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Γ-Convergence and Chattering Limits in Optimal Control Theory

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A formula relating two different variational limits of sequences of optimal control problems governed by ordinary differential equations is found out. It is employed to obtain a new representation theorem for the Γ-limit of a sequence of optimal control problems.

1. Introduction

In this paper we deal with sequences of optimal control problems of the form

$$\min_{(u,y)} \left\{ \int_0^1 f_h\big(t, y(t), u(t)\big)\, dt \ : \ y'(t) = g_h\big(t, y(t), u(t)\big),\ y(0) = y_0 \right\} \qquad (P_h)$$

where the state variable y varies in the Sobolev space $Y = W^{1,1}(0,1;\mathbb{R}^n)$, the control variable u varies in a space U of measurable functions, and the functions f_h, g_h satisfy suitable conditions. The variational convergence problem for such a sequence consists in finding a limit problem (P_∞) such that if (u_h, y_h) is an optimal pair of (P_h), up to extracting subsequences, we have $(u_h, y_h) \to (u_\infty, y_\infty)$ in a suitable sense, where (u_∞, y_∞) is an optimal pair of problem (P_∞). This has been done by many authors using two major approches. The first one, which uses the theory of Γ-convergence and its tools, has been developed in Buttazzo and Dal Maso [11], Buttazzo and Cavazzuti [10], Buttazzo and Freddi [12], [13], Denkowski and Migorski [16], Freddi [20], Migorski [24], [25]; the second one, which is based on the notion of chattering parameter functions and on the classical one of relaxed controls as parametrized measures in the sense of L. C. Young, has been studied by Artstein [1], [2], [3]. In Section 3 we briefly describe this method and refer for details and generalizations to [3]. In Section 4 we recall the definition and the main features of Γ-convergence, referring, for a more complete treatment, to the books of Attouch [4], Buttazzo [9] and Dal Maso [15]. In Section 7 we investigate on the relationship between the two kind of variational limits. Although in [10], [11], [12] and [13] the space of controls is an L^p space and even in the papers of Artstein U is the class of all measurable functions, we are able to compare the two approaches only when U is the space L^∞ of bounded functions. In this treatment, which uses classical Young's measures, such a restriction seems technically unavoidable. However, it could be considered as a first step towards a more complete study which should involve other classes of parametrized measures.

When the sequence of optimal control problems is independent of h, we are in the framework of relaxation, and the two methods provide two different relaxed formulations. Their

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relationship has been studied by Ioffe and Tihomirov [21], Berliocchi and Lasry [6] and by Mascolo and Migliaccio [23]. In spite of the apparent difference in formulation, they proved that the two relaxed problems are closely connected. Starting from some ideas about the relaxation case of [22] and [23] we prove a similar result in the more general setting of the variational convergence. More precisely, under uniform boundedness conditions on the set of controls we obtain, in Section 6, a formula which relates the two variational limits each other (Theorem 6.2) and in Section 7 we give a representation of the variational limit in the form of a new control problem with a formulation where parametrized measures do not appear. This last result can be sketched as follows. As Artstein shows in [3] (Section 11) it is not restrictive to assume that the control problems (P_h) can be written in a parametric form which, if the controls are uniformly bounded, that is $U = L^\infty(0,1;K)$ where K is a compact subset of $\mathbb{R}^m$, is given by

$$\min\Big\{\int_0^1 f\big(t,y(t),u(t),\rho_h(t)\big)\,dt \ :\ y'(t)=g\big(t,y(t),u(t),\rho_h(t)\big),\ y(0)=y_0,\ u(t)\in K\Big\}$$

where Γ is a compact metric space, and $\rho_h : [0,1] \to \Gamma$ are given parameter functions. Functions f and g are measurable in t and continuous with respect to the other variables and satisfy other suitable but quite reasonable assumptions (namely (a)...(e) of Section 3). By $\mu(t)$ we denote the weak limit (in the sense of parametrized measures) of the sequence $\delta_{\rho_h(t)}$. We shall prove that when U and Y are endowed respectively with the weak* topology and the strong one of L^∞, then the variational limit (P_∞) can be written in the form

$$\min\Big\{\int_0^1 \check f\big(t,y(t),u(t),y'(t)\big)\,dt \ :\ y(0)=y_0\Big\}$$

where the integrand $\check f$ is obtained in the following way: setting for every $(t,y,\lambda,\gamma,v)\in[0,1]\times\mathbb{R}^n\times\mathbb{R}^m\times\Gamma\times\mathbb{R}^n$

$$\varphi(t,y,\lambda,\gamma,v)) = f(t,y,\lambda,\gamma) + \chi_{\{v=g(t,y,u,\gamma),\ \lambda\in K\}}$$

