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On the Lavrentieff Phenomenon for Some Classes of Dirichlet Minimum Points

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Starting from the results of [27],the Lavrentieff phenomenon for the functional $F(\Omega,\varphi_O,\cdot)\colon\ u\in BV(\Omega)\mapsto \int_\Omega f(\nabla u)dx+\int_\Omega f^\infty(\frac{dD^s u}{d|D^s u|})d|D^s u|+\int_{\partial\Omega} f^\infty((\varphi_0-\gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1}$ between $BV(\Omega)$ and $BV(\Omega)\cap C^1(\Omega)$ is studied, where $f\colon\mathbb{R}^n\to[0,+\infty[$ is convex, f^∞ is its recession function, $\varphi_0\in L^1(\partial\Omega)$, and γ_Ω is the trace operator on $\partial\Omega$. The occurrence of the phenomenon is first discussed by means of an example, and then completely characterized. Sufficient conditions implying the absence of the phenomenon are also proved, and some relaxation properties of $F(\Omega,\varphi_\Omega,\cdot)$ are also established.

1. Introduction

In a paper of 1926, (cf. [36]), in connection with Tonelli's partial regularity theorem for the minimizers of one dimensional Dirichlet variational problems, (cf. [42]), M. Lavrentieff observed the occurrence of the surprising feature of some Dirichlet variational problems for integral functionals to depend critically on slight variations of the set of admissible functions. He produced an example of a rather elaborated one dimensional integral functional of the type

$$u\in W^{1,1}(]0,1[)\mapsto\int_0^1 f_L(x,u(x),u'(x))dx$$

whose minimum on the class $\{u\in W^{1,1}(]0,1[):u(0)=0,\ u(1)=1\}$ is strictly smaller than its infimum on sets of smooth functions, for example on $\{u\in C^1([0,1]):u(0)=0,\ u(1)=1\}$.

It is to be emphasized that this feature is surprising since f_L is globally continuous and strictly convex in the last variable, the integral $u\in W^{1,1}(]0,1[)\mapsto\int_0^1 f_L(x,u(x),u'(x))dx$ is sequentially lower semicontinuous with respect to the weak $W^{1,1}(]0,1[)$-topology, and $C^1([0,1])$ is dense in $W^{1,1}(]0,1[)$ endowed with its strong topology.

Starting from Lavrentieff's work, many paper have been devoted to the study of the phenomenon in order to simplify the original example, (cf. [38, 35, 8]), and to give sufficient conditions for its non occurrence, (cf. [5, 17, 20]).

More recently Buttazzo and Mizel, (cf. [13], and also [25]), proposed an abstract interpretation of the Lavrentieff phenomenon by means of relaxation. Given a topolog-

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ical space (U,τ), a τ-dense subset V of U, and a τ-lower semicontinuous functional $F\colon U \to]-\infty,+\infty]$, they considered the τ-lower semicontinuous envelope $\overline{F_V}$ of

$$F_V\colon u \in U \mapsto \begin{cases} F(u) & \text{if } u \in V \\ +\infty & \text{otherwise} \end{cases}$$

defined by

$$\overline{F_V}\colon u \in U \mapsto \sup\{G(u) : G\colon U \to]-\infty,+\infty],\, G \ \tau\text{-lower semicontinuous},\, G \le F_V\},$$

and observed that, being $\inf\{F(u) : u \in V\} = \inf\{\overline{F_V}(u) : u \in U\}$, the non occurrence of the Lavrentieff phenomenon for F, U and V, i.e. the equality $\inf\{F(u) : u \in U\} = \inf\{F(u) : u \in V\}$, can be deduced by the equality $\overline{F_V} = F$.

In this framework the occurrence of the Lavrentieff phenomenon for Neumann minimum problems has been studied in many papers also for multiple integrals of the Calculus of Variations defined in Sobolev and BV spaces, (cf. for example [1, 2, 9, 14, 19, 21, 25, 26, 44]). We recall that, given an open set Ω, $BV(\Omega)$ is defined as the set of the functions in $L^1(\Omega)$ having distributional partial first derivatives that are Borel measures with finite total variations in Ω, and that, for every $u \in BV(\Omega)$, ∇u is defined as the density of the absolutely continuous part of the vector measure Du with respect to Lebesgue measure, $D^s u$ as its singular part, and $\nabla^s u$ $\left(= \frac{dD^s u}{d|D^s u|}\right)$ as the Radon-Nikodym derivatives of $D^s u$ with respect to its total variation $|D^s u|$.

For example, in [21], the case has been treated in which Ω is a smooth bounded open subset of $\mathbb{R}^n$, $f\colon \mathbb{R}^n \to [0,+\infty[$ is convex, $U = BV(\Omega)$, τ is the $L^1(\Omega)$-topology, $V = C^1(\mathbb{R}^n)$, and F is the classical Goffman-Serrin integral defined as

$$F\colon u \in BV(\Omega) \mapsto \int_\Omega f(\nabla u)dx + \int_\Omega f^\infty(\nabla^s u)d|D^s u|$$

(as usual we have denoted by f^∞ the recession function of f defined by $f^\infty\colon z \in \mathbb{R}^n \mapsto \lim_{t\to 0} tf(\frac{z}{t})$), and it has been proved the non occurrence of the Lavrentieff phenomenon for F, $BV(\Omega)$ and $C^1(\mathbb{R}^n)$.

The study of relaxed Dirichlet minimum problems in BV spaces has been first performed in [32] where, also in more general settings, given $f\colon \mathbb{R}^n \to [0,+\infty[$ convex and verifying

$$|z| \le f(z) \le M(1+|z|) \quad \text{for some } M \ge 1 \text{ and every } z \in \mathbb{R}^n, \tag{1.1}$$

a smooth bounded open set Ω, and $\varphi_0 \in L^1(\partial\Omega)$, the functional

$$F(\Omega,\varphi_0,\cdot)\colon u \in BV(\Omega) \mapsto \int_\Omega f(\nabla u)dx + \int_\Omega f^\infty(\nabla^s u)d|D^s u| + \\ + \int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1} \tag{1.2}$$

has been introduced in connection with the minimization in $\{v \in W^{1,1}(\Omega) : \gamma_\Omega(v) = \varphi_0\}$ of the variational integral $u \mapsto \int_\Omega f(\nabla u)dx$, (in (1.2) we have denoted by $\mathbf{n}$ the exterior unit vector normal to $\partial\Omega$, by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure, and, for every $u \in BV(\Omega)$, by $\gamma_\Omega(u)$ the interior trace of u on $\partial\Omega$).

More recently in [27] the relaxation properties of the functional in (1.2) have been investigated under general assumptions on the data, and it has been proved that, if f is just convex, Ω has Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and $u_0 \in W^{1,1}_{loc}(\mathbb{R}^n)$ is such that $\gamma_\Omega(u_0) = \varphi_0$ and

$$f(t\nabla u_0) \in L^1_{loc}(\mathbb{R}^n) \quad \text{for every } t \in \mathbb{R}, \tag{1.3}$$

then $F(\Omega, \varphi_0, \cdot)$ is the $L^1(\Omega)$-lower semicontinuous envelope on the whole $BV(\Omega)$ of the functional

$$u \in BV(\Omega) \mapsto \begin{cases} \int_\Omega f(\nabla u)dx & \text{if } u \in u_0 + C^\infty_0(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

(cf. Theorem 2.5 in [27]). Moreover, if in addition $u_0 \in C^\infty(\Omega)$, it has also been proved that $F(\Omega, \varphi_0, \cdot)$ is the $L^1(\Omega)$-lower semicontinuous envelope on the whole $BV(\Omega)$ of the functional

$$u \in BV(\Omega) \mapsto \begin{cases} \int_\Omega f(\nabla u)dx & \text{if } u \in W^{1,1}(\Omega) \cap C^\infty(\Omega),\ \gamma_\Omega(u) = \varphi_0 \\ +\infty & \text{otherwise,} \end{cases}$$

(cf. Theorem 3.1 in [27]).

In the present paper we want to study some aspects of the Lavrentieff phenomenon for the functional in (1.2), BV spaces and sets of smooth functions, under general assumptions on the data and, in particular, when (1.1) is dropped.

To do this we consider a convex function $f\colon \mathbb{R}^n \to [0, +\infty[$, a bounded open set with Lipschitz boundary Ω, $\varphi_0 \in L^1(\partial\Omega)$, $u_0 \in W^{1,1}_{loc}(\mathbb{R}^n)$ verifying $\gamma_\Omega(u_0) = \varphi_0$, and, first of all, we observe that, since $F(\Omega, \varphi_0, u) = \int_\Omega f(\nabla u)dx$ for every $u \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$ with $\gamma_\Omega(u) = \varphi_0$, by the above recalled results of [27] the non occurrence of the Lavrentieff phenomenon for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $u_0 + C^\infty_0(\Omega)$, or for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $\{u \in W^{1,1}(\Omega) \cap C^\infty(\Omega) : \gamma_\Omega(u) = \varphi_0\}$ follows, provided the relative assumptions are fulfilled.

On the other side, the presence in (1.2) of the boundary integral, that allows $F(\Omega, \varphi_0, \cdot)$ to be defined and possibly finite on the whole $BV(\Omega)$, suggests the consideration of the Lavrentieff phenomenon also for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and sets of smooth functions with no fixed boundary traces.

A first choice in this direction could be $C^1(\mathbb{R}^n)$ as in [21]. This choice however seems to be not interesting from the point of view of Lavrentieff phenomenon, since in general one could have $\int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1} = +\infty$ for every $u \in C^1(\mathbb{R}^n)$.

On the other hand if $v \in L^1(\partial\Omega)$ is such that $\int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} < +\infty$, it is well known that there exist functions in $W^{1,1}(\Omega) \cap C^\infty(\Omega)$, (that clearly agrees with $BV(\Omega) \cap C^\infty(\Omega)$), whose traces on $\partial\Omega$ are v, (cf. [30]). This actually avoids a too violent influence of the boundary integral on $F(\Omega, \varphi_0, \cdot)$, and therefore suggests the study of the Lavrentieff phenomenon for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $BV(\Omega) \cap C^\infty(\Omega)$.

In this framework we first of all propose an example exhibiting a surprising and unexpected phenomenon, namely that Lavrentieff phenomenon may be produced in an integral on a bounded open set Ω, having functional dependences on the admissible functions only through their gradients in Ω and their values on $\partial\Omega$, just by weak local summability constraints on the values of the admissible functions and not by the properties of their gradients. For every $n \geq 3$, $p \in \left]\frac{n-1}{2}, n-1\right[$, and $q \in \left]\frac{(n-1)p}{n-1-p}, +\infty\right]$ we produce a bounded open set with Lipschitz boundary $\Omega \subseteq \mathbb{R}^n$, a convex function $f\colon \mathbb{R}^n \to [0, +\infty[$, and $\varphi_0 \in L^1(\partial\Omega)$ so that

$$\begin{aligned}&\min\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\} = \min\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega)\} < \\ &< \inf\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega) \cap L^q_{loc}(\Omega)\} = \\ &= \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap L^q_{loc}(\Omega)\} = +\infty,\end{aligned}$$

(cf. Example 3.3).

