K(X^*, w) \cap B^*(0, M\delta) \neq \emptyset$. By appropriate shrinkage of V_0 we may assume, from the epi-distance convergence of X_v to X_w (Lemma 4.2) that for all v in V_0 we have additionally:

$$X_v \cap \overline{B}(0, 1) \subseteq X_w + B(0, \frac{1}{2M}), \text{ and } X_w \cap \overline{B}(0, 1) \subseteq X_v + B(0, \frac{1}{2M})\,, \tag{4.7}$$

as well as (see Lemma 3.6) for some $\mu \in \mathbb{R}$,

$$(\forall v \in V_0)(\forall \|x^*\| \leq M)(f_v^*(x^*) \geq \mu \text{ and } g_v^*(x^*) \geq \mu)\,. \tag{4.8}$$

Let $\beta > \max\{|\mu|, |\frac{K}{\delta} - \mu|, M\}$, $v \in V_0$, and $(x_1^*, x_2^*) \in H_{\frac{K}{\delta}}(X^*, v) \cap B^*(0, M)$. Then $f_v^*(x_1^*) + g_v^*(x_2^*) < \frac{K}{\delta}$, $\|(x_1^*, x_2^*)\| < M$, so from (4.8), have that $\mu \leq g_v^*(x_2^*) < \frac{K}{\delta} - f_v^*(x_1^*) \leq \frac{K}{\delta} - \mu$ so $|g_v^*(x_2^*)| < \beta$ and hence

$$(x_2^*, g_v^*(x_2^*)) \in \operatorname{epi} g_v^* \cap B^*(0, \beta).$$

Similarly,

$$(x_1^*, f_v^*(x_1^*)) \in \operatorname{epi} f_v^* \cap B^*(0, \beta).$$

By the epi-distance convergence $f_v^* \to f_w^*$, $g_v^* \to g_w^*$, there is a neighborhood V' of w such that for all $v \in V'$,

$$\operatorname{epi} g_v^* \cap B^*(0, \beta) \subseteq \operatorname{epi} g_w^* + B^*(0, 1/3),$$

and similarly for the f_v^*, so for $v \in V_0 \cap V'$, and $(x_1^*, x_2^*) \in H_{\frac{K}{\delta}}(X^*, v) \cap B^*(0, M)$, we obtain that $(x_1^*, f_v^*(x_1^*)) \in \operatorname{epi} f_w^* + B^*(0, 1/3)$ and $(x_2^*, g_v^*(x_2^*)) \in \operatorname{epi} g_w^* + B^*(0, 1/3)$. There then exist $\bar{x}_1^*$, $\bar{x}_2^*$, β_1, β_2 with: $\|\bar{x}_i^* - x_i^*\| < 1/3$ $(i = 1, 2)$; $\beta_1 \geq f_w^*(\bar{x}_1^*)$; $\beta_2 \geq g_w^*(\bar{x}_2^*)$, with also

$$|\beta_1 - f_v^*(x_1^*)| < 1/3; \; |\beta_2 - g_v^*(x_2^*)| < 1/3,$$

so that

$$f_w^*(\bar{x}_1^*) + g_w^*(\bar{x}_2^*) \leq \beta_1 + \beta_2 < 1/2 + f_v^*(x_1^*) + 1/2 + g_v^*(x_2^*) < \frac{K}{\delta} + 1.$$

Also, $\|\bar{x}_1^* + \bar{x}_2^*\| \leq \|x_1^* + x_2^*\| + 1 < \frac{K}{\delta} + 1$ and we have $(x_1^*, x_2^*) \in H_{\frac{K}{\delta}+1}(X^*, w) + B^*(0, 1/3)$. It has now been proved that there exists $V' \in \mathcal{N}(w)$ such that for all $v \in V'$

$$H_{\frac{K}{\delta}}(X^*, v) \cap B^*(0, M) \subseteq H_{\frac{K}{\delta}+1}(X^*, w) + B^*(0, 1/3). \tag{4.9}$$

By another contraction of V_0, if required, we can ensure that (4.9) is also satisfied on V_0, as well as all the conclusions of Lemma 4.2.

Let $v \in V_0$ and $(y_1^*, y_2^*) \in \widetilde{H_K}(X_v^*, v) \cap B^*(0, M)$. Then by Remark 4.4, $\|y_1^* + y_2^*\| \leq \frac{1}{\delta}\|y_1^*|_{X_v} + y_2^*|_{X_v}\|_{X_v^*} < K/\delta$, so by (4.9),

$$(y_1^*, y_2^*) \in H_{\frac{K}{\delta}}(X^*, v) \cap B^*(0, M) \subseteq H_{\frac{K}{\delta}+1}(X^*, w) + B^*(0, 1/3).$$

Taking restrictions to X_w, we obtain

$$\|(y_1^*|_{X_w}, y_2^*|_{X_w})\|_{X_w^*} < \hat{\gamma}(\frac{K}{\delta} + 1) + 1/3\,,$$

since $(y_1^*|_{X_w}, y_2^*|_{X_w})$ is within distance 1/3 of $H_{\frac{K}{\delta}+1}(X_w^*, w)$ in $X_w^* \times X_w^*$, and from (4.5) the latter set is norm-bounded by $\hat{\gamma}(\frac{K}{\delta} + 1)$.