(χ_E denotes here the indicator function of the set E, that is the function which takes the value 0 on E and $+\infty$ elsewhere) and denoting by

$$\varphi^*(t,y,\lambda^*,\gamma,v^*) = \sup\{\lambda^*\lambda + v^*v - \varphi(t,y,\lambda,\gamma,v)\ :\ \lambda\in\mathbb{R}^m,\ v\in\mathbb{R}^n\}$$

the Fenchel duality transform of the function φ with respect to the variables λ and v, the others being frozen, then

$$\check f(t,y,\lambda,v) = \sup\{\lambda^*\lambda + v^*v - \int_\Gamma \varphi^*(t,y,\lambda^*,\gamma,v^*)\,\mu(t)(d\gamma)\ :\ \lambda^*\in\mathbb{R}^m,\ v^*\in\mathbb{R}^n\}. \tag{1}$$

In other words:

$$\check f(t,y,\lambda,v) = \left(\int_\Gamma \Big(f(t,y,\lambda^*,\gamma) + \chi_{\left\{\begin{array}{l} v^* = g(t,y,\lambda^*,\gamma)\\ \lambda^*\in K\end{array}\right.}\Big)^*\mu(t)(d\gamma)\right)^*(\lambda,v) \tag{2}$$

that is, we have to take the conjugate of the function φ with respect to the control pair (λ,v), integrate with respect to $\mu(t)$ in the parameter variable γ and, finally, take the conjugate again. We shall see in Section 8 that our result extends those obtained by Buttazzo and Cavazzuti in [10] when the right hand side of the state equation had the particular form

$$g_h(t, y, u) = a_h(t, y) + B_h(t, y)b_h(t, u).$$

When applied to the case of relaxation, these results reduce to those obtained by Mascolo and Migliaccio in [22] and [23].

One of the most important features of [3] is that it deals with fully nonlinear constraints in the sense that the state and control variables are not separable in the state equation. Thanks to the work of Artstein, and to the representation formula above, we are now able to write explicitely the Γ-limit of a sequence of problems with fully nonlinear state equations depending on highly oscillating parameters. This fact is enlightened by a concluding example which, due to this nonlinearity, cannot be developed into the framework of the previous papers [10], [11], [12], [13], [20].

2. Notation and preliminary notions

This section is devoted to recall some general notions and to explain notation.

Let $\mathcal{L}(0, 1)$ be the σ-algebra of Lebesgue measurable subsets of the interval $[0, 1]$, while $\mathcal{B}(S)$ denotes the Borel σ-algebra of a given complete separable metric space S and $\mathcal{P}(S)$ the set of probability measures on S which will be always endowed with the topology of weak convergence of measures.

Given a measurable space $(\Omega, \mathcal{F})$, and a map

$$\begin{aligned} \alpha : \quad & \Omega \to \mathcal{P}(S) \\ & \omega \mapsto \alpha(\omega) \end{aligned}$$

then the following propositions are equivalent (see for instance Valadier [27], Lemma A2 and Neveu [26], Proposition III-2-1)

1. for every open set $A \subseteq S$ the function $\omega \mapsto \alpha(\omega)(A)$ is $\mathcal{F}$-measurable;
2. for every Borel set $B \subseteq S$ the function $\omega \mapsto \alpha(\omega)(A)$ is $\mathcal{F}$-measurable;
3. for every bounded continuous function $\varphi : S \to \mathbb{R}$ the function $\omega \mapsto \int_S \varphi(s)\, \alpha(\omega)(ds)$ is $\mathcal{F}$-measurable;
4. for every positive $\mathcal{B}(S)$-measurable function ψ defined on S the function $\omega \mapsto \int_S \psi(s)$ $\alpha(\omega)(ds)$ is $\mathcal{F}$-measurable.

$\alpha(\omega)(ds)$ denotes integration in the variable s with respect to the measure $\alpha(\omega)$. When one of the above mentioned properties holds then the map α is said to be measurable and it is called a parametrized measure.

The following spaces of functions will be used throughout the paper:

$L^1(0, 1)$, the space of Lebesgue integrable real-valued functions defined on $[0, 1]$;

$L^1(0, 1; \mathbb{R}^m)$, $m \in \mathbb{N}$, the space of Lebesgue measurable functions $f : [0, 1] \to \mathbb{R}^m$ such that $|f| \in L^1(0, 1)$;

$W^{1,1}(0, 1; \mathbb{R}^n)$, $n \in \mathbb{N}$, the space of absolutely continuous functions $f : [0, 1] \to \mathbb{R}^n$;

$C^b(S)$, the space of bounded, continuous, real-valued functions defined on S;

$C_0(S)$, the space of continuous real functions f defined on S which vanish at infinity, in the sense that for any $\varepsilon > 0$ there exists a compact subset K of S such that $|f(x)| < \varepsilon$ for every $x \in S \setminus K$;

$\mathcal{M}^b(S)$, the space of all real bounded Borel measures on the space S.

Moreover, if $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ are measurable spaces then we denote by $\mathfrak{M}(\Omega_1; \Omega_2)$ the space of all $(\mathcal{F}_1, \mathcal{F}_2)$ measurable functions from Ω_1 to Ω_2. Namely, if K is a subset of $\mathbb{R}^m$, $\mathfrak{M}([0,1]; K)$ is the space of all $\big(\mathcal{L}(0,1), \mathcal{B}(\mathbb{R}^m)\big)$-measurable functions from $[0,1]$ to K, while $L^\infty(0,1;K)$ is the space of all $\big(\mathcal{L}(0,1), \mathcal{B}(\mathbb{R}^m)\big)$-measurable functions form $[0,1]$ to $\mathbb{R}^n$ such that $f(x) \in K$ for almost every $x \in [0,1]$.