Obviously, being $C^1(\Omega) \subseteq L^\infty_{loc}(\Omega)$, the same example also proves the occurrence of the Lavrentieff phenomenon for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $BV(\Omega) \cap C^1(\Omega)$, (cf. Example 3.4).

It is clear that Lavrentieff phenomenon is strictly linked to the regularity properties of the solutions of variational problems, and the examples in section 3 also show that the proposed functional cannot have solutions whose q-th powers are locally summable. Nevertheless it is to be pointed out that actually we prove even more, namely that there is an infinite gap between the infima taken into account. We refer to [31] and [39] for examples on the non regularity of the solutions of some partial differential equations connected to functionals of the type in (1.2).

Finally, in section 4, given a convex function $f\colon \mathbb{R}^n \to [0, +\infty[$, a bounded open set with Lipschitz boundary Ω, and $\varphi_0 \in L^1(\partial\Omega)$, the study of the non occurrence of Lavrentieff phenomenon for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $BV(\Omega) \cap C^\infty(\Omega)$ is carried out, and it is proved that it can be characterized in terms of the finiteness of $F(\Omega, \varphi_0, \cdot)$ on suitable subspaces of $BV(\Omega)$.

First of all it is observed that obviously

$$\inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} = \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\} = +\infty$$

if and only if

$$\inf\{F(\Omega, \varphi_0, v) : v \in BV(\Omega)\} = +\infty. \tag{1.4}$$

Then, in order to treat the nontrivial case of the functionals not identically equal to $+\infty$, it is proved that, under the following assumption on f

$$\begin{aligned}&\text{for every } \omega \in (L^1(]0,1[^n))^n \text{ verifying } f(\omega) \in L^1(]0,1[^n) \\ &\text{there exists } t_\omega \in]1, +\infty[\text{ such that } f(t_\omega \omega) \in L^1(]0,1[^n),\end{aligned} \tag{1.5}$$

it results that

$$\inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} = \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\} < +\infty$$

if and only if

$$\inf\{F(\Omega, \varphi_0, v) : v \in BV(\Omega) \cap L^\infty_{loc}(\Omega)\} < +\infty, \tag{1.6}$$

(cf. Theorem 4.8).

As corollary it turns out that, if (1.5) holds, the Lavrentieff phenomenon for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $BV(\Omega) \cap C^\infty(\Omega)$ occurs if and only if $\inf\{F(\Omega, \varphi_0, v) : v \in BV(\Omega)\} < +\infty$ and $\inf\{F(\Omega, \varphi_0, v) : v \in BV(\Omega) \cap C^\infty(\Omega)\} = +\infty$, (cf. Theorem 4.9).

Such results are deduced by working in the above described framework of relaxation by performing the following steps.

First of all it is observed that both $F(\Omega, \varphi_0, \cdot)$ agrees on the whole $BV(\Omega)$ with the $L^1(\Omega)$-lower semicontinuous envelope of the functional

$$u \in L^1(\Omega) \mapsto \begin{cases} F(\Omega, \varphi_0, u) & \text{if } u \in BV(\Omega) \cap C^\infty(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{1.7}$$

and is identically equal to $+\infty$ if and only if (1.4) holds.

Then it is proved that, under assumption (1.5), both $F(\Omega, \varphi_0, \cdot)$ agrees on the whole $BV(\Omega)$ with the $L^1(\Omega)$-lower semicontinuous envelope of the functional in (1.7) and is not identically equal to $+\infty$ if and only if (1.6) holds, (cf. Theorem 4.7).

We also point out that, if $p \in [1, +\infty[$ and

$$f(z) \leq M(1 + |z|^p) \quad \text{for some } M \geq 0 \text{ and every } z \in \mathbb{R}^n, \tag{1.8}$$

then in theorems 4.7 and 4.8 condition (1.6) can be replaced by

$$\inf\{F(\Omega, \varphi_0, v) : v \in BV(\Omega) \cap L^p_{loc}(\Omega)\} < +\infty. \tag{1.9}$$

Finally some sufficient conditions implying (1.6) are proposed, and the consequent results on relaxation and absence of Lavrentieff phenomenon are proved. For example if $n = 1$, or if $\varphi_0 \in L^\infty(\partial\Omega)$, or if the right-hand side of (1.1) holds, or if there exists $u_0 \in BV(\Omega) \cap L^\infty_{loc}(\Omega)$ such that $\gamma_\Omega(u_0) = \varphi_0$ and $\int_\Omega f(\nabla u_0)dx + \int_\Omega f^\infty(\nabla^s u_0)d|D^s u_0| < +\infty$, then the Lavrentieff phenomenon for $F(\Omega, \varphi_0, v)$, $BV(\Omega)$ and $BV(\Omega) \cap C^\infty(\Omega)$ cannot occur. In particular it cannot occur if there exists $u_0 \in C^\infty(\Omega)$ with $\gamma_\Omega(u_0) = \varphi_0$ verifying (1.3).

2. Notations and preliminary results

For every $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B_r(x)$ the open ball centred at x with radius r.

For every Lebesgue measurable subset E of $\mathbb{R}^n$ we denote by $\mathcal{L}^n(E)$ the n-dimensional Lebesgue measure of E.

Given two open subsets of $\mathbb{R}^n$ A and B, we say that $A \subset\subset B$ if $\overline{A}$ is a compact subset of B.

We now recall some properties of BV spaces, we refer to [29], [33] and [45] for complete references on the subject.

For every open subsets Ω of $\mathbb{R}^n$ we denote by $BV_{loc}(\Omega)$ the set of the functions in $L^1_{loc}(\Omega)$ that belong to $BV(A)$ for every open set A such that $A \subset\subset \Omega$.

Let Ω be an open set, we recall that $W^{1,1}(\Omega) \subseteq BV(\Omega)$, that $D^s u \equiv 0$ for every $u \in W^{1,1}(\Omega)$ and that, consequently, $\int_\Omega f^\infty(\nabla^s u)d|D^s u| = 0$ for every $u \in W^{1,1}(\Omega)$ and every convex function $f\colon \mathbb{R}^n \to [0, +\infty[$.

We also recall that $BV(\Omega) \subseteq L^{\frac{n}{n-1}}(\Omega)$, ($BV(\Omega) \subseteq L^\infty(\Omega)$ if $n = 1$).

If $u \in BV(\Omega)$, we denote by $|Du|$ the total variation of the vector measure Du, and by S_u the set of the points in Ω where u has not an approximate limit, i.e. $x \in \Omega \setminus S_u$ if and only if there exists $\tilde{u}(x) \in \mathbb{R}$ such that for every $\varepsilon > 0$

$$\lim_{r \to 0} \frac{\mathcal{L}^n(\{y \in B_r(x) : |u(y) - \tilde{u}(x)| > \varepsilon\})}{r^n} = 0.$$

It can be proved that (cf. Theorem 15.2 in [43], and 3.2.29 in [29]) S_u is $\mathcal{H}^{n-1}$-rectifiable, and that (cf. 2.9.13 in [29]) the function $\tilde{u}$ is Borel and equal to u $\mathcal{L}^n$-a.e. in Ω. Moreover the vector measure Du can be splitted as $Du = \tilde{D}u + Ju$, where $\tilde{D}u(B) = Du(B \setminus S_u)$ and $Ju(B) = Du(B \cap S_u)$ for every Borel subset B of Ω.

For $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ it is possible to define (cf. Theorem 9.2 in [43] and 3.2.26 in [29]) two real numbers $u^+(x)$ and $u^-(x)$, called the upper and the lower approximate limits of u at x, and a unit vector $\nu_u(x) \in \mathbb{R}^n$ such that for every $\varepsilon > 0$

$$\lim_{r \to 0} \frac{\mathcal{L}^n(\{y \in B_r(x) : (y - x) \cdot \nu_u(x) > 0, |u(y) - u^+(x)| > \varepsilon\})}{r^n} = 0,$$
$$\lim_{r \to 0} \frac{\mathcal{L}^n(\{y \in B_r(x) : (y - x) \cdot \nu_u(x) < 0, |u(y) - u^-(x)| > \varepsilon\})}{r^n} = 0.$$

For $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ the triplet $(u^+(x), u^-(x), \nu_u(x))$ turns out to be uniquely determined up to an interchange of $u^+(x)$ with $u^-(x)$ and to a change of sign of $\nu_u(x)$.

The upper and lower approximate limits of u at point x which is not in S_u coincide with $\tilde{u}(x)$, moreover, (cf. Theorem 15.1 in [43])

$$Ju(B) = \int_{B \cap S_u} (u^+ - u^-) \nu_u d\mathcal{H}^{n-1} \quad \text{for every Borel subset } B \text{ of } \Omega. \tag{2.1}$$

If Ω is an open set with Lipschitz boundary, $u \in BV(\Omega)$ we denote by $u_{(0)}$ the null extension of u to $\mathbb{R}^n$ defined by

$$u_{(0)}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega} \end{cases} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n,$$

then it turns out that $u_{(0)} \in BV(\mathbb{R}^n)$, and we define (cf. Definition 5.10.5 in [45]) the trace $\gamma_\Omega(u)$ of u on $\partial\Omega$ as

$$\gamma_\Omega(u) = (u_{(0)})^+ + (u_{(0)})^-.$$

It turns out that (cf. Theorem 5.9.6, Remark 5.10.6 and Remark 5.8.3 in [45]) for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ the vector $\nu_{u_{(0)}}(x)$ agrees with the exterior (interior) normal $\mathbf{n}(x)$ to $\partial\Omega$ at x, moreover $(u_{(0)})^+(x) = 0$ or $(u_{(0)})^-(x) = 0$ and $\gamma_\Omega(u)(x) = (u_{(0)})^+(x)$ or $\gamma_\Omega(u)(x) = (u_{(0)})^-(x)$.

Finally we recall that (cf. [45] Theorem 5.14.4) it results that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{\Omega \cap B_r(x_0)} |u(x) - \gamma_\Omega(u)(x_0)|^{\frac{n}{n-1}} dx = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in \partial\Omega, \tag{2.2}$$

from which it also follows that, when $u \in W^{1,1}(\Omega)$, $\gamma_\Omega(u)$ agrees $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$ with the usual Sobolev trace of u.

The following result is proved in [4].

Proposition 2.1. *Let $u \in BV(\Omega)$, and let $g\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz with $g(0) = 0$, then $g(u) \in BV(\Omega)$ and*

$$Jg(u)(B) = \int_{B\cap S_u} (g(u^+) - g(u^-))\nu_u d\mathcal{H}^{n-1}.$$

By Proposition 2.1 we deduce the following result.