For each $v \in V_0$, there are $\bar{x}_1$, $\bar{x}_2$ in $B(0,1) \cap X_v$ such that $\|(y_1^*, y_2^*)|_{X_v}\|_{X_v^*} \leq \langle y_1^*, \bar{x}_1\rangle + \langle y_2^*, \bar{x}_2\rangle + 1$, and from (4.7) there are $\bar{y}_1$, $\bar{y}_2$ in X_w within $\frac{1}{2M}$ of $\bar{x}_1$, $\bar{x}_2$ respectively, so that

$$|\langle (y_1^*, y_2^*), (\bar{x}_1, \bar{x}_2) - (\bar{y}_1, \bar{y}_2)\rangle| \leq \|(y_1^*, y_2^*)\| \left(\|\bar{x}_1 - \bar{y}_1\| + \|\bar{x}_2 - \bar{y}_2\|\right) \leq 1.$$

Thus

$$\begin{aligned}
\|(y_1^*, y_2^*)|_{X_v}\|_{X_v^*} &\leq \langle y_1^*, \bar{x}_1\rangle + \langle y_2^*, \bar{x}_2\rangle + 1 \\
&\leq |\langle (y_1^*, y_2^*), (\bar{x}_1, \bar{x}_2) - (\bar{y}_1, \bar{y}_2)\rangle| + \langle y_1^*, \bar{y}_1\rangle + \langle y_2^*, \bar{y}_2\rangle + 1 \\
&\leq 2 + 3\|(y_1^*|_{X_w}, y_2^*|_{X_w})\|_{X_w^*} \quad \text{since } \|\bar{y}_i\| \leq 1 + \frac{1}{2M} \leq 3/2 \\
&< 2 + 3(\hat{\gamma}(\frac{K}{\delta} + 1) + 1/3) = 3(\hat{\gamma}(\frac{K}{\delta} + 1) + 1)
\end{aligned}$$

and since $y_i^* = \widetilde{y_i^*|_{X_v}}$, we obtain (on recalling the definition (4.6) for γ)

$$\|(y_1^*, y_2^*)\| \leq \frac{1}{\delta}\|(y_1^*, y_2^*)|_{X_v}\|_{X_v^*} < \frac{3}{\delta}(\hat{\gamma}(\frac{K}{\delta} + 1) + 1) = \gamma\,.$$

Hence it has been shown for $v \in V_0$, and γ and $M > \gamma$ as above, that

$$\widetilde{H_K}(X_v^*, v) \cap B^*(0, M) \subseteq B^*(0, \gamma)\,. \tag{4.10}$$

This implies that

$$\widetilde{H_K}(X_v^*, v) \cap B^*(0, M) = \widetilde{H_K}(X_v^*, v) \cap B^*(0, \gamma)\,,$$

and further, by convexity, *whenever* $\widetilde{H_K}(X_v^*, v) \cap B^*(0, M)$ *is nonempty*, that

$$\widetilde{H_K}(X_v^*, v) \subseteq B^*(0, M)\,,$$

for if $\widetilde{H_K}(X_v^*, v)$ contains points outside $B^*(0, M)$ then we can join these with line-segments to points of $\widetilde{H_K}(X_v^*, v) \cap B^*(0, M) = \widetilde{H_K}(X_v^*, v) \cap B^*(0, \gamma)$, yielding elements of $\widetilde{H_K}(X_v^*, v) \cap B^*(0, M)$ of norm greater than γ, contradicting (4.10).

To complete the proof of our result, then, we need to demonstrate for all v in some neighborhood V' of w, that $\widetilde{H_K}(X_v^*, v) \cap B^*(0, M)$ is indeed nonempty. Once this is done, it will follow that for all $v \in V_0 \cap V'$,

$$\widetilde{H_K}(X_v^*, v) \subseteq B^*(0, M)\,,$$

so from (4.10),

$$\widetilde{H_K}(X_v^*, v) \subseteq B^*(0, \gamma)\,.$$

We now attend to this final task. From the choice of M, recall that $H_K(X^*, w) \cap B^*(0, M\delta) \neq \emptyset$. We first show that for every v in some neighborhood V',

$$H_K(X^*, v) \cap B^*(0, M\delta) \neq \emptyset\,, \tag{4.11}$$

from which the required result will follow, on taking restrictions to X_v. Set $N := M\delta$, and take $(x_1^*, x_2^*) \in H_K(X^*, w) \cap B^*(0, N)$. Since the f_v^* strongly epi-converge to f_w^*, as do the g_v^* to g_w^*, there follows

$$\begin{aligned} K &> f_w^*(x_1^*) + g_w^*(x_2^*) = (\text{e-ls}_{v\to w} f_v^*)(x_1^*) + (\text{e-ls}_{v\to w} g_v^*)(x_2^*) \\ &= \sup_{U_1 \in \mathcal{N}_s(x_1^*)} \limsup_{v\to w} \inf_{U_1} f_v^* + \sup_{U_2 \in \mathcal{N}_s(x_2^*)} \limsup_{v\to w} \inf_{U_2} g_v^* \\ &\geq \limsup_{v\to w} \inf_{U_1} f_v^* + \limsup_{v\to w} \inf_{U_2} g_v^* \end{aligned}$$

for any (strong) neighborhoods U_1, U_2 of x_1^* and x_2^* respectively. Select these neighborhoods to be so small that $\|\bar{x}_1^* + \bar{x}_2^*\| < K$ and $\|(\bar{x}_1^*, \bar{x}_2^*)\| < N$ whenever $(\bar{x}_1^*, \bar{x}_2^*) \in U_1 \times U_2$. Thus, since neither $\limsup_{v\to w} \inf_{U_1} f_v^*$ nor $\limsup_{v\to w} \inf_{U_2} g_v^*$ equals $+\infty$, there is $V \in \mathcal{N}(w)$ such that for each $v \in V$, $\inf_{U_1} f_v^* \neq +\infty$, $\inf_{U_2} g_v^* \neq +\infty$, and

$$\limsup_{v\to w} \inf_{U_1} f_v^* + \limsup_{v\to w} \inf_{U_2} g_v^* \geq \limsup_{v(\in V)\to w} \left[\inf_{U_1} f_v^* + \inf_{U_2} g_v^*\right].$$

Hence there is $V' \subseteq V_0$ such that $K > \sup_{v\in V'} [\inf_{U_1} f_v^* + \inf_{U_2} g_v^*]$, that is, for each $v \in V'$, there is $(\bar{x}_1^*, \bar{x}_2^*) \in U_1 \times U_2$ such that $f_v^*(\bar{x}_1^*) + g_v^*(\bar{x}_2^*) < K$, so from the choice of U_i, we have $(\bar{x}_1^*, \bar{x}_2^*) \in H_K(X^*, v) \cap B^*(0, N)$ and so (4.11) is proved. For such $(\bar{x}_1^*, \bar{x}_2^*)$ we immediately have $(\bar{x}_1^*|_{X_v}, \bar{x}_2^*|_{X_v}) \in H_K(X_v^*, v)$ and $\|(\bar{x}_1^*, \bar{x}_2^*)|_{X_v}\|_{X_v^*} \leq \|(\bar{x}_1^*, \bar{x}_2^*)\| < N$, so that

$$(\widetilde{\bar{x}_1^*|_{X_v}}, \widetilde{\bar{x}_2^*|_{X_v}}) \in \widetilde{H_K}(X_v^*, v) \cap B^*(0, N/\delta) = \widetilde{H_K}(X_v^*, v) \cap B^*(0, M),$$

proving nonemptiness of the latter set. $\square$

We are now in a position to prove the main result of this paper.