Given a set E, let's denote by χ_E the indicator function of the set E, which takes the value 0 on E and $+\infty$ elsewhere.

Remark 2.1. If the space S is also locally compact then the properties 1., 2., 3. and 4. above are equivalent to measurability with respect to $\mathcal{B}(\Omega)$ and the Borel σ-algebra of the $\sigma\big(M^b(S), C_0(S)\big)$ topology on $\mathcal{P}(S)$ (see Valadier [27], Remark 1, page 157). In this sense the mapping α belongs to the space $\mathfrak{M}(\Omega, \mathcal{P}(S))$.

3. Chattering variational limits

In this section we briefly describe a tool developed by Artstein in [1], [2], [3]. Starting from his idea of chattering parameter function which issue from, and interact with, the classical one of relaxed control, he explained how to study the variational convergence of optimal control problems with ordinary state equations. We give here only a quick and simple description for the purposes of the present paper only, and refer to [3] for other results, details and generalizations.

Given a complete separable metric space Γ, let ρ be a $\mathcal{B}(\Gamma)$-measurable map

$$\rho : [0,1] \to \Gamma$$

which will be called a *parameter function*. Let moreover

$$f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \Gamma \to \mathbb{R},$$
$$g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \Gamma \to \mathbb{R}^n$$

be two $\mathcal{L}(0,1) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\Gamma)$-measurable functions which are continuous in (y, u, γ) and satisfying the following assumptions:

(a) for every compact subset K of $\mathbb{R}^m$ the function

$$t \mapsto \sup_{(u,\gamma) \in K \times \Gamma} |f(t, 0, u, \gamma)|$$

belongs to $L^1(0,1)$;

(b) there exist two constants $\alpha > 0$ and $\beta \geq 0$ such that $f(t, y, u, \gamma) \geq \alpha|u| - \beta$ for every $y \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\gamma \in \Gamma$ and almost every $t \in [0,1]$;

(c) there exists a continuous function $\omega : [0, +\infty[\to [0, +\infty[$, $\omega(0) = 0$, such that

$$|f(t, x, u, \gamma) - f(t, y, u, \gamma)| \leq \omega(|x - y|)(1 + |u|)$$

for every $x, y \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\gamma \in \Gamma$ and almost every $t \in [0,1]$;

(d) there exist $C \in L^1(0,1)$ and $D > 0$ such that

$$|g(t,0,u,\gamma)| \le C(t) + D|u|,$$

uniformly in $\gamma \in \Gamma$, for every $u \in \mathbb{R}^m$ and almost every $t \in [0,1]$;

(e) there exists $k \in L^1(0,1)$ such that

$$|g(t,y,u,\gamma) - g(t,x,u,\gamma)| \le k(t)|y-x|$$

for every $x, y \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\gamma \in \Gamma$ and almost every $t \in [0,1]$.

Remark 3.1. As the aim of the paper is the comparison between chattering variational limits and Γ-limits, our assumptions (a)–(e) deviate slightly from hypotheses 2A–2D of [3]. In fact it is easy to see that the former are slightly milder and have been introduced because they fit better to the framework of Γ-convergence. Moreover the fundamental results of Artstein that are used in the sequel, namely Theorems 2.3 and 2.5 below, can be easily proved to hold true when 2A–2D are replaced by (a)–(e). In exchange for this slight extension, in the sequel we will be constrained to restrict to bounded controls, even if Artstein deals with controls which are only measurable functions.

We are interested by the study of the asymptotic behaviour (as $h \to \infty$) of the optimal pair of control problems

$$\min\Big\{\int_0^1 f\big(t,y(t),u(t),\rho_h(t)\big)\,dt \ :\ y'(t) = g\big(t,y(t),u(t),\rho_h(t)\big),\ y(0)=y_0\Big\} \qquad (P_h)$$

where $u \in L^1(0,1;\mathbb{R}^m)$ and $y \in W^{1,1}(0,1;\mathbb{R}^n)$ and $\rho_h : [0,1] \to \Gamma$ is a sequence of parameter functions.

Denote by $\mathcal{P}(\Gamma)$ the probability measures on Γ with the topology of the weak convergence of measures and by $\mathcal{M}$ the space of functions

$$\mu : [0,1] \to \mathcal{P}(\Gamma)$$

which are measurable in the sense stated in Section 2. The elements of $\mathcal{M}$ are called *chattering parameters*. We say that a sequence of chattering parameters (μ_h) *converges weakly* to a chattering parameter μ if

$$\int_0^1 \Big[\int_\Gamma \psi(t,\gamma)\,\mu_h(t)(d\gamma)\Big]dt \to \int_0^1 \Big[\int_\Gamma \psi(t,\gamma)\,\mu(t)(d\gamma)\Big]dt \tag{3}$$

as $h \to \infty$ for all $\psi \in C^b([0,1]\times\Gamma)$. The space of classical parameter functions can be viewed as a subspace of $\mathcal{M}$ by the identification of ρ with $\delta_{\rho(t)}$ that is the Dirac measure supported at $\{\rho(t)\}$.