Proposition 2.2. *Let Ω be a bounded open set with Lipschitz boundary, $u \in BV(\Omega)$, and let $g\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz with $g(0) = 0$, then $\gamma_\Omega(g(u)) = g(\gamma_\Omega(u))$.*

Proof. Let $E \subseteq \partial\Omega$ be a Borel set, and let $u_{(0)}$ and $g(u)_{(0)}$ be the null extensions of u and $g(u)$ to $\mathbf{R}^n$.

By using the fact that $g(0) = 0$, it is soon verified that $g(u)_{(0)} = g(u_{(0)})$, hence, by virtue of this and by Proposition 2.1, we get that $g(u)_{(0)} \in BV(\mathbb{R}^n)$ and that

$$Jg(u)_{(0)} = Jg(u_{(0)}) = \int_{E\cap S_{u_{(0)}}} \left(g((u_{(0)})^+) - g((u_{(0)})^-)\right) \nu_{u_{(0)}} d\mathcal{H}^{n-1}. \tag{2.3}$$

Let us recall now that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ the vector $\nu_{u_{(0)}}(x)$ agrees with the exterior (interior) normal $\mathbf{n}(x)$ to $\partial\Omega$ at x. Moreover, if $x \in E \setminus S_{u_{(0)}}$, $(u_{(0)})^+(x) = (u_{(0)})^-(x) = 0$, consequently, being also $g(0) = 0$, it follows that $g((u_{(0)})^+(x)) = g((u_{(0)})^-(x)) = 0$ and, by (2.3), that

$$Jg(u)_{(0)} = \int_E \left(g((u_{(0)})^+(x)) - g((u_{(0)})^-(x))\right) \mathbf{n} d\mathcal{H}^{n-1}. \tag{2.4}$$

On the other hand, by (2.1), and arguing as above we get that

$$Jg(u)_{(0)} = \int_E ((g(u)_{(0)})^+ - (g(u)_{(0)})^-)\mathbf{n} d\mathcal{H}^{n-1}, \tag{2.5}$$

therefore from (2.4) and (2.5), being E an arbitrary Borel subset of $\partial\Omega$, we conclude that

$$\left(g((u_{(0)})^+(x)) - g((u_{(0)})^-(x))\right) \mathbf{n}(x) = ((g(u)_{(0)})^+(x) - (g(u)_{(0)})^-(x))\mathbf{n}(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega. \tag{2.6}$$

Finally, being $g(0) = 0$, it turns out that $g((u_{(0)})^+) - g((u_{(0)})^-) = g(\gamma_\Omega(u))$ and $(g(u)_{(0)})^+ - (g(u)_{(0)})^- = \gamma_\Omega(g(u))$, and the thesis follows by (2.6). □

We now recall the following lower semicontinuity results, (cf. [41] and [27] respectively).

Proposition 2.3. *Let $f\colon \mathbf{R}^n \to [0,+\infty[$ be convex, then for every bounded open set Ω the functional*

$$u \in BV_{loc}(\Omega) \mapsto \int_\Omega f(\nabla u)dx + \int_\Omega f^\infty(\nabla^s u)d|D^s u|$$

is $L^1_{loc}(\Omega)$-lower semicontinuous.

Proposition 2.4. *Let $f\colon \mathbf{R}^n \to [0,+\infty[$ be convex, Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and let $F(\Omega,\varphi_0,\cdot)$ be given by (1.2), then $F(\Omega,\varphi_0,\cdot)$ is $L^1(\Omega)$-lower semicontinuous.*

Given a mollifier ρ, i.e. a function in $C_0^\infty(B_1(0))$ such that $\rho \geq 0$ and $\int_{B_1(0)} \rho(y)dy = 1$, we denote, for every open set Ω, every $u \in L^1_{loc}(\Omega)$, $\varepsilon > 0$, and $x \in \Omega$ with $\mathrm{dist}(x,\partial\Omega) > \varepsilon$, by $(\rho_\varepsilon * u)(x)$ the regularization of u at x defined by $(\rho_\varepsilon * u)(x) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \rho(\frac{x-y}{\varepsilon})u(y)dy$. We recall that $u_\varepsilon \in C^\infty(\{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > \varepsilon\})$, and that $\rho_\varepsilon * u \to u$ in $L^1_{loc}(\Omega)$ as $\varepsilon \to 0$.

Let us recall the following result, (cf. Lemma 3.3 in [15]).

Lemma 2.5. *Let Ω_0 be a bounded open set, Ω an open set with $\Omega \subset\subset \Omega_0$, $\varepsilon \in]0, \mathrm{dist}(\Omega,\partial\Omega_0)[$, $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex, and $u \in BV_{loc}(\Omega_0)$, then*

$$\int_\Omega f(\nabla(\rho_\varepsilon * u))dx \leq \int_{\Omega_0} f(\nabla u)dx + \int_{\Omega_0} f^\infty(\nabla^s u)d|D^s u|.$$

By Lemma 2.5 we deduce the following result.

Lemma 2.6. *Let Ω_0 be a bounded open set, Ω an open set with $\Omega \subset\subset \Omega_0$, $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex, and $u \in BV_{loc}(\Omega_0)$ such that*

$$\int_{\Omega_0} f(\nabla u)dx + \int_{\Omega_0} f^\infty(\nabla^s u)d|D^s u| < +\infty,$$

then

$$\limsup_{\varepsilon\to 0} \int_\Omega f(\nabla(\rho_\varepsilon * u))dx \leq \int_{\overline{\Omega}} f(\nabla u)dx + \int_{\overline{\Omega}} f^\infty(\nabla^s u)d|D^s u|.$$

Proof. Let Ω' be an open set such that $\Omega \subset\subset \Omega' \subset\subset \Omega_0$ then, by Lemma 2.5 applied to Ω' and Ω, we get

$$\lim_{\varepsilon\to 0} \int_\Omega f(\nabla(\rho_\varepsilon * u))dx \leq \int_{\Omega'} f(\nabla u)dx + \int_{\Omega'} f^\infty(\nabla^s u)d|D^s u|.$$

If Ω' decreases to Ω the thesis follows. □

Finally we recall some results of technical nature, the first one being proved in [22], (cf. Lemma 2.2 in [22]).

Lemma 2.7. *Let Ω be a bounded open set, $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex, $\{\omega_h\}$ be bounded in $L^\infty(\Omega;\mathbb{R}^n)$, and $\omega \in L^\infty(\Omega;\mathbb{R}^n)$ be such that $\omega_h \to \omega$ $\mathcal{L}^n$-a.e. in Ω, then*

$$\lim_{h\to+\infty} \int_\Omega f(\omega_h)dx = \int_\Omega f(\omega)dx.$$

Lemma 2.8. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex, E be a measurable set with $\mathcal{L}^n(E) < +\infty$, and $\omega\colon E \to \mathbb{R}^n$ be measurable, then the limit $\lim_{t\to 1^+} \int_E f(t\omega)dx$ exists.*

If in addition $\int_E f(t_0\omega)dx < +\infty$ for some $t_0 \in]1,+\infty[$, then

$$\lim_{t\to 1^+} \int_E f(t\omega)dx = \int_E f(\omega)dx.$$

Proof. If $\int_E f(t\omega)dx = +\infty$ for every $t \in]1,+\infty[$ then clearly $\lim_{t\to 1^+} \int_E f(t\omega)dx$ exists. If $\int_E f(t_0\omega)dx < +\infty$ for some $t_0 \in]1,+\infty[$ then, by the convexity inequality

$$f(t\omega) \le \frac{t}{t_0} f(t_0\omega) + (1 - \frac{t}{t_0})f(0) \quad \text{for every } t \in]1, t_0],$$

and Lebesgue Dominated Convergence Theorem, the thesis follows. □

Lemma 2.9. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex and verifying* (1.5)*, then for every bounded measurable set E, and every $\omega \in (L^1(E))^n$ verifying $f(\omega) \in L^1(E)$ there exists $t_{\omega,E} \in]1,+\infty[$ such that $f(t_{\omega,E}\omega) \in L^1(E)$.*

Proof. Let E, ω be as above, and extend ω to the whole $\mathbb{R}^n$ by setting $\omega(x) = 0$ for every $x \in \mathbb{R}^n \setminus E$.

Since E is bounded, we can find $x_1, \dots, x_m \in \mathbb{R}^n$ such that the cubes $x_1+]0,1[^n, \dots, x_m+]0,1[^n$ are pairwise disjoint and $\mathcal{L}^n(E \setminus \cup_{j=1}^m (x_j+]0,1[^n)) = 0$, consequently set $\omega_1 = \omega(\cdot + x_1), \dots, \omega_m = \omega(\cdot + x_m)$, let $t_{\omega_1}, \dots, t_{\omega_m} \in]1,+\infty[$ be given by (1.5), and define $t_{\omega,E} = \min\{t_{\omega_1}, \dots, t_{\omega_m}\}$.

By the convexity of f it follows that

$$\begin{aligned}\int_E f(t_{\omega,E}\omega)dx &\le \int_{\cup_{j=1}^m (x_j+]0,1[^n)} f(t_{\omega,E}\omega)dx = \sum_{j=1}^m \int_{x_j+]0,1[^n} f(t_{\omega,E}\omega)dx = \\ &= \sum_{j=1}^m \int_{]0,1[^n} f(t_{\omega,E}\omega_j)dx = \sum_{j=1}^m \int_{]0,1[^n} f\left(\frac{t_{\omega,E}}{t_{\omega_j}} t_{\omega_j}\omega_j + (1 - \frac{t_{\omega,E}}{t_{\omega_j}})0\right) dx \le \\ &\le \sum_{j=1}^m \left(\frac{t_{\omega,E}}{t_{\omega_j}} \int_{]0,1[^n} f(t_{\omega_j}\omega_j)dx + (1 - \frac{t_{\omega,E}}{t_{\omega_j}})f(0)\right) < +\infty,\end{aligned}$$

from which the thesis follows. □

3. The example

In the present section we show by an example that, provided $n \ge 3$, for some $p \in \left]\frac{n-1}{2}, n-1\right[$, $q \in \left]\frac{(n-1)p}{n-1-p}, +\infty\right[$, some bounded open set with Lipschitz boundary Ω, some convex function $f\colon \mathbb{R}^n \to [0,+\infty[$, and $\varphi_0 \in L^1(\partial\Omega)$ the Lavrentieff phenomenon may occur for the functional $F(\Omega, \varphi_0, \cdot)$ in (1.2), $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega) \cap L^q_{loc}(\Omega)$. More precisely that

$$\min\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega)\} < \inf\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega) \cap L^q_{loc}(\Omega)\} = +\infty.$$

By virtue of this we will deduce that the same phenomenon may occur for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $BV(\Omega) \cap L^q_{loc}(\Omega)$, for $F(\Omega, \varphi_0, \cdot)$, $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega) \cap C^\infty(\Omega)$, and for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $BV \cap C^\infty(\Omega)$.