Theorem 4.9. *Assume that W is topological and that $\{f_v\}_{v\in W}$ and $\{g_v\}_{v\in W}$ are families of proper closed convex extended-real-valued functions on a Banach space X, which are epi-distance convergent (as $v \to w$) to f_w and g_w respectively. Moreover suppose that*

(i) $0 \in \operatorname{sqri}(\operatorname{dom} f_w - \operatorname{dom} g_w)$,

(ii) *the algebraic complement Y_w of X_w is closed and $X_v \cap Y_w = \{0\}$ for all v near w, where $X_v := \overline{\operatorname{span}}(\operatorname{dom} f_v - \operatorname{dom} g_v)$.*

Then $\{f_v + g_v\}_{v\in W}$ is epi-distance convergent to $f_w + g_w$ as $v \to w$.

Proof. We first prove the result under the additional assumption that for all v in a neighborhood of w, at least one of $\operatorname{dom} f_v$, $\operatorname{dom} g_v$ intersects the subspace $X_v := \overline{\operatorname{span}}(\operatorname{dom} f_v - \operatorname{dom} g_v)$, which in fact implies that X_v contains both $\operatorname{dom} f_v$ and $\operatorname{dom} g_v$. Subsequently we will discard this assumption.

By Lemma 4.2, $0 \in \operatorname{sqri}(\operatorname{dom} f_v - \operatorname{dom} g_v)$ for all v in a neighborhood of w, so by Proposition 2.11, $f_v^* \square g_v^*$ is exact and $f_v^* \square g_v^* = (f_v + g_v)^* \in \Gamma^*(X^*)$. If we assume that the family of functions $\{f_v^* \square g_v^*\}_{v\in W}$ epi-distance converge (as $v \to w$) to $f_w^* \square g_w^*$ then since epi-distance convergence τ_d renders Young-Fenchel conjugation $f \mapsto f^*$ a homeomorphism from $(\Gamma(X), \tau_d)$ to $(\Gamma^*(X^*), \tau_d)$ (see [9]) we may argue as follows:

$$\begin{aligned} \tau_d\text{-}\lim_{v\to w}(f_v + g_v) &= \tau_d\text{-}\lim_{v\to w}(f_v^* \square g_v^*)^* = (\tau_d\text{-}\lim_{v\to w} f_v^* \square g_v^*)^* \\ &= (f_w^* \square g_w^*)^* = f_w + g_w. \end{aligned}$$

The required task is then to prove the epi-distance convergence of $f_v^*\Box g_v^*$ to $f_w^*\Box g_w^*$. To establish this, we first derive (3.7) (for $f_v^*\Box g_v^*$), then as a corollary of that argument we shall obtain (3.6) also.

As argued in the proof of Lemma 4.8, the map $v \mapsto f_v^*\Box g_v^*$ is (norm-)lower-semicontinuous at w. Let K_0 be such that $B^*(0, K_0) \cap \operatorname{epi}(f_w^*\Box g_w^*) \neq \emptyset$. Then the semicontinuity implies that for all v in some neighborhood V_0 of w, $B^*(0, K_0) \cap \operatorname{epi} f_v^*\Box g_v^* \neq \emptyset$. Let $K > K_0$, $\epsilon > 0$. We now shrink V_0 further so that additionally, the indicated hypotheses of the theorem hold thereon, along with the conclusions of Lemmas 4.8 and 4.2.

Let $v \in V_0$ and $(x^*, \alpha) \in \operatorname{epi}(f_v^*\Box g_v^*) \cap B^*(0, K)$. From the exactness of the inf-convolution there exists $y^* \in X^*$ such that $(f_v^*\Box g_v^*)(x^*) = f_v^*(y^*) + g_v^*(x^* - y^*) \leq \alpha < K$. Thus $(y^*, x^* - y^*) \in H_K(X^*, v)$. Let x_1^* denote the extension (in the sense of Definition 4.3) of $y^*|_{X_v}$ to X. Place $x_2^* := x^* - x_1^*$ (so $x_1^* + x_2^* = x^*$). Since X_v contains both $\operatorname{dom} f_v$ and $\operatorname{dom} g_v$, there follows that

$$f_v^*(x_1^*) + g_v^*(x_2^*) = f_v^*(y^*) + g_v^*(x^* - y^*) \leq \alpha\,.$$

Let $z^* := \widetilde{x_2^*|_{X_v}}$ (the extension of $x_2^*|_{X_v} \in X_v^*$). Then since

$$\|x_1^*|_{X_v} + z^*|_{X_v}\|_{X_v^*} = \|(x_1^* + x_2^*)|_{X_v}\|_{X_v^*} \leq \|x_1^* + x_2^*\| = \|x^*\| < K$$

and

$$(f_v|_{X_v})^*(x_1^*|_{X_v}) + (g_v|_{X_v})^*(x_2^*|_{X_v}) = f_v^*(x_1^*) + g_v^*(x_2^*) \leq \alpha < K\,,$$

we have

$$(x_1^*, z^*) = (\widetilde{y^*|_{X_v}}, \widetilde{x_2^*|_{X_v}}) = (\widetilde{x_1^*|_{X_v}}, \widetilde{x_2^*|_{X_v}}) \in \widetilde{H_K}(X_v^*, v)\,.$$