Remark 3.2. The space $\mathcal{M}$ might be endowed by another widely used notion of convergence, which is the so-called narrow convergence of Young measures: a sequence of chattering parameters (μ_h) is said to be *narrowly converging* to a chattering parameter μ if (3) holds for every test function $\psi \in L^1(0,1;C_0(\Gamma)) = \{\psi : [0,1]\times\Gamma \to \mathbb{R} \ :\ \psi$ is measurable, $\psi(t,\cdot) \in C_0(\Gamma)$ for a.e. $t \in [0,1]$, $\int_0^1 \sup|\psi(t,\cdot)|\,dt < +\infty\}$. It is known that if Γ is compact then the narrow convergence coincides with the weak convergence stated above (see Valadier [27]).

Analogously, denote by $\mathcal{U}$ the space of measurable mappings

$$\nu : [0,1] \times \Gamma \to \mathcal{P}(\mathbb{R}^m)$$

where $\mathcal{P}(\mathbb{R}^m)$ are the probability measures over $\mathbb{R}^m$ and once again measurability is intended in the sense stated in Section 2. They are called *chattering relaxed controls* (shortly *relaxed controls*). Classical control functions $u(t)$ are embedded into $\mathcal{U}$ by the identification with $\delta_{u(t)}$.

We are now in a position to introduce the fundamental objects of this section, that are the *chattering relaxed control problems*

$$\min \Big\{ \int_0^1 \big[\int_\Gamma \big[\int_{\mathbb{R}^m} f\big(t, y(t), \lambda, \gamma\big)\, \nu(t,\gamma)(d\lambda) \big] \mu(t)(d\gamma) \big]\, dt \ :$$

$$y'(t) = \int_\Gamma \big[\int_{\mathbb{R}^m} g\big(t, y(t), \lambda, \gamma\big)\, \nu(t,\gamma)(d\lambda) \big] \mu(t)(d\gamma),\ y(0) = y_0 \Big\} \quad (C_\mu)$$

where the chattering parameter μ is fixed and the minimum is taken over all the relaxed controls ν. Moreover the cost is assumed to be $+\infty$ if either the state equation does not have a unique solution defined on the entire interval $[0,1]$ or the integrand in the cost functional is not integrable. Anyway, here and in the rest of the paper it will be assumed always that the involved parametrized measures are of first order, that is

$$t \mapsto \int_\Gamma \big[\int_{\mathbb{R}^m} |\lambda|\, \nu(t,\gamma)(d\lambda) \big] \mu(t)(d\gamma) \ \in \ L^1(0,1). \tag{4}$$

This object is well defined due to the measurability assumption on the parametrized measures μ and ν. Indeed, from 4. of Section 2, by taking $(\Omega, \mathcal{F}) = \big([0,1]\times\Gamma, \mathcal{L}(0,1) \otimes \mathcal{B}(\Gamma)\big)$, $S = \mathbb{R}^m$ and $\mathcal{P}(S) = \mathcal{P}(\mathbb{R}^m)$, the function $\varphi : (t,\gamma) \mapsto \int_{\mathbb{R}^m} |\lambda|\, \nu(t,\gamma)(d\lambda)$ turns out to be $\mathcal{L}(0,1) \otimes \mathcal{B}(\Gamma)$-measurable. Afterwards, since the positive function $(t,\gamma) \mapsto \varphi(t,\gamma)$ is $\mathcal{L}(0,1) \otimes \mathcal{B}(\Gamma)$-measurable, the function $\int_\Gamma \varphi(t,\gamma)\mu(t)(d\gamma)$ is $\mathcal{L}(0,1)$-measurable by Fubini's theorem for parametrized measures (see e.g. Valadier [27]). Moreover the assumptions we made on the functions f and g imply that the maps $(t,\gamma) \mapsto \int_{\mathbb{R}^m} f\big(t, y(t), \lambda, \gamma\big)$ $\nu(t,\gamma)(d\lambda)$ and $(t,\gamma) \mapsto \int_{\mathbb{R}^m} g\big(t, y(t), \lambda, \gamma\big)\, \nu(t,\gamma)(d\lambda)$ are $\mathcal{L}(0,1) \otimes \mathcal{B}(\Gamma)$-measurable and the cost functional and the right hand side of the state equation are well defined (see for instance Neveu [26], Proposition III-2-1). Moreover condition (4), together with the assumptions (d) and (e) on the function g, implies global existence and uniqueness of the solution to the state equation.