Example 3.1. Let $n \geq 3$, $\Omega =]-1,1[^n$, $p \in \left]\frac{n-1}{2}, n-1\right[$, then $0 < \frac{n-1-p}{p} < 1$ and $\frac{(n-1)p}{n-1-p} > n-1$, therefore, taken $q \in \left]\frac{(n-1)p}{n-1-p}, +\infty\right[$, it results that $q > n-1$ and $0 < \frac{n-1}{q} < \frac{n-1-p}{p} < 1$.

Let $\alpha \in \left[\frac{n-1}{q}, \frac{n-1-p}{p}\right[$, and set

$$f\colon (z_1, \ldots, z_n) \in \mathbb{R}^n \mapsto |z_1|^q + |z_2|^p + \ldots + |z_n|^p,$$

$$\varphi_0\colon (x_1, \ldots, x_n) \in \partial(]-1,1[^n) \setminus \{(1,0,\ldots,0), (-1,0,\ldots,0)\} \mapsto (x_2^2 + \ldots + x_n^2)^{-\frac{\alpha}{2}},$$

$$u_0\colon (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus (\mathbb{R} \times \{(0,\ldots,0)\}) \mapsto (x_2^2 + \ldots + x_n^2)^{-\frac{\alpha}{2}},$$

then $\varphi_0 \in L^1(\partial\Omega)$ and $u_0 \in W^{1,p}_{loc}(\mathbb{R}^n)$. Moreover, being $p > 1$, it results that $f^\infty(z_1, \ldots, z_n) = 0$ if $(z_1, \ldots, z_n) = (0, \ldots, 0)$, $f^\infty(z_1, \ldots, z_n) = +\infty$ if $(z_1, \ldots, z_n) \neq (0, \ldots, 0)$.

Obviously we have that

$$\min\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega)\} \leq F(\Omega, \varphi_0, u_0) = \int_\Omega f(\nabla u_0) dx < +\infty. \tag{3.1}$$

We now want to show that

$$\inf\left\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega) \cap L^q_{loc}(\Omega)\right\} = +\infty. \tag{3.2}$$

On the contrary, let us assume that there exists $v \in W^{1,p}(\Omega) \cap L^q_{loc}(\Omega)$ such that

$$\int_\Omega f(\nabla v) dx + \int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n}) d\mathcal{H}^{n-1} < +\infty, \tag{3.3}$$

then clearly $\gamma_\Omega(v) = \varphi_0$.

Let us fix $a \in]0,1[$ and set $P = \{1\} \times [-a,a]^{n-1}$, $K = [-a,a]^n$, then it is easy to prove that

$$\varphi_0(1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n) + \int_{x_1}^1 \nabla_1 v(t, x_2, \ldots, x_n) dt$$
$$\text{for every } x_1 \in]-a,a[\text{ and } \mathcal{L}^{n-1}\text{-a.e. } (x_2, \ldots, x_n) \in]-1,1[^{n-1},$$

from which, by Holder inequality, we deduce that

$$|\varphi_0(1, x_2, \ldots, x_n)|^q \leq$$
$$\leq 2^{q-1}|v(x_1, x_2, \ldots, x_n)|^q + 2^{q-1}(1-x_1)^{q-1} \int_{-1}^1 |\nabla_1 v(t, x_2, \ldots, x_n)|^q dt$$
$$\text{for every } x_1 \in]-a,a[\text{ and } \mathcal{L}^{n-1}\text{-a.e. } (x_2, \ldots, x_n) \in]-1,1[^{n-1}. \tag{3.4}$$

We integrate both sides of (3.4) with respect to x_1 in $[-a, a]$ getting

$$\begin{aligned}
&2a|\varphi_0(1, x_2, \ldots, x_n)|^q \leq \\
&\leq 2^{q-1}\int_{-a}^{a}|v(x_1, x_2, \ldots, x_n)|^q dx_1 + 2^{q-1}\int_{-a}^{a}(1-x_1)^{q-1}\int_{-1}^{1}|\nabla_1 v(t, x_2, \ldots, x_n)|^q dt dx_1 \leq \\
&\qquad \leq 2^{q-1}\int_{-a}^{a}|v(x_1, x_2, \ldots, x_n)|^q dx_1 + 2^{2q-1}a\int_{-1}^{1}|\nabla_1 v(t, x_2, \ldots, x_n)|^q dt \\
&\qquad\qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } (x_2, \ldots, x_n) \in]-1, 1[^{n-1},
\end{aligned}$$

from which, by performing an integration over $[-a, a]^{n-1}$, we infer

$$\begin{aligned}
2a\int_P |\varphi_0|^q d\mathcal{H}^{n-1} &\leq 2^{q-1}\int_K |v|^q dx + 2^{2q-1}a\int_\Omega |\nabla_1 v|^q dx \leq \\
&\leq 2^{q-1}\int_K |v|^q dx + 2^{2q-1}a\int_\Omega f(\nabla v)dx. \quad (3.5)
\end{aligned}$$

By (3.5) and (3.3), being $v \in L^q(K)$, we deduce that $\varphi_0 \in L^q_{loc}(P)$, contrary to the fact that, being $\alpha \geq \frac{n-1}{q}$, φ_0 cannot be in $L^q_{loc}(P)$.

By virtue of this (3.2) holds.

In conclusion both (3.1) and (3.2) are fulfilled, and the described Lavrentieff phenomenon actually occurs.

Remark 3.2. It is clear that if in Example 3.1 we would take q smaller than the Sobolev exponent of p, namely $q \in \left[1, \frac{np}{n-p}\right]$, then, by Sobolev Embedding Theorem, there would not be any Lavrentieff phenomenon for $F(\Omega, \varphi_0, \cdot)$, $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega) \cap L^q(\Omega)$. On the contrary the example shows that the phenomenon occurs if q is greater than the Sobolev exponent of p but in the space dimension $n-1$, namely $q \in \left]\frac{(n-1)p}{n-1-p}, +\infty\right[$.

Example 3.3. Let n, Ω, p, q, f, φ_0 be as in Example 3.1, and $F(\Omega, \varphi_0, \cdot)$ be given by (1.2), then, once observed that, by using Poincaré-Wirtinger inequality, it is not difficult to prove that $\min\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\} = \min\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega)\}$, and that $\inf\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega) \cap L^q_{loc}(\Omega)\} = \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap L^q_{loc}(\Omega)\}$, by Example 3.1 we conclude that

$$\begin{aligned}
&\min\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\} = \min\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega)\} < \\
&< \inf\{F(\Omega, \varphi_0, u) : u \in W^{1,p}(\Omega) \cap L^q_{loc}(\Omega)\} = \\
&= \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap L^q_{loc}(\Omega)\} = +\infty,
\end{aligned}$$

and hence that the Lavrentieff phenomenon occurs also for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $BV(\Omega) \cap L^q_{loc}(\Omega)$.

Example 3.4. Let n, Ω, p, q, f, φ_0 be as in Example 3.1, and $F(\Omega, \varphi_0, \cdot)$ be given by (1.2), then, once observed that $C^1(\Omega) \subseteq L^q_{loc}(\Omega)$, by Example 3.3 we deduce that the Lavrentieff phenomenon occurs also for $F(\Omega, \varphi_0, \cdot)$, $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega) \cap C^1(\Omega)$, and for $F(\Omega, \varphi_0, \cdot)$, $BV(\Omega)$ and $BV(\Omega) \cap C^1(\Omega)$.

4. Conditions for the non occurrence of Lavrentieff phenomenon

Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex, Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and let $F(\Omega,\varphi_0,\cdot)$ be the functional defined in (1.2).

In the present section we study some conditions in order to forestall Lavrentieff phenomenon for $F(\Omega,\varphi_0,\cdot)$, $BV(\Omega)$ and $BV(\Omega)\cap C^\infty(\Omega)$.

We start by recalling that the $L^1(\Omega)$-lower semicontinuous envelope $\overline{F}(\Omega,\varphi_0,\cdot)$ of the functional in (1.7) is also given by

$$\overline{F}(\Omega,\varphi_0,\cdot)\colon u\in L^1(\Omega)\mapsto$$
$$\inf\left\{\liminf_{h\to+\infty} F(\Omega,\varphi_0,u_h) : \{u_h\}\subseteq BV(\Omega)\cap C^\infty(\Omega),\ u_h\to u \text{ in } L^1(\Omega)\right\}. \quad (4.1)$$

Consequently, by Proposition 2.4, we deduce that

$$F(\Omega,\varphi_0,u)\le \overline{F}(\Omega,\varphi_0,u) \quad \text{for every } u\in BV(\Omega). \quad (4.2)$$

Remark 4.1. If $f\colon \mathbb{R}^n \to [0,+\infty[$ is convex, Ω is a bounded open set with Lipschitz boundary, $\varphi_0\in L^1(\partial\Omega)$, $F(\Omega,\varphi_0,\cdot)$ is defined in (1.2) and $\overline{F}(\Omega,\varphi_0,\cdot)$ by (4.1), then it is clear that

$$F(\Omega,\varphi_0,u)=\overline{F}(\Omega,\varphi_0,u)=+\infty \quad \text{for every } u\in BV(\Omega)$$

if and only if (1.4) holds. Moreover

$$\inf\{F(\Omega,\varphi_0,u) : u\in BV(\Omega)\cap C^\infty(\Omega)\}=\inf\{F(\Omega,\varphi_0,u) : u\in BV(\Omega)\}=+\infty$$

if and only if (1.4) holds.

In order to treat the remaining case, in which $F(\Omega,\varphi_0,\cdot)$ is not identically $+\infty$, by using the relaxation arguments exposed in the introduction, we will first introduce a general framework in which there is no Lavrentieff phenomenon for $F(\Omega,\varphi_0,\cdot)$, $BV(\Omega)\cap L^p_{loc}(\Omega)$ and $BV(\Omega)\cap C^\infty(\Omega)$ for some $p\in[1,+\infty]$, and then we will give some conditions in order to prove the same result for $F(\Omega,\varphi_0,\cdot)$, $BV(\Omega)$ and $BV(\Omega)\cap L^p_{loc}(\Omega)$.

Also in this case we will furnish a characterization of the non occurrence of the Lavrentieff phenomenon for $F(\Omega,\varphi_0,\cdot)$, $BV(\Omega)$ and $BV(\Omega)\cap C^\infty(\Omega)$.