Hence, from Lemma 4.8 follows that $\|x_1^*\| \leq \|(x_1^*, z^*)\| \leq \gamma$ (where γ is defined in the said Lemma, and is independent of $v \in V_0$, but dependent on K), and so $\|x_2^*\| \leq \|x^*\| + \|x_1^*\| < K + \gamma := \bar{\gamma}$. Thus we have

$$\begin{aligned}
(x_1^*, x_2^*) &\in H_K(X^*, v) \cap B^*(0, \bar{\gamma}) \ \text{ and} && (4.12)\\
(x^*, \alpha) &= (x_1^*, \alpha - g_v^*(x_2^*)) + (x_2^*, g_v^*(x_2^*)) \in \operatorname{epi} f_v^* \cap (B_{X^*}(0, \bar{\gamma}) \times \mathbb{R}) \\
&\qquad + \operatorname{epi} g_v^* \cap (B_{X^*}(0, \bar{\gamma}) \times \mathbb{R}), && (4.13)
\end{aligned}$$

where $B_{X^*}(0, \bar{\gamma})$ denotes the open ball of radius $\bar{\gamma}$ in X^*. We now need to find bounds on the $g_v^*(x_2^*)$. From Lemma 3.6, we may contract V_0 if necessary so that for some real constant μ we have

$$(\forall v \in V_0)(\forall \|x_2^*\| \leq \bar{\gamma})(g_v^*(x_2^*) \geq \mu).$$

To obtain an upper bound, define, for $v \in V_0$,

$$F_v := \{x_2^* \in X^* \mid \exists x_1^* \in X^* \,\text{such that}\; (x_1^*, x_2^*) \in H_K(X^*, v) \cap B^*(0, \bar{\gamma})\}.$$

Note for each $v \in V_0$, that $F_v \neq \emptyset$ since from (4.12), $H_K(X^*, v) \cap B^*(0, \bar{\gamma})$ is nonempty. We claim now that

$$(\exists V' \in \mathcal{N}(w))(\exists \rho \in \mathbb{R})(\forall v \in V')(\forall x_2^* \in F_v)(g_v^*(x_2^*) \leq \rho). \tag{4.14}$$

To see this, suppose the contrary. Then there are nets $v_\beta \to w$ (in V_0), $x^*_{2_\beta} \in F_{v_\beta}$ with $\lim_\beta g^*_{v_\beta}(x^*_{2_\beta}) = +\infty$. If $x^*_{1_\beta}$ is such that $(x^*_{1_\beta}, x^*_{2_\beta}) \in H_K(X^*, v_\beta) \cap B^*(0,\bar{\gamma})$, then as $\|x^*_{1_\beta}\| \le \bar{\gamma}$, we may, by taking a weak*-convergent subnet if necessary, assume that it has a weak* limit, to be denoted by x^*_1.

Let $\lambda \in \mathbb{R}$. Then eventually $g^*_{v_\beta}(x^*_{2_\beta}) > K - \lambda$ so for such β, $f^*_{v_\beta}(x^*_{1_\beta}) < K - g^*_{v_\beta}(x^*_{2_\beta}) < \lambda$. Thus $(x^*_1, \lambda) \in bw^*\text{-}\limsup_{v\to w} \operatorname{epi} f^*_v \subseteq \operatorname{epi}(bw^*\text{-e-li}_{v\to w} f^*_v)$ and as $\lambda \in \mathbb{R}$ is arbitrary, $(bw^*\text{-e-li}_{v\to w} f^*_v)(x^*_1) = -\infty$, so by Lemma 3.5, as f_v strongly epi-converges to f_w,

$$f_w \ge (bw^*\text{-e-li}_{v\to w} f^*_v)^* \equiv +\infty \text{ on } X\,,$$

contradicting the properness of f_w. This proves (4.14).

Hence, if $V_0 \in \mathcal{N}(w)$ is taken small enough, we have also for some $\rho > 0$ (dependent on K, ϵ) that

$$(\forall v \in V_0)(\forall x^*_2 \in F_v)(|g^*_v(x^*_2)| \le \rho)\,.$$

Combining this with (4.13), it follows for all $v \in V_0$, that

$$\operatorname{epi}(f^*_v \Box g^*_v) \cap B^*(0,K) \subseteq \operatorname{epi} f^*_v \cap B^*(0,\hat{K}) + \operatorname{epi} g^*_v \cap B^*(0,\hat{K}) \tag{4.15}$$

where $\hat{K}(K,\epsilon) := K + \max(\rho, \bar{\gamma})$.

Finally, by the epi-distance convergence $f^*_v \to f^*_w$ and $g^*_v \to g^*_w$ as $v \to w$, there is $V_1 \in \mathcal{N}(w)$ so that for all $v \in V_1$, $\operatorname{epi} f^*_v \cap B^*(0,\hat{K}) \subseteq \operatorname{epi} f^*_w + B^*(0,\epsilon/2)$ and similarly for the g^*_v at w. It then follows, for all $v \in V_0 \cap V_1$, that

$$\begin{aligned}\operatorname{epi}(f^*_v \Box g^*_v) \cap B^*(0,K) &\subseteq \operatorname{epi} f^*_v \cap B^*(0,\hat{K}) + \operatorname{epi} g^*_v \cap B^*(0,\hat{K})\\ &\subseteq \operatorname{epi} f^*_w + \operatorname{epi} g^*_w + B^*(0,\epsilon).\end{aligned}$$

This yields (3.7). To finish, we verify (3.6). This follows immediately from (4.15) at $v = w$ and the epi-distance convergence $f^*_v \to f^*_w$ and $g^*_v \to g^*_w$, since for all v in a neighborhood of w,

$$\begin{aligned}\operatorname{epi}(f^*_w \Box g^*_w) \cap B^*(0,K) &\subseteq \operatorname{epi} f^*_w \cap B^*(0,\hat{K}) + \operatorname{epi} g^*_w \cap B^*(0,\hat{K})\\ &\subseteq \operatorname{epi} f^*_v + \operatorname{epi} g^*_v + B^*(0,\epsilon/2) + B^*(0,\epsilon/2)\\ &\subseteq \operatorname{epi}(f^*_v \Box g^*_v) + B^*(0,\epsilon)\,.\end{aligned}$$

We now have proved the result, under the additional assumption that for each v, at least one of $\operatorname{dom} f_v$ or $\operatorname{dom} g_v$ intersects X_v. We will now remove this assumption. Take $\rho > \inf_X f_w$. From the epi-distance convergence of level-sets [14, Proposition 7.1.7] we have $v \mapsto \{f_v \le \rho\}$ non-empty and lower-semicontinuous at w. Thus on taking $x_w \in \{f_w \le \rho\}$ we have some $x_v \in \{f_v \le \rho\}$ strongly converging to x_w. Place $\hat{f}_v(\cdot) := f_v(\cdot + x_v)$ and $\hat{g}_v(\cdot) := g_v(\cdot + x_v)$. Then $\{\hat{f}_v\}_{v\in W}$ and $\{\hat{g}_v\}_{v\in W}$ epi-distance converge to $\hat{f}_w$ and $\hat{g}_w$, respectively.