Following Artstein [3], by a standard construction, we can associate with a chattering parameter μ the unique probability measure D_μ on $\mathcal{L}(0,1) \otimes \mathcal{B}(\Gamma)$ such that

$$D_\mu(E \times G) = \int_E \mu(t)(G)\, dt \tag{5}$$

for any $E \in \mathcal{L}(0,1)$ and $G \in \mathcal{B}(\Gamma)$. To simplify notation, when ρ is a parameter function and $\delta_{\rho(t)}$ is the associated chattering parameter, we shall write D_ρ instead of D_{δ_ρ}. Moreover (see again Neveu [26], Proposition III-2-1) for every $\varphi \in C^b([0,1]\times\Gamma)$ we have

$$\int_{[0,1]\times\Gamma} \varphi\, dD_\mu = \int_0^1 \big[\int_\Gamma \varphi(t,\gamma)\, \mu(t)(d\gamma) \big]\, dt. \tag{6}$$

Remark 3.3. An immediate consequence of (6) is that, given a sequence (μ_h) of chattering parameters, $\mu_h \to \mu$ weakly if and only if $D_{\mu_h} \to D_\mu$ weakly (as probability measures).

In an analogous way, we can associate with a relaxed control ν and a chattering parameter μ the unique probability measure $D_\mu(\nu)$ on $\mathcal{L}(0,1) \otimes \mathcal{B}(\Gamma) \otimes \mathcal{B}(\mathbb{R}^m)$ such that

$$D_\mu(\nu)(E \times G \times B) = \int_E \Big[\int_G \nu(t,\gamma)(B)\,\mu(t)(d\gamma) \Big] dt \tag{7}$$

for any $E \in \mathcal{L}(0,1)$, $G \in \mathcal{B}(\Gamma)$ and $B \in \mathcal{B}(\mathbb{R}^m)$; moreover, for every $\varphi \in C^b([0,1] \times \Gamma \times \mathbb{R}^m)$ we have

$$\int_{[0,1]\times\Gamma\times\mathbb{R}^m} \varphi\, dD_\mu(\nu) = \int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\nu(t,\gamma)(d\lambda) \Big] \mu(t)(d\gamma) \Big]\, dt. \tag{8}$$

We usually write $D_\rho(\nu)$ instead of $D_{\delta_\rho}(\nu)$ and, similarly, if $\nu(t,\gamma) = \delta_{u(t)}$ we write $D_\rho(u)$ instead of $D_\rho(\nu)$.

Definition 3.4. Let $Q : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \Gamma \to \mathbb{R}$ be a measurable function, continuous with respect to the variables $(y,u,\gamma) \in \mathbb{R}^n \times \mathbb{R}^m \times \Gamma$. Let $\mathcal{W} = (\mu_h, \nu_h)$ be a sequence of chattering parameters and relaxed controls. We say that $\mathcal{W}$ is Q-tight if

(i) the sequence $t \mapsto \displaystyle\int_\Gamma \Big[\int_{\mathbb{R}^m} |\lambda|\, \nu_h(t,\gamma)(d\lambda) \Big] \mu_h(t)(d\gamma)$ is weakly relatively compact in $L^1(0,1)$;

(ii) for every $\varepsilon > 0$ there exists a compact subset K_ε of $\mathbb{R}^m$ such that

$$\int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m \setminus K_\varepsilon} |Q(t, y_h(t), \lambda, \gamma)|\, \nu_h(t,\gamma)(d\lambda) \Big] \mu_h(t)(d\gamma) \Big] dt < \varepsilon \quad \forall h \in \mathbb{N}$$

where y_h is the solution (unique) of the state equation

$$y'(t) = \int_\Gamma \Big[\int_{\mathbb{R}^m} g\big(t, y(t), \lambda, \gamma\big)\, \nu_h(t,\gamma)(d\lambda) \Big] \mu_h(t)(d\gamma), \quad y(0) = y_0.$$

Proposition 3.5. *If* (μ_h, ν_h) *satisfies condition* (i) *of the* Definition 3.4 *then the sequence* $\big(D_{\mu_h}(\nu_h)\big)$ *is relatively compact with respect to the weak convergence of probability measures.*

Proof. By the Dunford-Pettis theorem, condition (i) of Definition 3.4 implies uniform integrability, that is for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$A \in \mathcal{L}(0,1),\ |A| < \delta_\varepsilon \quad \Longrightarrow \quad \int_A \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} |\lambda|\, \nu_h(t,\gamma)(d\lambda) \Big] \mu_h(t)(d\gamma) \Big] dt < \varepsilon$$

where the modulus denotes Lebesgue's measure. Then, by taking E_ε a compact subset of $[0,1]$ such that $|[0,1] \setminus E_\varepsilon| < \delta_\varepsilon$ and B_1 the closed unit ball of $\mathbb{R}^m$, we have

$$\int_{[0,1]\setminus E_\varepsilon} \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m \setminus B_1} \nu_h(t,\gamma)(d\lambda) \Big] \mu_h(t)(d\gamma) \Big] dt \le$$

$$\le \int_{[0,1]\setminus E_\varepsilon} \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} |\lambda|\, \nu_h(t,\gamma)(d\lambda) \Big] \mu_h(t)(d\gamma) \Big] dt < \varepsilon$$