First of all we recall that, being f^∞ 1-homogeneous, it results that

$$\int_\Omega f^\infty(\nabla^s(tu))d|D^s(tu)| = t\int_\Omega f^\infty(\nabla^s u)d|D^s u|$$
$$\text{for every } u\in BV(\Omega),\ t\in[0,+\infty[. \quad (4.3)$$

Lemma 4.2. *Let $f\colon \mathbb{R}^n\to[0,+\infty[$ be convex, $p\in[1,+\infty]$, and assume that, if $p<+\infty$, (1.8) holds. Then for every bounded open set with Lipschitz boundary Ω, $u\in BV(\Omega)\cap L^p_{loc}(\Omega)$ there exists $\{u_h\}\subseteq BV(\Omega)\cap C^\infty(\Omega)$ such that $u_h\to u$ in $L^1(\Omega)$, $\gamma_\Omega(u_h)=\gamma_\Omega(u)$ for every $h\in\mathbb{N}$, and*

$$\lim_{h\to+\infty}\int_\Omega f(\nabla u_h)dx \le \lim_{t\to 1^+}\int_\Omega f(t\nabla u)dx + \int_\Omega f^\infty(\nabla^s u)d|D^s u|.$$

Proof. Let Ω, u be as above, and let us preliminarly observe that we can clearly assume that there exists $\eta_0 \in]0, +\infty[$ such that

$$\int_\Omega f((1+\eta)\nabla u)dx + \int_\Omega f^\infty(\nabla^s u)d|D^s u| < +\infty \quad \text{for every } \eta \in]0, \eta_0[. \tag{4.4}$$

For every $\eta > 0$ let $f_\eta = f + \eta|\cdot|$, and set $f_\eta^\infty = (f_\eta)^\infty$.

Let $\eta \in]0, \eta_0[$ and, by virtue of (4.4), for every $j \in \mathbb{N} \cup \{0\}$ let A_j be an open set such that $A_j \subset\subset A_{j+1} \subset\subset \Omega$, $\cup_{j=0}^\infty A_j = \Omega$, $\text{dist}(A_j, \partial\Omega) \geq \frac{\text{dist}(A_0,\partial\Omega)}{j+1}$, $\mathcal{L}^n(\partial A_j) = |Du|(\partial A_j) = 0$, and

$$\int_\Omega f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_\Omega f_\eta^\infty(\nabla^s u)d|D^s u| - \frac{\eta}{2^{j+2}} \leq$$
$$\leq \int_{A_j} f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_{A_j} f_\eta^\infty(\nabla^s u)d|D^s u|. \tag{4.5}$$

Let us also consider a partition of unity $\{\psi_j\}_{j\in\mathbb{N}\cup\{0\}}$ relative to the covering $\{A_{j+1} \setminus \overline{A_{j-1}}\}_{j\in\mathbb{N}\cup\{0\}}$, (where we have set $A_{-1} = \emptyset$), i.e. for every $j \in \mathbb{N} \cup \{0\}$ a function $\psi_j \in C_0^\infty(A_{j+1} \setminus \overline{A_{j-1}})$ with $0 \leq \psi_j \leq 1$ in Ω and $\sum_{j=0}^\infty \psi_j = 1$ in Ω.

Let $\{\varepsilon_h\}$ be a decreasing sequence of positive numbers such that $\lim_{h\to+\infty} \varepsilon_h = 0$, and let, for every $h \in \mathbb{N}$, $\rho_{\varepsilon_h} * u$ be the regularization of u defined in section 2.

Let us fix $j \in \mathbb{N} \cup \{0\}$, then

$$\rho_{\varepsilon_h} * u \to u \quad \text{in } L^p(A_j) \text{ and } \mathcal{L}^n\text{-a.e. in } A_j. \tag{4.6}$$

By the properties of A_j, Lemma 2.6 and (4.3) we get

$$\limsup_{h\to+\infty} \int_{A_j} f_\eta((1+\eta)\nabla(\rho_{\varepsilon_h} * u))dx = \limsup_{h\to+\infty} \int_{A_j} f_\eta(\nabla(\rho_{\varepsilon_h} * ((1+\eta)u)))dx \leq$$
$$\leq \int_{A_j} f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_{A_j} f_\eta^\infty(\nabla^s u)d|D^s u|. \tag{4.7}$$

In addition, by Proposition 2.3 and (4.3), we have that

$$\int_{A_j} f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_{A_j} f_\eta^\infty(\nabla^s u)d|D^s u| \leq$$
$$\leq \liminf_{h\to+\infty} \int_{A_j} f_\eta((1+\eta)\nabla(\rho_{\varepsilon_h} * u))dx. \tag{4.8}$$

By (4.6) we deduce the existence of $h_1(\eta, j) \in \mathbb{N}$ such that $\varepsilon_{h_1(\eta,j)} < \text{dist}(A_{j+1}, \partial\Omega)$ and

$$\int_{A_{j+1}\setminus\overline{A_{j-1}}} |(\rho_{\varepsilon_h} * u) - u|dx \leq \frac{\eta}{2^j} \quad \text{for every } h \geq h_1(\eta, j). \tag{4.9}$$

By (4.7) we deduce the existence of $h_2(\eta, j) \in \mathbb{N}$ such that $\varepsilon_{h_2(\eta,j)} < \mathrm{dist}(A_{j+1}, \partial\Omega)$ and

$$\int_{A_{j+1}} f_\eta((1+\eta)\nabla(\rho_{\varepsilon_h} * u))dx \leq$$
$$\leq \int_{A_{j+1}} f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_{A_{j+1}} f_\eta^\infty(\nabla^s u)d|D^s u| + \frac{\eta}{2^j}$$
$$\text{for every } h \geq h_2(\eta, j). \quad (4.10)$$

By (4.8) we deduce the existence of $h_3(\eta, j) \in \mathbb{N}$ such that $\varepsilon_{h_3(\eta,j)} < \mathrm{dist}(A_{j-1}, \partial\Omega)$ and

$$\int_{A_{j-1}} f_\eta((1+\eta)\nabla(\rho_{\varepsilon_h} * u))dx \geq$$
$$\geq \int_{A_{j-1}} f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_{A_{j-1}} f_\eta^\infty(\nabla^s u)d|D^s u| - \frac{\eta}{2^{j+1}}$$
$$\text{for every } h \geq h_3(\eta, j), \quad (4.11)$$

moreover, by collecting together (4.11) and (4.5), we obtain

$$\int_{A_{j-1}} f_\eta((1+\eta)\nabla(\rho_{\varepsilon_h} * u))dx \geq$$
$$\geq \int_{\Omega} f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_{\Omega} f_\eta^\infty(\nabla^s u)d|D^s u| - \frac{\eta}{2^j}$$
$$\text{for every } h \geq h_3(\eta, j). \quad (4.12)$$

Let us observe now that, if $j \neq 0$, it results $\sum_{i=0}^\infty \psi_i = \psi_{j-1} + \psi_j = 1$ in $A_j \setminus \overline{A_{j-1}}$, consequently

$$\nabla\psi_{j-1} + \nabla\psi_j = 0 \quad \text{in } A_j \setminus \overline{A_{j-1}}. \quad (4.13)$$

Let us prove that

$$\lim_{(h,k)\to(+\infty,+\infty)} \int_{A_j\setminus\overline{A_{j-1}}} f_\eta\left(\frac{1+\eta}{\eta}[(\rho_{\varepsilon_h} * u)\nabla\psi_{j-1} + (\rho_{\varepsilon_k} * u)\nabla\psi_j]\right)dx =$$
$$= f(0)\mathcal{L}^n(A_j \setminus \overline{A_{j-1}}). \quad (4.14)$$

On the contrary there would exist $\overline{\varepsilon} > 0$ and two increasing sequences $\{h_i\}$, $\{k_i\} \subseteq \mathbb{N}$ such that

$$\left|\int_{A_j\setminus\overline{A_{j-1}}} f_\eta\left(\frac{1+\eta}{\eta}\left[(\rho_{\varepsilon_{h_i}} * u)\nabla\psi_{j-1} + (\rho_{\varepsilon_{k_i}} * u)\nabla\psi_j\right]\right)dx - f(0)\mathcal{L}^n(A_j \setminus \overline{A_{j-1}})\right| \geq \overline{\varepsilon}$$
$$\text{for every } i \in \mathbb{N},$$

contrary to the fact that by (4.6), (4.13) and Lemma 2.7 if $p = +\infty$, or by (4.6), (4.13) and Lebesgue Dominated Convergence Theorem if $p < +\infty$, it would result

$$\lim_{i\to+\infty}\int_{A_j\setminus\overline{A_{j-1}}} f_\eta\left(\frac{1+\eta}{\eta}\left[(\rho_{\varepsilon_{h_i}} * u)\nabla\psi_{j-1} + (\rho_{\varepsilon_{k_i}} * u)\nabla\psi_j\right]\right)dx = f(0)\mathcal{L}^n(A_j\setminus\overline{A_{j-1}}).$$

By (4.14) we deduce the existence of $h_4(\eta, j) \in \mathbb{N}$ such that $\varepsilon_{h_4(\eta,j)} < \mathrm{dist}(A_j, \partial\Omega)$ and

$$\int_{A_j\setminus\overline{A_{j-1}}} f_\eta\left(\frac{1+\eta}{\eta}\left[(\rho_{\varepsilon_h} * u)\nabla\psi_{j-1} + (\rho_{\varepsilon_k} * u)\nabla\psi_j\right]\right)dx \leq$$
$$\leq f(0)\mathcal{L}^n(A_j\setminus\overline{A_{j-1}}) + \frac{\eta}{2^j} \quad \text{for every } h,\, k \geq h_4(\eta, j). \tag{4.15}$$

Finally, for every $j \in \mathbb{N}\cup\{0\}$, let $h(\eta, j) \in \mathbb{N}$ be such that $h(\eta, j) \geq \max\{h_1(\eta, j), h_2(\eta, j), h_3(\eta, j), h_4(\eta, j+1)\}$ and $h(\eta, j+1) \geq h(\eta, j)$, and define

$$w_\eta = \sum_{j=0}^{\infty}\psi_j(\rho_{\varepsilon_{h(\eta,j)}} * u). \tag{4.16}$$

Since $\varepsilon_{h(\eta,j)} < \mathrm{dist}(A_{j+1}, \partial\Omega)$ for every $j \in \mathbb{N}\cup\{0\}$, and for every $x \in \Omega$ the series in (4.16) has at most a finite number of non zero terms, it turns out that $w_\eta \in C^\infty(\Omega)$.

By using the properties of $\{\psi_j\}$ and (4.9), we have

$$\|w_\eta - u\|_{L^1(\Omega)} = \left\|\sum_{j=0}^{\infty}\psi_j(\rho_{\varepsilon_{h(\eta,j)}} * u) - \sum_{j=0}^{\infty}\psi_j u\right\|_{L^1(\Omega)} \leq$$
$$\leq \sum_{j=0}^{\infty}\int_{A_{j+1}\setminus\overline{A_{j-1}}} |(\rho_{\varepsilon_{h(\eta,j)}} * u) - u|dx \leq 2\eta, \tag{4.17}$$

from which we conclude that $w_\eta \in L^1(\Omega)$.