As $0 \in \operatorname{dom} \hat{f}_v$ and $X_v = \overline{\operatorname{span}}(\operatorname{dom} \hat{f}_v - \operatorname{dom} \hat{g}_v)$, we may apply the previous argument to infer the epi-distance convergence of $\hat{f}_v + \hat{g}_v$ to $\hat{f}_w + \hat{g}_w$. Translating the sum by $-x_v$, we get $f_v + g_v = (\hat{f}_v + \hat{g}_v)(\cdot - x_v)$ epi-distance converging to $f_w + g_w$. $\square$

In [8] we find a rather complicated condition ensuring epi-distance convergence of a sum (see condition (1) in Proposition 4.10 below). The form of these conditions facilitates estimation of the epi-distance metrics. Putting aside the issue of these estimates we observe that this condition is equivalent to an interiority condition. Place $\Delta(X) = \{(x,x) \mid x \in X\}$. We note that the mapping $P : X \times X \to X$ defined by $P(x,y) = y - x$ is open, as is P^{-1}.

Proposition 4.10. *Let f and g be proper closed convex functions on a Banach space X. Then the following are equivalent:*

(i) *there exists $t > 0$, $r > 0$ and $s > 0$ such that*

$$(B(0,s))^2 \subseteq \Delta(X) \cap (B(0,r))^2 - \{f \le r\} \times \{g \le r\} \cap (B(0,t))^2; \tag{4.16}$$

(ii) $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$.

Proof. Clearly (4.16) implies $0 \in \operatorname{int}(\Delta(X) - \operatorname{dom} f \times \operatorname{dom} g)$. Then

$$\begin{aligned} 0 &\in \operatorname{int} P\left(\Delta(X) - \operatorname{dom} f \times \operatorname{dom} g\right) \\ &= \operatorname{int} P\{(x,x) - (u,v) \mid x \in X,\, u \in \operatorname{dom} f,\, v \in \operatorname{dom} g\} \\ &= \operatorname{int}\{(x-v) - (x-u) \mid x \in X,\, u \in \operatorname{dom} f,\, v \in \operatorname{dom} g\} \\ &= \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g). \end{aligned}$$

For the converse we use Corollary 2.7 to deduce from $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$ that

$$0 \in \operatorname{int}(\{f \le t\} \cap B(0,t) - \{g \le t\} \cap B(0,t))$$

for some $t > 0$. Define $D := \Delta(X) - \{f \le t\} \times \{g \le t\} \cap (B(0,t))^2$. Then $PD = \{f \le t\} \cap B(0,t) - \{g \le t\} \cap B(0,t)$ so $0 \in \operatorname{int} PD$ and hence $0 \in \operatorname{int} P^{-1}PD$. But since

$$\begin{aligned} P^{-1}PD &= D + \ker P = D + \Delta(X) \\ &= \Delta(X) - \{f \le t\} \times \{g \le t\} \cap (B(0,t))^2 + \Delta(X) = D, \end{aligned}$$

it follows that $0 \in \operatorname{int}\left(\Delta(X) - \{f \le t\} \times \{g \le t\} \cap \left(\overline{B}(0,t)\right)^2\right)$. As $\Delta(X)$ and $\{f \le t\} \times \{g \le t\} \cap \left(\overline{B}(0,t)\right)^2$ are closed convex sets they may be viewed as the domains of their indicator functions. Thus applying Corollary 2.7 again we get for some $r > 0$ that

$$0 \in \operatorname{int}\left(\Delta(X) \cap (B(0,r))^2 - \{f \le t\} \times \{g \le t\} \cap \left(\overline{B}(0,r)\right)^2\right).$$

□

5. Saddle-point Convergence in Fenchel Duality

Convex-concave bivariate functions are related to convex bivariate functions through partial conjugation (i.e. conjugation with respect to one of the variables). In this context we are led to the introduction of equivalence classes of saddle-functions which are uniquely associated with concave or convex parents (depending on the which variable is partially conjugated). We direct the reader to the excellent texts of Rockafellar [22] and [23] for a detailed treatment of this phenomenon. Two bivariate functions are said to belong to the same equivalence class if they have the same convex and concave parents. Such members of the same equivalence class not only have the same saddle-point but so do all linear perturbations of these two functions. Thus when discussing the variational convergence of saddle-functions one is necessarily led to the study of the convergence of the equivalence class. When referring to a saddle-function (or convex-concave bivariate function) we are by association referring to its equivalence class via the selection of a member. The following is taken from [7] from which we adapt results and proofs.

Definition 5.1. Suppose that (X,τ) and (Y,σ) are two topological spaces and $\{K^n : X \times Y \to \overline{\mathbb{R}},\ n \in \mathbb{N}\}$ is a sequence of bi-variate functions. Define:

$$\begin{aligned} e_\tau/h_\sigma\text{-ls}_{n\to\infty} K^n(x,y) &= \sup_{\{y_n \to^\sigma y\}} \inf_{\{x_n \to^\tau x\}} \limsup_{n\to\infty} K^n(x_n,y_n) \\ h_\sigma/e_\tau\text{-li}_{n\to\infty} K^n(x,y) &= \inf_{\{x_n \to^\tau x\}} \sup_{\{y_n \to^\sigma y\}} \liminf_{n\to\infty} K^n(x_n,y_n) \ . \end{aligned}$$

Definition 5.2. Suppose that (X,τ) and (Y,σ) are two topological spaces and $\{K^n : X \times Y \to \overline{\mathbb{R}},\ n \in \mathbb{N}\}$ is a sequence of bivariate functions.