and, by (7), given any compact subset G_ε of Γ, the set $K_\varepsilon = E_\varepsilon \times G_\varepsilon \times B_1$ is a compact subset of $[0,1]\times\Gamma\times\mathbb{R}^m$ such that

$$D_{\mu_h}(\nu_h)\big([0,1]\times\Gamma\times\mathbb{R}^m \setminus K_\varepsilon\big) = \int_{[0,1]\setminus E_\varepsilon} \Big[\int_{\Gamma\setminus G_\varepsilon} \Big[\int_{\mathbb{R}^m\setminus B_1} \nu_h(t,\gamma)(d\lambda)\Big]\mu_h(t)(d\gamma)\Big]\,dt < \varepsilon.$$

and the sequence of probability measures $\big(D_{\mu_h}(\nu_h)\big)$ turns out to be tight and eventually (see Billingsley [8], page 37) relatively compact with respect to the weak convergence. □

The following closure result concernes variations in both controls and chattering parameters.

Theorem 3.6 ([3], Theorem 9.3)). *Let $Q : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \Gamma \to \mathbb{R}$ be a measurable function, continuous with respect to the variables $(y,u,\gamma) \in \mathbb{R}^n\times\mathbb{R}^m\times\Gamma$, and satisfying conditions* (a), (b), (c) *above. Let μ_h converge weakly to μ_∞ and let (ν_h) be relaxed controls such that the sequence $(\mu_h,\nu_h)_{h\in\mathbb{N}\cup\{\infty\}}$ is Q-tight. Let y_h be the solution (unique) of the state equation*

$$y'(t) = \int_\Gamma \Big[\int_{\mathbb{R}^m} g\big(t,y(t),\lambda,\gamma\big)\,\nu_h(t,\gamma)(d\lambda)\Big]\mu_h(t)(d\gamma), \quad y(0)=y_0.$$

If $D_{\mu_h}(\nu_h)$ weakly converges to $D_{\mu_\infty}(\nu_\infty)$, then

$$\lim_{h\to\infty}\int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} Q\big(t,y_h(t),\lambda,\gamma\big)\,\nu_h(t,\gamma)(d\lambda)\Big]\mu_h(t)(d\gamma)\Big]\,dt =$$

$$= \int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} Q\big(t,y_\infty(t),\lambda,\gamma\big)\,\nu_\infty(t,\gamma)(d\lambda)\Big]\mu_\infty(t)(d\gamma)\Big]\,dt.$$

Remark 3.7. Theorem 3.6 holds true when in particular $Q = f$.

The next Theorem 3.8 gives the continuous dependence of optimal controls.

Theorem 3.8 ([3], Propositions 10.1 and 10.2). *Let μ_h converge weakly to μ_∞ and let ν_h be an optimal (possibly relaxed) control of (C_{μ_h}). Assume that $(\mu_h,\nu_h)_{h\in\mathbb{N}}$ is f-tight.*

(i) *If ν_∞ is a relaxed control such that $D_{\mu_h}(\nu_h)$ converges weakly to $D_{\mu_\infty}(\nu_\infty)$, then ν_∞ is an optimal solution of (C_{μ_∞}).*

(ii) *There exists an optimal control ν_∞ of (C_{μ_∞}), and a subsequence (ν_{h_j}) such that $D_{\mu_{h_j}}(\nu_{h_j})$ converges weakly to $D_{\mu_\infty}(\nu_\infty)$. If, in particular, (C_{μ_∞}) has a unique solution ν_∞ then the whole sequence $D_{\mu_h}(\nu_h)$ converges weakly to $D_{\mu_\infty}(\nu_\infty)$.*

4. Gamma-convergence

In 1982 Buttazzo and Dal Maso started the study of limits of optimal control problems by using the theory of Γ-convergence. In this section we recall the definition and the main features of sequential Γ-limits and refer for applications to the papers cited in the introduction.

Let us denote by U and Y two topological spaces and let $F_h : U \times Y \to \overline{\mathbb{R}}$ be a sequence of functions. For every $u \in U$ and $y \in Y$ we define

$$\Gamma_{\mathrm{seq}}(U^-, Y^-) \liminf_{h\to\infty} F_h(u,y) = \inf_{u_h \to u} \inf_{y_h \to y} \liminf_{h\to\infty} F_h(u_h, y_h)$$

and

$$\Gamma_{\mathrm{seq}}(U^-, Y^-) \limsup_{h\to\infty} F_h(u,y) = \inf_{u_h \to u} \inf_{y_h \to y} \limsup_{h\to\infty} F_h(u_h, y_h).$$

If the two Γ-limits coincide, then their common value will be indicated by

$$\Gamma_{\mathrm{seq}}(U^-, Y^-) \lim_{h\to\infty} F_h(u,y).$$

Remark 4.1. In the subsequent sections U will be the space $L^\infty(0,1;\mathrm{co}(K))$ of all measurable functions which take values into the convex hull of the compact subset K of $\mathbb{R}^m$, endowed with the weak* topology, while Y will be the space of absolutely continuous functions $W^{1,1}(0,1;\mathbb{R}^n)$ with the metric of uniform convergence. Hence the topology on the product space $U \times Y$ is metrizable and the infima which appear in the definitions of the Γ-limits are in fact minima. Moreover, the sequential definition of Γ-convergence given above coincides with the topological one and Γ-limits do not change if we replace the functions F_h by their lower semicontinuous envelopes (see Dal Maso [15], Propositions 6.11 and 8.1).