By the convexity of f_η we obtain

$$\int_\Omega f_\eta(\nabla w_\eta)dx =$$
$$= \int_\Omega f_\eta\left(\frac{1}{1+\eta}\sum_{j=0}^{\infty}(1+\eta)\psi_j\nabla(\rho_{\varepsilon_{h(\eta,j)}} * u) + \frac{\eta}{1+\eta}\sum_{j=0}^{\infty}\frac{1+\eta}{\eta}(\rho_{\varepsilon_{h(\eta,j)}} * u)\nabla\psi_j\right)dx \leq$$
$$\leq \frac{1}{1+\eta}\int_\Omega f_\eta\left(\sum_{j=0}^{\infty}\psi_j(1+\eta)\nabla(\rho_{\varepsilon_{h(\eta,j)}} * u)\right)dx +$$
$$+ \frac{\eta}{1+\eta}\int_\Omega f_\eta\left(\frac{1+\eta}{\eta}\sum_{j=0}^{+\infty}(\rho_{\varepsilon_{h(\eta,j)}} * u)\nabla\psi_j\right)dx \leq$$

$$\leq \frac{1}{1+\eta}\int_\Omega \sum_{j=0}^{+\infty} \psi_j f_\eta((1+\eta)\nabla(\rho_{\varepsilon_{h(\eta,j)}} * u))dx +$$
$$+ \frac{\eta}{1+\eta}\int_\Omega f_\eta\left(\frac{1+\eta}{\eta}\sum_{j=0}^{+\infty}(\rho_{\varepsilon_{h(\eta,j)}} * u)\nabla\psi_j\right)dx \leq$$

$$\leq \frac{1}{1+\eta}\int_{A_1} f_\eta((1+\eta)\nabla(\rho_{\varepsilon_{h(\eta,0)}} * u))dx +$$
$$+ \frac{1}{1+\eta}\sum_{j=1}^{\infty}\int_{A_{j+1}\setminus\overline{A_{j-1}}} f_\eta((1+\eta)\nabla(\rho_{\varepsilon_{h(\eta,j)}} * u))dx +$$
$$+ \frac{\eta}{1+\eta}\int_\Omega f_\eta\left(\frac{1+\eta}{\eta}\sum_{j=0}^{+\infty}(\rho_{\varepsilon_{h(\eta,j)}} * u)\nabla\psi_j\right)dx. \tag{4.18}$$

Let us observe that, for all $j \in \mathbb{N}\cup\{0\}$, we have $\nabla\psi_j = 0$ in A_0, $\sum_{i=1}^{+\infty}(\rho_{\varepsilon_{h(\eta,i)}} * u)\nabla\psi_i = (\rho_{\varepsilon_{h(\eta,j-1)}} * u)\nabla\psi_{j-1} + (\rho_{\varepsilon_{h(\eta,j)}} * u)\nabla\psi_j$ in $A_j \setminus \overline{A_{j-1}}$. Consequently

$$\int_\Omega f_\eta\left(\frac{1+\eta}{\eta}\sum_{j=0}^{+\infty}(\rho_{\varepsilon_{h(\eta,j)}} * u)\nabla\psi_j\right)dx =$$
$$= f(0)\mathcal{L}^n(A_0) + \sum_{j=1}^{\infty}\int_{A_j\setminus\overline{A_{j-1}}} f_\eta\left(\frac{1+\eta}{\eta}\left[(\rho_{\varepsilon_{h(\eta,j-1)}} * u)\nabla\psi_{j-1} + (\rho_{\varepsilon_{h(\eta,j)}} * u)\nabla\psi_j\right]\right)dx. \tag{4.19}$$

By (4.18), (4.4), (4.10), (4.12), (4.19), and (4.15) we get

$$\int_\Omega f_\eta(\nabla w_\eta)dx \leq$$
$$\leq \int_\Omega f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_\Omega f_\eta^\infty(\nabla^s u)d|D^s u| + \eta +$$
$$+ \sum_{j=1}^{\infty}\left[\int_\Omega f_\eta((1+\eta)\nabla u)dx + (1+\eta)\int_\Omega f_\eta^\infty(\nabla^s u)d|D^s u| + \frac{\eta}{2^j} -\right.$$
$$\left. - \int_\Omega f_\eta((1+\eta)\nabla u)dx - (1+\eta)\int_\Omega f_\eta^\infty(\nabla^s u)d|D^s u| + \frac{\eta}{2^j}\right] +$$
$$+ \frac{\eta}{1+\eta}\left\{f(0)\mathcal{L}^n(\Omega) + \sum_{j=1}^{\infty}\left[f(0)\mathcal{L}^n(A_j\setminus\overline{A_{j-1}}) + \frac{\eta}{2^j}\right]\right\} =$$
$$= \int_\Omega f((1+\eta)\nabla u)dx + (1+\eta)\int_\Omega f^\infty(\nabla^s u)d|D^s u| +$$
$$+ \eta(1+\eta)|Du|(\Omega) + 4\eta + \frac{\eta}{1+\eta}f(0)\left\{\mathcal{L}^n(\Omega) + \mathcal{L}^n(\Omega - A_0)\right\} < +\infty. \tag{4.20}$$

Therefore by (4.20), (4.4), and the first part of Lemma 2.8 we conclude that

$$\int_\Omega |\nabla w_\eta| dx \leq \frac{1}{\eta} \int_\Omega f_\eta(\nabla w_\eta) dx < +\infty,$$

i.e. $w_\eta \in BV(\Omega) \cap C^\infty(\Omega)$, and that

$$\limsup_{\eta\to 0} \int_\Omega f(\nabla w_\eta) dx \leq \limsup_{\eta\to 0} \int_\Omega f_\eta(\nabla w_\eta) dx \leq \lim_{t\to 1^+} \int_\Omega f(t\nabla u) dx + \int_\Omega f^\infty(\nabla^s u) d|D^s u|,$$

from which the thesis will follow once we prove that $\gamma_\Omega(w_\eta) = \gamma_\Omega(u)$ for every $\eta \in]0, \eta_0[$. To do this let us take $\eta \in]0, \eta_0[$, and observe that, since u, $w_\eta \in BV(\Omega)$, by (2.2) it follows that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{\Omega\cap B_r(x_0)} |u - \gamma_\Omega(u)(x_0)| dx = \lim_{r\to 0} \frac{1}{r^n} \int_{\Omega\cap B_r(x_0)} |w_\eta - \gamma_\Omega(w_\eta)(x_0)| dx = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in \partial\Omega,$$

from which we deduce that

$$|\gamma_\Omega(w_\eta)(x_0) - \gamma_\Omega(u)(x_0)| \leq \lim_{r\to 0} \frac{1}{\mathcal{L}^n(\Omega \cap B_r(x_0))} \int_{\Omega\cap B_r(x_0)} |w_\eta - u| dx \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in \partial\Omega.$$

Therefore, to complete the proof, it suffices prove that

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^n(\Omega \cap B_r(x_0))} \int_{\Omega\cap B_r(x_0)} |w_\eta - u| dx = 0 \ \text{ for every } x_0 \in \partial\Omega. \tag{4.21}$$

Let $x_0 \in \partial\Omega$. For every $r > 0$ let us set $j(r) = \min\{j \in \mathbb{N} : A_j \cap B_r(x_0) \neq \emptyset\}$, then, by the properties of $\{A_j\}_{j\in\mathbb{N}\cup\{0\}}$, it follows that $r > \operatorname{dist}(A_{j(r)}, \partial\Omega) \geq \frac{\operatorname{dist}(A_0,\partial\Omega)}{j(r)+1}$ and therefore that $j(r) > \frac{\operatorname{dist}(A_0,\partial\Omega)}{r} - 1$. By arguing as in (4.17) and by using (4.9), we have that

$$\int_{\Omega\cap B_r(x_0)} |w_\eta - u| dx \leq \sum_{j=j(r)}^{\infty} \int_{(A_{j+1}\setminus\overline{A_{j-1}})\cap B_r(x_0)} |(\rho_{\varepsilon_{h(\eta,j)}} * u) - u| dx \leq$$

$$\leq \frac{\eta}{2^{j(r)-1}} \leq \frac{\eta}{2^{\frac{\operatorname{dist}(A_0,\partial\Omega)}{r}-2}}, \tag{4.22}$$

therefore, once recalled that Ω has Lipschitz boundary and hence that there exists $C_\Omega > 0$ such that $\mathcal{L}^n(\Omega \cap B_r(x_0)) \geq C_\Omega r^n$ for every r small enough, by (4.22) we deduce (4.21) and the thesis. □

Lemma 4.3. *Let $f\colon \mathbb{R}^n \to [0, +\infty[$ be convex and verifying (1.5), $p \in [1, +\infty]$, and assume that, if $p < +\infty$, (1.8) holds. Then for every bounded open set with Lipschitz boundary Ω, $u \in BV(\Omega) \cap L^p_{loc}(\Omega)$ there exists $\{u_h\} \subseteq BV(\Omega) \cap C^\infty(\Omega)$ such that $u_h \to u$ in $L^1(\Omega)$, $\gamma_\Omega(u_h) = \gamma_\Omega(u)$ for every $h \in \mathbb{N}$ and*

$$\lim_{h\to+\infty} \int_\Omega f(\nabla u_h) dx = \int_\Omega f(\nabla u) dx + \int_\Omega f^\infty(\nabla^s u) d|D^s u|.$$

Proof. Follows by Lemma 4.2, (1.5), Lemma 2.9, Lemma 2.8, and Proposition 2.3. □

Proposition 4.4. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex and verifying (1.5), $p \in [1,+\infty]$, and assume that, if $p < +\infty$, (1.8) holds. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, $F(\Omega,\varphi_0,\cdot)$ and $\overline{F}(\Omega,\varphi_0,\cdot)$ be defined by (1.2) and (4.1), then*

$$\overline{F}(\Omega,\varphi_0,u) = F(\Omega,\varphi_0,u) \quad \text{for every } u \in BV(\Omega) \cap L^p_{loc}(\Omega).$$

Proof. Follows immediately by (4.2), and Lemma 4.3. □

In the sequel we will make use of the following assumption.

$$\inf\{F(\Omega,\varphi_0,v) : v \in W^{1,1}(\Omega) \cap L^p_{loc}(\Omega)\} < +\infty. \tag{4.23}$$

Lemma 4.5. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex, $p \in [1,+\infty]$, Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and $F(\Omega,\varphi_0,\cdot)$ be defined by (1.2). Let us assume that (4.23) holds, then for every $u \in BV(\Omega)$ there exists $\{w_h\} \subseteq BV(\Omega) \cap L^p_{loc}(\Omega)$ such that $w_h \to u$ in $L^1(\Omega)$ and*

$$\lim_{h\to+\infty} F(\Omega,\varphi_0,w_h) = F(\Omega,\varphi_0,u).$$

Proof. Let $u \in BV(\Omega)$, $\varepsilon > 0$.