(1) We say that they epi/hypo-converge in the extended sense to a function $K : X\times Y \to \overline{\mathbb{R}}$ if

$$\underline{\mathrm{cl}}_x(e_\tau/h_\sigma\text{-ls}_{n\to\infty} K^n) \le K \le \overline{\mathrm{cl}}^y(h_\sigma/e_\tau\text{-li}_{n\to\infty} K^n)$$

where $\underline{\mathrm{cl}}_x$ denotes the extended lower closure with respect to x (and therefore w.r.t. τ) for fixed y and $\overline{\mathrm{cl}}^y$ denotes the extended upper closure with respect to y (and therefore w.r.t. σ) for fixed x, (where generally, $\overline{\mathrm{cl}} f := -\underline{\mathrm{cl}}(-f)$).

(2) A point $(\bar{x},\bar{y})$ is a saddle-point of a bivariate function $K : X \times Y \to \overline{\mathbb{R}}$ if for all $(x,y) \in X \times Y$ we have $K(\bar{x},y) \le K(\bar{x},\bar{y}) \le K(x,\bar{y})$.

The interest in this kind of convergence stems from the following result (see [7, Thm 2.4]).

Proposition 5.3. *Let us assume that $\{K^n, K : (X,\tau) \times (Y,\sigma) \to \overline{\mathbb{R}},\ n \in \mathbb{N}\}$ are such that they epi/hypo-converge in the extended sense. Assume also that $(\bar{x}_k, \bar{y}_k^*)$ are saddle points of K^{n_k} for all k and $\{n_k\}$ is an increasing sequence of integers, such that $\bar{x}_{n_k} \xrightarrow{\tau} \bar{x}$ and $\bar{y}^*_{n_k} \xrightarrow{\sigma} \bar{y}^*$. Then $(\bar{x},\bar{y}^*)$ is a saddle point of K and*

$$K(\bar{x},\bar{y}^*) = \lim_{k\to\infty} K^{n_k}(\bar{x}_k,\bar{y}_k^*) \ .$$

The next result from [7] uses sequential forms of the epi-limit functions, as per the following

Definition 5.4 ([7, p. 541]). Let (X,τ) be topological, $f_n : X \to \overline{\mathbb{R}}$. Then

$$\begin{aligned} (\tau\text{-seq-e-ls}_{n\to\infty} f_n)(x) &:= \inf_{\{x_n\} \xrightarrow{\tau} x} \limsup_{n\to\infty} f_n(x_n) \\ (\tau\text{-seq-e-li}_{n\to\infty} f_n)(x) &:= \inf_{\{x_n\} \xrightarrow{\tau} x} \liminf_{n\to\infty} f_n(x_n) \end{aligned}$$

It can be shown that these reduce to the usual (topologically defined) forms if (X,τ) is first-countable, and that the above infima are achieved. We will need these alternate forms, for generally weak topologies on normed spaces are not first-countable.

Definition 5.5. Let (X, τ) and (X^*, τ^*) be topological vector spaces. We shall say they are paired if there is a bilinear map $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ such that the maps $x^* \mapsto \langle \cdot, x^* \rangle$ and $x \mapsto \langle x, \cdot \rangle$ are (algebraic) isomorphisms such that $X^* \cong (X, \tau)^*$ and $X \cong (X^*, \tau^*)^*$ respectively.

It is readily checked that if (X, τ) and (X^*, τ^*) are paired, and so are (Y, σ) and (Y^*, σ^*), then $(X \times Y, \tau \times \sigma)$ is paired with $(X^* \times Y^*, \tau^* \times \sigma^*)$, with the pairing

$$\langle (x, y), (x^*, y^*) \rangle = \langle x, x^* \rangle + \langle y, y^* \rangle,$$

and similarly for other combinations of product spaces.

For any convex-concave saddle function $K : X \times Y^* \to \overline{\mathbb{R}}$, that is, where K is convex in the first argument and concave in the second, we may associate a *convex-* and *concave-parent*. These play a fundamental role in convex duality (see [22]) and are defined respectively as:

$$\begin{aligned} F(x, y) &= \sup\nolimits_{y^* \in Y^*} [K(x, y^*) + \langle y, y^* \rangle] \\ G(x^*, y^*) &= \inf\nolimits_{x \in X} [K(x, y^*) - \langle x, x^* \rangle]. \end{aligned}$$

(In Fenchel duality we have $F(x, y) = f(x) + g(x + y)$). One can show that for any closed convex function $F : X \times Y \to \overline{\mathbb{R}}$, if $G := -F^*$ relative to the natural pairing of $X \times Y$ with $X^* \times Y^*$, (these yielding the primal objective $F(\cdot, 0)$ and dual objective $G(0, -\cdot)$), we have an interval of saddle functions (all equivalent in the sense that they possess the same saddle points) given by

$$[\underline{K}, \overline{K}] := \{K : X \times Y^* \to \overline{\mathbb{R}} \mid K \text{ convex-concave, } \underline{K} \leq K \leq \overline{K} \text{ on } X \times Y^*\},$$

where

$$\begin{aligned} \underline{K}(x, y^*) &= \sup\nolimits_{x^* \in X^*} [G(x^*, y^*) + \langle x, x^* \rangle] \\ \overline{K}(x, y^*) &= \inf\nolimits_{y \in Y} [F(x, y) - \langle y, y^* \rangle] \ . \end{aligned}$$

The following result is taken from [7] and requires no additional assumption.