The following proposition has been proved in [11].

Proposition 4.2. *Let (u_h, y_h) be a minimum point for F_h, or simply a pair such that*

$$\lim_{h\to\infty} F_h(u_h, y_h) = \lim_{h\to\infty} \big[\inf_{U\times Y} F_h \big].$$

Assume that (u_h, y_h) converges to (u_∞, y_∞) in $U \times Y$ and that there exists

$$F_\infty(u,y) = \Gamma_{\mathrm{seq}}(U^-, Y^-) \lim_{h\to\infty} F_h(u,y) \quad \forall (u,y) \in U\times Y.$$

Then we have

(i) (u_∞, y_∞) *is a minimum point for F_∞ on $U \times Y$;*
(ii) $\lim_{h\to\infty} \big[\inf_{U\times Y} F_h \big] = \min_{U\times Y} F_\infty$.

The Γ-convergence theory can be used to study the behaviour as $h \to \infty$ of optimal pairs of sequences of the optimal control problems (P_h) of Section 3 by setting

$$\begin{aligned} &J_h(u,y) = \int_0^1 f\big(t, y(t), u(t), \rho_h(t)\big)\, dt, \\ &\Lambda_h = \{(u,y) \in U \times Y \ : \ y'(t) = g\big(t, y(t), u(t), \rho_h(t)\big),\ y(0) = y_0, u(t) \in K\}, \\ &F_h(u,y) = J_h(u,y) + \chi_{\Lambda_h}(u,y) \end{aligned} \tag{9}$$

and using Proposition 4.2. Many papers on the subject are devoted to identify the Γ-limit of a sequence and to try to represent it into a (P_h)-like form, that is into the sum of a new cost functional and the indicator of a new set of admissible pairs.

5. Continuous extensions of parametrized measures

In this section we state a theorem which extends to parametrized measures the well-known Tietze-Uryshon extension theorem. It will be applied in the next section to prove the main theorem on the relationship between chattering limits and Γ-limits.

Let Ω be a metric space with its Borel σ-algebra $\mathcal{B}(\Omega)$. A parametrized measure over $\mathbb{R}^m$ is a measurable mapping

$$\begin{aligned} \nu :& \Omega \to \mathcal{P}(\mathbb{R}^m) \\ & \omega \mapsto \nu(\omega) \end{aligned}$$

where $\mathcal{P}(\mathbb{R}^m)$ are probability measures over $\mathbb{R}^m$ and measurability is intended in the sense stated in Section 2. Anyway, as observed in Remark 2.1, this is the case when this measurability concept coincides with measurability with respect to $\mathcal{B}(\Omega)$ and the Borel σ-algebra of the $\sigma\big(\mathcal{M}^b(\mathbb{R}^m), C_0(\mathbb{R}^m)\big)$ topology on $\mathcal{P}(\mathbb{R}^m)$. In the sequel also continuity is taken with respect to the metric of Ω and the $\sigma\big(\mathcal{M}^b(\mathbb{R}^m), C_0(\mathbb{R}^m)\big)$ topology on $\mathcal{P}(\mathbb{R}^m)$. The chattering relaxed controls of Section 3 are in fact parametrized measures over $\mathbb{R}^m$ where $\Omega = [0,1] \times \Gamma$.

Theorem 5.1. *Let $\nu : \Omega \to \mathcal{P}(\mathbb{R}^m)$ be a parametrized measure and H be a closed subset of Ω such that the restriction of ν to H is continuous. Then there exists a parametrized measure $\hat{\nu} : \Omega \to \mathcal{P}(\mathbb{R}^m)$ such that*

(1) *$\hat{\nu}$ is continuous on Ω;*
(2) *$\hat{\nu}(\omega) = \nu(\omega)$ for every $\omega \in H$;*
(3) *if K is a subset of $\mathbb{R}^m$ such that $\operatorname{supp} \nu(\omega) \subseteq K$ for every $\omega \in H$ then $\operatorname{supp} \hat{\nu}(\omega) \subseteq K$ for every $\omega \in \Omega$.*

This theorem is in fact a specialization of a general theorem of Dugundji (see [17], Theorem 4.1), which we recall here for convenience of the reader.

Theorem 5.2. *Let Ω be a metric space, H a closed subset of Ω, L a locally convex linear space, and $f : H \to L$ a continuous map. Then there exists a continuous extension $\hat{f} : \Omega \to L$ of f; furthermore, $\hat{f}(\Omega)$ is contained into the convex hull of $f(H)$.*

To obtain Theorem 5.1 by Theorem 5.2 we can choose $L = \mathcal{M}^b(\mathbb{R}^m)$ endowed with the $\sigma\big(\mathcal{M}^b(\mathbb{R}^m), C_0(\mathbb{R}^m)\big)$ topology. With this topology L is a locally convex linear space. To conclude is then enough to observe that the set of probability measures with support in K is a convex subset of $\mathcal{M}^b(\mathbb{R}^m)$.