If $A \subset\subset \Omega$ we have by Lemma 2.5 that

$$\limsup_{\varepsilon\to 0} \int_A f(\nabla(\rho_\varepsilon * u))dx \le \int_\Omega f(\nabla u)dx + \int_\Omega f^\infty(\nabla^s u)d|D^s u|. \tag{4.24}$$

By (4.23) let $v \in W^{1,1}(\Omega) \cap L^p_{loc}(\Omega)$ be such that

$$\int_\Omega f(\nabla v)dx + \int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} < +\infty, \tag{4.25}$$

and let us set, for every $h \in \mathbb{N}$, $w_{\varepsilon,h} = \max\{\min\{\rho_\varepsilon * u, v+h\}, v-h\}$, $w_h = \max\{\min\{u, v+h\}, v-h\}$. It result that $w_h \in BV(\Omega) \cap L^p_{loc}(\Omega)$ for every $h \in \mathbb{N}$, $w_{\varepsilon,h} \to w_h$ in $L^1(A)$ and

$\mathcal{L}^n$-a.e. in A, therefore by Proposition 2.3 and (4.24) we have

$$\begin{aligned}
&\int_A f(\nabla w_h)dx + \int_A f^\infty(\nabla^s w_h)d|D^s w_h| \leq \\
&\leq \liminf_{\varepsilon\to 0}\int_A f(\nabla w_{\varepsilon,h})dx \leq \\
&\leq \limsup_{\varepsilon\to 0}\int_A f(\nabla(\rho_\varepsilon * u))dx + \\
&\qquad + \limsup_{\varepsilon\to 0}\int_{\{y\in A:(\rho_\varepsilon*u)(y)>v(y)+h\}} f(\nabla v)dx + \\
&\qquad + \limsup_{\varepsilon\to 0}\int_{\{y\in A:(\rho_\varepsilon*u)(y)<v(y)-h\}} f(\nabla v)dx \leq \\
&\leq \int_\Omega f(\nabla u)dx + \int_\Omega f^\infty(\nabla^s u)d|D^s u| + \\
&\qquad + \int_{\{y\in\Omega:u(y)\geq v(y)+h\}} f(\nabla v)dx + \int_{\{y\in\Omega:u(y)\leq v(y)-h\}} f(\nabla v)dx \\
&\qquad \text{for every } h\in\mathbb{N},
\end{aligned}$$

from which, taking into account also (4.25), we deduce that

$$\limsup_{h\to+\infty}\left\{\int_\Omega f(\nabla w_h)dx + \int_\Omega f^\infty(\nabla^s w_h)d|D^s w_h|\right\} \leq \\ \leq \int_\Omega f(\nabla u)dx + \int_\Omega f^\infty(\nabla^s u)d|D^s u|. \quad (4.26)$$

In order to treat the boundary integrals $\int_{\partial\Omega} f^\infty((\varphi_0-\gamma_\Omega(w_h))\mathbf{n})d\mathcal{H}^{n-1}$, let us set, for every $h\in\mathbb{N}$, $A_h=\{x\in\partial\Omega:\gamma_\Omega(u)(x)\geq\gamma_\Omega(v)(x)+h\}$, $B_h=\{x\in\partial\Omega:\gamma_\Omega(u)(x)\leq\gamma_\Omega(v)(x)-h\}$, and observe that by Proposition 2.2 we have

$$\varphi_0(x)-\gamma_\Omega(w_h)(x)=\begin{cases}\varphi_0(x)-\gamma_\Omega(u)(x) & \text{if } x\in\partial\Omega\setminus(A_h\cup B_h)\\ \varphi_0(x)-\gamma_\Omega(v)(x)-h & \text{if } x\in A_h\\ \varphi_0(x)-\gamma_\Omega(v)(x)+h & \text{if } x\in B_h\end{cases}$$
$$\text{for every } h\in\mathbb{N} \text{ and } \mathcal{H}^{n-1}\text{-a.e. } x\in\partial\Omega,$$

therefore

$$\begin{aligned}
&\int_{\partial\Omega} f^\infty((\varphi_0-\gamma_\Omega(w_h))\mathbf{n})d\mathcal{H}^{n-1} = \\
&\quad = \int_{A_h} f^\infty((\varphi_0-(\gamma_\Omega(v)+h))\mathbf{n})d\mathcal{H}^{n-1} + \int_{B_h} f^\infty((\varphi_0-(\gamma_\Omega(v)-h))\mathbf{n})d\mathcal{H}^{n-1} + \\
&\qquad + \int_{\partial\Omega\setminus(A_h\cup B_h)} f^\infty((\varphi_0-\gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1} \quad \text{for every } h\in\mathbb{N}. \quad (4.27)
\end{aligned}$$

Let us estimate the first term in the right-hand side of (4.27). By the 1-homogeneity of f^∞ we have

$$\begin{aligned}
&\int_{A_h} f^\infty((\varphi_0 - (\gamma_\Omega(v)+h))\mathbf{n})d\mathcal{H}^{n-1} = \\
&\qquad = \int_{A_h\cap\{y\in\partial\Omega:\varphi_0(y)\geq\gamma_\Omega(v)(y)+h\}} f^\infty((\varphi_0 - (\gamma_\Omega(v)+h))\mathbf{n})d\mathcal{H}^{n-1} + \\
&\qquad + \int_{A_h\cap\{y\in\partial\Omega:\varphi_0(y)<\gamma_\Omega(v)(y)+h\}} f^\infty((\varphi_0 - (\gamma_\Omega(v)+h))\mathbf{n})d\mathcal{H}^{n-1} = \\
&= \int_{A_h\cap\{y\in\partial\Omega:\varphi_0(y)\geq\gamma_\Omega(v)(y)+h\}} \frac{\varphi_0 - (\gamma_\Omega(v)+h)}{\varphi_0 - \gamma_\Omega(v)} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} + \\
&+ \int_{A_h\cap\{y\in\partial\Omega:\varphi_0(y)<\gamma_\Omega(v)(y)+h\}} \frac{\varphi_0 - (\gamma_\Omega(v)+h)}{\varphi_0 - \gamma_\Omega(u)} f^\infty((\varphi_0 - \gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1} \leq \\
&\qquad \leq \int_{A_h} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} + \int_{A_h} f^\infty((\varphi_0 - \gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1} \\
&\qquad\qquad \text{for every } h\in\mathbb{N}. \qquad (4.28)
\end{aligned}$$

Analogously we also have

$$\begin{aligned}
&\int_{B_h} f^\infty((\varphi_0 - (\gamma_\Omega(v)-h))\mathbf{n})d\mathcal{H}^{n-1} \leq \\
&\leq \int_{B_h} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} + \int_{B_h} f^\infty((\varphi_0 - \gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1} \quad \text{for every } h\in\mathbb{N}. \qquad (4.29)
\end{aligned}$$

Therefore, by (4.27)–(4.29), it follows that

$$\begin{aligned}
&\int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(w_h))\mathbf{n})d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1} + \\
&\qquad + \int_{A_h} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} + \int_{B_h} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} \\
&\qquad\qquad \text{for every } h\in\mathbb{N}. \qquad (4.30)
\end{aligned}$$

By (4.30) and (4.25) we conclude, as h diverges, that

$$\limsup_{h\to+\infty} \int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(w_h))\mathbf{n})d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(u))\mathbf{n})d\mathcal{H}^{n-1}. \qquad (4.31)$$

Finally, by (4.26), (4.31), and Proposition 2.4 the thesis follows. □

The following result yields conditions in order to fulfil (4.23).

Lemma 4.6. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex and verifying* (1.5)*, $p \in [1,+\infty]$, and assume that, if $p < +\infty$,* (1.8) *holds. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and let us assume that* (1.9) *is fulfilled, then* (4.23) *holds.*

Proof. Follows directly by Lemma 4.3. □

We can prove now the main results of this section.

Theorem 4.7. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex and verifying* (1.5)*, $p \in [1,+\infty]$, and assume that, if $p < +\infty$,* (1.8) *holds. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, $F(\Omega,\varphi_0,\cdot)$ and $\overline{F}(\Omega,\varphi_0,\cdot)$ be defined by* (1.2) *and* (4.1)*, then both the conditions*

$$\overline{F}(\Omega,\varphi_0,u) = F(\Omega,\varphi_0,u) \quad \text{for every } u \in BV(\Omega) \tag{4.32}$$

and $F(\Omega,\varphi_0,\cdot)$ is not identically $+\infty$ hold if and only if (1.9) *is fulfilled.*

Proof. Let us assume that (1.9) is fulfilled then clearly $F(\Omega,\varphi_0,\cdot)$ is not identically $+\infty$.

Let $u \in BV(\Omega)$, then by (1.9), Lemma 4.6, Lemma 4.5, Proposition 4.4, and the $L^1(\Omega)$-lower semicontinuity of $\overline{F}(\Omega,\varphi_0,\cdot)$ we deduce the existence of $\{w_h\} \subseteq BV(\Omega) \cap L^p_{loc}(\Omega)$ such that $w_h \to u$ in $L^1(\Omega)$ and

$$F(\Omega,\varphi_0,u) = \lim_{h\to+\infty} F(\Omega,\varphi_0,w_h) = \lim_{h\to+\infty} \overline{F}(\Omega,\varphi_0,w_h) \geq \overline{F}(\Omega,\varphi_0,u),$$

from which, together with (4.2), identity (4.32) follows.

Let us assume now that (4.32) holds and that $F(\Omega,\varphi_0,\cdot)$ is not identically $+\infty$, then being $C^\infty(\Omega) \subseteq L^p_{loc}(\Omega)$, it soon follows that (1.9) is fulfilled. □

Theorem 4.8. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex and verifying* (1.5)*, $p \in [1,+\infty]$, and assume that, if $p < +\infty$,* (1.8) *holds. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and $F(\Omega,\varphi_0,\cdot)$ be defined by* (1.2)*, then*

$$\inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} = \inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega)\} < +\infty$$

if and only if (1.9) *holds.*

Proof. Follows by Theorem 4.7. □

By the previous result we deduce the following description of the circumstances in which Lavrentieff phenomenon for $F(\Omega,\varphi_0,\cdot)$, $BV(\Omega)$ and $BV(\Omega) \cap C^\infty(\Omega)$ occurs.

Theorem 4.9. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex and verifying* (1.5)*. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and $F(\Omega,\varphi_0,\cdot)$ be defined by* (1.2)*, then*

$$\inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega)\} < \inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} \tag{4.33}$$

if and only if

$$\inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega)\} < +\infty, \ \inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} = +\infty. \tag{4.34}$$

Proof. It is clear that (4.34) implies (4.33).

Conversely, if (4.33) holds, it must necessarily result $\inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega)\} < +\infty$, moreover, by Theorem 4.8 applied with $p = +\infty$, $\inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega)\cap L^\infty_{loc}(\Omega)\} = +\infty$ and consequently $\inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega)\cap C^\infty(\Omega)\} = +\infty$. □

Remark 4.10. We emphasize that the function f in the examples of section 3 verifies condition (1.5).