Proposition 5.6. *Let $(X, \tau), (X^*, \tau^*)$ and $(Y, \sigma), (Y^*, \sigma^*)$ be paired topological vector spaces, with the pairings sequentially continuous; let $\{F^n, F : X \times Y \to \overline{\mathbb{R}}, n \in \mathbb{N}\}$ be a family of bivariate $(\tau \times \sigma)$-closed convex functions. Then, if K^n, K are members of the corresponding equivalence classes of bivariate convex-concave saddle functions,*

(i)

$$\begin{aligned} (\tau \times \sigma)\text{-seq-e-ls}_{n \to \infty} F^n &\leq F \quad \text{on } X \times Y \\ \textit{implies} \quad \underline{\mathrm{cl}}_x (e_\tau / h_{\sigma^*}\text{-ls } \overline{K}^n) &\leq \underline{K} \ ; \end{aligned}$$

(ii)

$$\begin{aligned} (\tau^* \times \sigma^*)\text{-seq-e-ls}_{n \to \infty} (F^n)^* &\leq (F)^* \quad \text{on } X \times Y \\ \textit{implies} \quad \overline{K} &\leq \overline{\mathrm{cl}}^{y^*} (h_{\sigma^*} / e_\tau\text{-li } \underline{K}^n) \ . \end{aligned}$$

Proposition 5.7. *Suppose that X is a Banach space and $\{f_n, f\}_{n=1}^{\infty}$ and $\{g_n, g\}_{n=1}^{\infty}$ be two families of proper closed, convex extended-real-valued functions epi-distance convergent to f and g, respectively. Then*

$$K^n(x, y^*) = \inf_{y \in X}[f_n(x) + g_n(x+y) - \langle y, y^* \rangle]$$

epi/hypo-converges (in the extended sense) to

$$K(x, y^*) = \inf_{y \in X}[f(x) + g(x+y) - \langle y, y^* \rangle]$$

with respect to the strong topology on X and the weak topology on X^*.*

Proof. Define proper closed convex functions on $X \times X$ by $\hat{f}_n(x, y) := f_n(x)$ and $\hat{g}_n(x, y) := g_n(x+y)$, and similarly define $\hat{f}$ and $\hat{g}$. Then we have the epi-distance convergence $\hat{f}_n \to \hat{f}$ and $\hat{g}_n \to \hat{g}$ on $X \times X$. Note that $\operatorname{dom} \hat{f} = \operatorname{dom} f \times X$ and $\operatorname{dom} \hat{g} = P^{-1} \operatorname{dom} g$ (and similarly for f_n, g_n), where $P : X \times X \to X$ is the addition operation. Consequently, $\operatorname{dom} \hat{f} - \operatorname{dom} \hat{g} = X \times X = \operatorname{dom} \hat{f}_n - \operatorname{dom} \hat{g}_n$ for all n, so from Theorem 4.9 (or [8]) follows the epi-distance convergence

$$F^n := \hat{f}_n + \hat{g}_n \to F := \hat{f} + \hat{g}.$$

Thus also obtains [9] the epi-distance convergence $(F^n)^* \to F^*$. From the resultant strong epi-convergence on $X \times X$ and on $X^* \times X^*$,

$$\begin{aligned} F \geq (s \times s)\text{-e-ls}_{n\to\infty} F^n &= (s \times s)\text{-seq-e-ls}_{n\to\infty} F^n \qquad \text{and} \\ F^* \geq (s^* \times s^*)\text{-e-ls}_{n\to\infty} (F^n)^* &= (s^* \times s^*)\text{-seq-e-ls}_{n\to\infty} (F^n)^* \\ &\geq (w^* \times w^*)\text{-seq-e-ls}_{n\to\infty} (F^n)^*, \end{aligned}$$

where s and s^* stand for the respective norm topologies on X and X^*. Now apply Proposition 5.6. □

We note the following for future reference. For $u \in X$, write

$$\varphi(u) := \inf_{x \in X}\{f(x) + g(x+u)\} = \widehat{(f \Box \widehat{g})}(u), \tag{5.1}$$

and similarly for φ_n, where for any function ψ, $\widehat{\psi}(x) := \psi(-x)$. Note that $\operatorname{dom} \varphi = \operatorname{dom} g - \operatorname{dom} f$ and similarly for φ_n. The operation $\psi \mapsto \widehat{\psi}$ commutes with conjugation and with epi-distance limits; the verification of this is an elementary exercise. From [22] we have the following: Calling $\inf_X(f+g)$ the primal problem, and $\inf_X(f_n + g_n)$ the approximate problems, then $-\varphi^*$ and $-\varphi_n^*$ are the associated dual objective functionals, and:

$$\begin{aligned} (\bar{x}, \bar{y}^*) \quad & \text{is a saddle-point of } K \text{ iff} \\ \varphi(0) \quad &= \quad (f+g)(\bar{x}) = \inf_X (f+g) = \sup_{X^*} -\varphi^* = -\varphi^*(\bar{y}^*), \end{aligned}$$

and similarly for φ_n and the saddle-points $(\bar{x}_n, \bar{y}_n^*)$ of K^n. On taking conjugates of φ_n we obtain

$$\varphi_n^* = (\widehat{f_n \Box \widehat{g_n}})^* = (\widehat{f_n^* + \widehat{g_n}^*}) = \widehat{f}_n^* + g_n^*$$

and so the dual problem becomes

$$\sup_{X^*} -\varphi_n^* = -\left(f_n^* \Box g_n^*\right)(0) = -\inf_{y^* \in X^*} \left(f_n^*(-y^*) + g_n^*(y^*)\right) . \tag{5.2}$$

The next result tackles the problem of finding convergent sequences of dual variables. (Note that Proposition 5.3 makes no claim about such existence). There are well-known criteria for the convergence of the primal solutions $\bar{x}_n$– for instance, if $f_n + g_n$ epi-distance converges to $f + g$, and the latter satisfies a well-posedness condition [14, Section 7.5] and [13]. From [19] we know that $0 \in \operatorname{sqri}(\operatorname{dom} f - \operatorname{dom} g)$ is a sufficient condition for strong duality to hold.

Corollary 5.8. *Suppose that X is a separable Banach space and $\{f_n, f\}_{n=1}^{\infty}$ and $\{g_n, g\}_{n=1}^{\infty}$ be two families of proper closed, convex extended-real-valued functions epi-distance convergent to f and g respectively. Let K^n, K be the associated saddle-functions as in* Proposition 5.7. *Assume also the following:*

(i) $0 \in \operatorname{sqri}(\operatorname{dom} f - \operatorname{dom} g)$;

(ii) *The algebraic complement Y of the subspace $M = \operatorname{cone}(\operatorname{dom} f - \operatorname{dom} g)$, is closed and $M_n \cap Y = \{0\}$ eventually, where M_n denotes the subspace $\overline{\operatorname{span}}(\operatorname{dom} f_n - \operatorname{dom} g_n)$.*

Then if $(\bar{x}_n, \bar{y}_n^*)$ are saddle-points of K^n for each n and a subsequence of the $\bar{x}_n$ has a strong limit $\bar{x}$, and the saddle-values along this subsequence are bounded below, then K has a saddle-point $(\bar{x}, \bar{y}^*)$ that is a $(s \times w^*)$ -limit of saddlepoints $(\bar{x}_n, \widetilde{(\bar{y}_n^*)|_{M_n}})$ of a subsequence of the K^n, with $K(\bar{x}, \bar{y}^*)$ the limit of the corresponding saddle-function values. (Here 's' stands for the norm topology on X and $\widetilde{(\bar{y}_n^*)|_{M_n}}$ is the extension to X^* of the restriction of $\bar{y}_n^*$ to M_n as defined in Definition 4.3*).*

Proof. The proof follows from Propositions 5.7 and 5.3, on showing that the $\widetilde{(\bar{y}_n^*)|_{X_n}}$ are norm-bounded in X^*, so that weak*-convergent subsequences are available and are the required dual variables.

As in the proof of Theorem 4.9, there are $x_n \in \operatorname{dom} f_n$ converging to some $x \in \operatorname{dom} f$. Place $\check{f}_n(\cdot) := f_n(x_n + \cdot)$, $\check{g}_n(\cdot) := g_n(x_n + \cdot)$, with analogous definitions for $\check{f}$ and $\check{g}$ as translates by x. Then $0 \in \operatorname{dom} \check{f}_n, \operatorname{dom} \check{f}$, and hence these functions satisfy the hypotheses of Lemma 4.8.

Let $\check{\varphi}_n$ be the value function corresponding to $\check{f}_n$ and $\check{g}_n$ via (5.1). Similarly, denote the corresponding saddle function by $\check{K}^n$. Then we immediately observe that $\check{\varphi}_n^* = \varphi_n^*$, from which follows that

$$(\bar{x}_n, \bar{y}_n^*) \text{ is a saddlepoint of } K^n \text{ iff } (\bar{x}_n - x_n, \bar{y}_n^*) \text{ is a saddlepoint of } \check{K}^n ,$$

since $(\bar{x}_n, \bar{y}_n^*)$ are an optimal pair for the primal and dual problems if and only if $(\bar{x}_n - x_n, \bar{y}_n^*)$ are optimal for the problems based on the translated functions $\check{f}_n$, $\check{g}_n$. Evidently the optimal values are not affected by this translation, so we also obtain that $\check{K}^n(\bar{x}_n - x_n, \bar{y}_n^*) = K^n(\bar{x}_n, \bar{y}_n^*)$. Hence the saddle-values of $\check{K}^n$ are also bounded below. As M_n contains both $\operatorname{dom} \check{f}_n$ and $\operatorname{dom} \check{g}_n$ (recall this follows from $0 \in \operatorname{dom} \check{f}_n$), we obtain

$$\check{K}^n(\bar{x}_n - x_n, \bar{y}_n^*) = \check{K}^n(\bar{x}_n - x_n, \widetilde{\bar{y}_n^*|_{M_n}}) ,$$

which follows from $\check{\varphi}_n^*(\bar{y}_n^*) = \check{f}_n^*(-\bar{y}_n^*) + \check{g}_n^*(\bar{y}_n^*) = \check{f}_n^*(-\widetilde{\bar{y}_n^*|_{M_n}}) + \check{g}_n^*(\widetilde{\bar{y}_n^*|_{M_n}}) = \check{\varphi}_n^*(\widetilde{\bar{y}_n^*|_{M_n}})$, since M_n contains the domains of $\check{f}_n$ and $\check{g}_n$. Letting $-\alpha \in \mathbb{R}$ be a lower bound for the saddle-values of K^n (and hence of $\check{K}^n$), we have for all n large, that

$$(-\widetilde{\bar{y}_n^*|_{M_n}}, \widetilde{\bar{y}_n^*|_{M_n}}) \in H_\alpha(\widetilde{M_n, n})$$

(where the latter set is defined relative to the translated functions $\check{f}_n$, $\check{g}_n$), since

$$(\check{f}_n|_{M_n})^*(-\bar{y}_n^*|_{M_n}) + (\check{g}_n|_{M_n})^*(\bar{y}_n^*|_{M_n}) = \check{f}_n^*(-\bar{y}_n^*) + \check{g}_n^*(\bar{y}_n^*) = -\check{K}^n(\bar{x}_n - x_n, \bar{y}_n^*) < \alpha$$

so that $(-\bar{y}_n^*|_{M_n}, \bar{y}_n^*|_{M_n}) \in H_\alpha(M_n, n)$. By Lemma 4.8, there is $\gamma > 0$ such that $\|\widetilde{\bar{y}_n^*|_{M_n}}\| \leq \gamma$ for all large n. Then $\bar{z}_n^* := \widetilde{\bar{y}_n^*|_{M_n}} \in X^*$ is norm-bounded and hence has a weakly* convergent subsequence $\bar{z}_n^* \to \bar{z}^*$. For each n, $(\bar{x}_n - x_n, \bar{z}_n^*)$ a saddlepoint for $\check{K}^n$, so $(\bar{x}_n, \bar{z}_n^*)$ is one for K^n. By Propositions 5.7 and 5.3, $(\bar{x}, \bar{z}^*)$ is a saddlepoint for K, with value the limit of the saddle-values along the sequence. □

Remark 5.9. A case in the above result where the saddle-values are bounded below, occurs for instance when the sum $f + g$ of the limit functions has bounded level-sets, for from the epi-distance convergence $f_n + g_n \to f + g$ (Theorem 4.9), and from [14, Corollary 7.5.3], the infima of $f_n + g_n$ converge to that of $f + g$, which is finite by properness, and these correspond to the saddle-values.

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