Remark 4.11. We remark that if $f : \mathbb{R}^n \to [0,+\infty[$ is convex, $p \in [1,+\infty]$, Ω is a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and $F(\Omega,\varphi_0,\cdot)$ is defined by (1.2), then it directly follows by Proposition 2.4 and Lemma 4.5 that if (4.23) holds, then $F(\Omega,\varphi_0,\cdot)$ is not identically $+\infty$ and

$$F(\Omega,\varphi_0,u) = \inf\left\{\liminf_{h\to+\infty} F(\Omega,\varphi_0,u_h) : \{u_h\} \subseteq BV(\Omega)\cap L^p_{loc}(\Omega),\ u_h \to u \text{ in } L^1(\Omega)\right\}$$
$$\text{for every } u \in BV(\Omega).$$

Consequently

$$\inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega)\} = \inf\{F(\Omega,\varphi_0,u) : u \in BV(\Omega)\cap L^p_{loc}(\Omega)\} < +\infty.$$

The following results yield sufficient conditions in order to fulfil (1.9).

Proposition 4.12. *Let $f : \mathbb{R}^n \to [0,+\infty[$ be convex and verifying (1.5), $p \in [1,+\infty]$, and assume that, if $p < +\infty$, (1.8) holds. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, $F(\Omega,\varphi_0,\cdot)$ and $\overline{F}(\Omega,\varphi_0,\cdot)$ be defined by (1.2) and (4.1), and let us assume that there exists $v \in BV(\Omega)\cap L^p_{loc}(\Omega)$ such that*

$$\gamma_\Omega(v) = \varphi_0, \qquad \int_\Omega f(\nabla v)dx + \int_\Omega f^\infty(\nabla^s v)d|D^s v| < +\infty, \tag{4.35}$$

then

$$\overline{F}(\Omega,\varphi_0,u) = F(\Omega,\varphi_0,u) \quad \text{for every } u \in BV(\Omega)$$

and $F(\Omega,\varphi_0,\cdot)$ is not identically $+\infty$.

Proof. Follows by Theorem 4.7, once observed that the described assumptions imply (1.9). □

Proposition 4.13. *Let $n = 1$, $f : \mathbb{R} \to [0,+\infty[$ be convex and verifying (1.5). Let a, b, α, $\beta \in \mathbb{R}$ with $a < b$, $\varphi_0 : x \in \{a,b\} \mapsto \begin{cases} \alpha & \text{if } x = a \\ \beta & \text{if } x = b \end{cases}$, $F(]a,b[,\varphi_0,\cdot)$ and $\overline{F}(]a,b[,\varphi_0,\cdot)$ be defined by (1.2) and (4.1), then*

$$\overline{F}(]a,b[,\varphi_0,u) = F(]a,b[,\varphi_0,u) \quad \text{for every } u \in BV(]a,b[)$$

and $F(]a,b[,\varphi_0,\cdot)$ is not identically $+\infty$.

Proof. Follows by Proposition 4.12 applied with $p = +\infty$, once observed that it is always possible to find an affine function v verifying (4.35). □

Theorem 4.14. *Let* $f\colon \mathbb{R}^n \to [0,+\infty[$ *be convex and verifying* (1.5)*. Let* Ω *be a bounded open set with Lipschitz boundary,* $\varphi_0 \in L^\infty(\partial\Omega)$*,* $F(\Omega,\varphi_0,\cdot)$ *and* $\overline{F}(\Omega,\varphi_0,\cdot)$ *be defined by* (1.2) *and* (4.1)*, then*

$$\overline{F}(\Omega,\varphi_0,u) = F(\Omega,\varphi_0,u) \quad \textit{for every } u \in BV(\Omega).$$

Proof. By virtue of Remark 4.1 we have to treat only the case in which $F(\Omega,\varphi_0,\cdot)$ is not identically $+\infty$.

Let $v \in BV(\Omega)$ be such that $F(\Omega,\varphi_0,v) < +\infty$, take $k \in [0,+\infty[$ with $k > \|\varphi_0\|_{L^\infty(\partial\Omega)}$, and set $v_k = \max\{\min\{v,k\},-k\}$, then by Proposition 2.1 we infer that $v_k \in BV(\Omega) \cap L^\infty(\Omega)$.

We first observe that, by using Lemma 2.5 and Proposition 2.3, it is easy to prove that $\int_\Omega f(\nabla v_k)dx + \int_\Omega f^\infty(\nabla^s v_k)d|D^s v_k| \le \int_\Omega f(\nabla v)dx + \int_\Omega f^\infty(\nabla^s v)d|D^s v| + f(0)|\Omega| < +\infty$.

Let us now set $A_k = \{x \in \partial\Omega : \gamma_\Omega(v)(x) > k\}$, $B_k = \{x \in \partial\Omega : \gamma_\Omega(v)(x) < -k\}$, then by Proposition 2.2 we have

$$\gamma_\Omega(v_k)(x) = \begin{cases} \gamma_\Omega(v)(x) & \text{if } x \in \partial\Omega \setminus (A_k \cup B_k) \\ k & \text{if } x \in A_k \\ -k & \text{if } x \in B_k. \end{cases}$$

Moreover, by the 1-homogeneity of f^∞, we also have that

$$\begin{aligned} \int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(v_k))\mathbf{n})d\mathcal{H}^{n-1} &= \int_{\partial\Omega\setminus(A_k\cup B_k)} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} + \\ &\quad + \int_{A_k} f^\infty((\varphi_0 - k))\mathbf{n})d\mathcal{H}^{n-1} + \int_{B_k} f^\infty((\varphi_0 + k)\mathbf{n})d\mathcal{H}^{n-1} = \\ &= \int_{\partial\Omega\setminus(A_k\cup B_k)} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} + \\ &\quad + \int_{A_k} \frac{\varphi_0 - k}{\varphi_0 - \gamma_\Omega(v)} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} + \\ &\quad + \int_{B_k} \frac{\varphi_0 + k}{\varphi_0 - \gamma_\Omega(v)} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} \le \\ &\le \int_{\partial\Omega} f^\infty((\varphi_0 - \gamma_\Omega(v))\mathbf{n})d\mathcal{H}^{n-1} < +\infty. \end{aligned}$$

By such properties we conclude that $F(\Omega,\varphi_0,v_k) < +\infty$, i.e. that (1.9) with $p = +\infty$ holds, and the thesis follows by Theorem 4.7 applied with $p = +\infty$. □

Proposition 4.15. *Let* $n > 1$*,* $f\colon \mathbb{R}^n \to [0,+\infty[$ *be convex and verifying* (1.5)*,* $p \in \left[1, \frac{n}{n-1}\right]$*, and assume that* (1.8) *holds. Let* Ω *be a bounded open set with Lipschitz boundary,* $\varphi_0 \in L^1(\partial\Omega)$*,* $F(\Omega,\varphi_0,\cdot)$ *and* $\overline{F}(\Omega,\varphi_0,\cdot)$ *be defined by* (1.2) *and* (4.1)*, then*

$$\overline{F}(\Omega,\varphi_0,u) = F(\Omega,\varphi_0,u) \quad \textit{for every } u \in BV(\Omega).$$

Proof. As at the beginning of Proposition 4.7 it is not restrictive to assume that $F(\Omega, \varphi_0, \cdot)$ is not identically $+\infty$.

By virtue of this, and by the embedding of $BV(\Omega)$ in $L^p(\Omega)$ we get that actually (1.9) holds, and the thesis follows by Theorem 4.7. □

Proposition 4.16. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex, and assume that* (1.8) *holds with $p = 1$. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, $F(\Omega, \varphi_0, \cdot)$ and $\overline{F}(\Omega, \varphi_0, \cdot)$ be defined by* (1.2) *and* (4.1)*, then*

$$\overline{F}(\Omega, \varphi_0, u) = F(\Omega, \varphi_0, u) \quad \textit{for every } u \in BV(\Omega)$$

and $F(\Omega, \varphi_0, \cdot)$ is not identically $+\infty$.

Proof. We observe that by using (1.8) with $p = 1$ it follows that (1.5) holds, moreover (1.9) is fulfilled and the thesis follows by Theorem 4.7. □

In the following theorems we apply the above results to the study of Lavrentieff phenomenon.

Proposition 4.17. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex and verifying* (1.5)*, $p \in [1,+\infty]$, and assume that, if $p < +\infty$,* (1.8) *holds. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, $F(\Omega, \varphi_0, \cdot)$ be defined by* (1.2)*, and let us assume that there exists $v \in BV(\Omega) \cap L^p_{loc}(\Omega)$ such that*

$$\gamma_\Omega(v) = \varphi_0, \qquad \int_\Omega f(\nabla v)dx + \int_\Omega f^\infty(\nabla^s v)d|D^s v| < +\infty,$$

then

$$\inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} = \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\} < +\infty.$$

Proof. Follows by Proposition 4.12. □

Proposition 4.18. *Let $n = 1$, $f\colon \mathbb{R} \to [0,+\infty[$ be convex and verifying* (1.5)*. Let a, b, α, $\beta \in \mathbb{R}$ with $a < b$, $\varphi_0\colon x \in \{a, b\} \mapsto \begin{cases} \alpha & \textit{if } x = a \\ \beta & \textit{if } x = b \end{cases}$, and $F(]a, b[, \varphi_0, \cdot)$ be defined by* (1.2)*, then*

$$\begin{aligned} &\inf\{F(]a, b[, \varphi_0, u) : u \in BV(]a, b[) \cap C^\infty(]a, b[)\} = \\ &\qquad = \inf\{F(]a, b[, \varphi_0, u) : u \in BV(]a, b[)\} < +\infty. \end{aligned}$$

Proof. Follows by Proposition 4.13. □

Theorem 4.19. *Let $f\colon \mathbb{R}^n \to [0,+\infty[$ be convex and verifying* (1.5)*. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^\infty(\partial\Omega)$, and $F(\Omega, \varphi_0, \cdot)$ be defined by* (1.2)*, then*

$$\inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} = \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\}.$$

Proof. Follows by Theorem 4.14. □

Theorem 4.20. *Let $n > 1$, $f\colon \mathbb{R}^n \to [0, +\infty[$ be convex and verifying* (1.5)*, $p \in \left[1, \frac{n}{n-1}\right]$, and assume that* (1.8) *holds. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and $F(\Omega, \varphi_0, \cdot)$ be defined by* (1.2)*, then*

$$\inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} = \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\}.$$

Proof. Follows by Theorem 4.8. □

Proposition 4.21. *Let $f\colon \mathbb{R}^n \to [0, +\infty[$ be convex, and assume that* (1.8) *holds with $p = 1$. Let Ω be a bounded open set with Lipschitz boundary, $\varphi_0 \in L^1(\partial\Omega)$, and $F(\Omega, \varphi_0, \cdot)$ be defined by* (1.2)*, then*

$$\inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega) \cap C^\infty(\Omega)\} = \inf\{F(\Omega, \varphi_0, u) : u \in BV(\Omega)\} < +\infty.$$

Proof. Follows by Proposition 4.16. □

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