6. Relationship between chattering limits and Γ-limits

In order to state a theorem relating Γ-limits of sequences of optimal control problems with their chattering variational limits, let us set

$$\begin{aligned}
&\widetilde{J}(\mu,\nu,y)=\int_0^1\Big[\int_\Gamma\Big[\int_{\mathbb{R}^m} f\big(t,y(t),\lambda,\gamma\big)\,\nu(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\Big]\,dt,\\
&\widetilde{\Lambda}=\Big\{(\mu,\nu,y)\in\mathcal{M}\times\mathcal{U}\times Y\ :\\
&\qquad :\ y'(t)=\int_\Gamma\Big[\int_{\mathbb{R}^m} g\big(t,y(t),\lambda,\gamma\big)\,\nu(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma),\ y(0)=y_0\Big\},\\
&\widetilde{F}(\mu,\nu,y)=\widetilde{J}(\mu,\nu,y)+\chi_{\widetilde{\Lambda}}(\mu,\nu,y).
\end{aligned}\tag{10}$$

Definition 6.1. Let K be a subset of $\mathbb{R}^m$. A relaxed control ν is said to be equi-supported in K if $\operatorname{supp}\nu(t,\gamma)\subseteq K$ for every $\gamma\in\Gamma$ and almost every $t\in[0,1]$.

Let K be a compact subset of $\mathbb{R}^m$ and U be the space $L^\infty(0,1;\operatorname{co}(K))$ endowed with its weak* topology and Y the space $W^{1,1}(0,1;\mathbb{R}^n)$ with the metric of uniform convergence. On the integrand f we assume conditions (a), (b), (c), (d), (e). Our main goal is to prove the following theorem.

Theorem 6.2. *If the sequence of parameter functions (ρ_h) converges to a chattering one, say μ in the space $\mathcal{M}$, that is*

$$\delta_{\rho_h(t)}\to\mu\quad\textit{weakly,}\tag{11}$$

then for every $u\in U$ and $y\in Y$

$$\Gamma_{\mathrm{seq}}(U^-,Y^-)\lim_{h\to\infty}F_h(u,y)=\min\{\widetilde{F}(\mu,\nu,y)\ :\ \nu\in\mathcal{B}(u)\}\tag{12}$$

where

$$\mathcal{B}(u)=\Big\{\nu\in\mathcal{U}\ :\ \ \nu\ \textit{is equi-supported in } K \textit{ and}\ \int_\Gamma\Big[\int_{\mathbb{R}^m}\lambda\,\nu(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)=u(t)\ \textit{a.e. } t\in[0,1]\Big\}.$$

Remark 6.3. If $\Gamma=\mathbb{R}^d$ then hypothesis (11) is satisfied by a subsequence if (ρ_h) is bounded in L^1 (see [3], Corollary 7.3). Moreover, if μ is associated with a measurable function, that is $\mu(t)=\delta_{\rho(t)}$ for a suitable measurable function ρ, then $\rho_h\to\rho$ in measure and hence in the L^1 norm. On the contrary, if (ρ_h) is weakly, but not norm converging in L^1, then μ is not associated to any measurable function (see Valadier [27], Theorems 19 and 20).

Remark 6.4. We remark that when the sequence of parameter functions ρ_h is constant, that is $\rho_h=\rho$ for any $h\in\mathbb{N}$, then for all $h\in\mathbb{N}$ it is $F_h=F$ and the Γ-limit

$$\Gamma_{\mathrm{seq}}(U^-,Y^-)\lim_{h\to\infty}F_h$$

is the lower semicontinuous envelope of F on the space $U\times Y$, and formula (12) reduces to the one discovered by Mascolo and Migliaccio [23].

The rest of the current section is devoted to prove Theorem 6.2. To this aim we need of the following approximation lemma.

Lemma 6.5. *Let ρ_h be a sequence of parameter functions weakly converging in $\mathcal{M}$ to a chattering one μ and let ν be a relaxed control equi-supported in K. Then there exists a sequence of (continuous) relaxed controls ν_h, equi-supported in K such that $D_{\rho_h}(\nu_h)$ converges weakly to $D_\mu(\nu)$.*

Proof. Let's begin by showing that there exists a family of relaxed controls $\{\nu^\varepsilon(t,\gamma)\}_\varepsilon$ continuous at every point $(t,\gamma) \in [0,1]\times\Gamma$ and equi-supported in K such that

$$D_\mu(\nu^\varepsilon) \to D_\mu(\nu) \text{ weakly as } \varepsilon \to 0, \tag{13}$$

that is to say (see (8)) that as ε goes to 0 the function of ε

$$\int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\,\nu^\varepsilon(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\Big]\,dt \tag{14}$$

converges to

$$\int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\,\nu(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\Big]\,dt \tag{15}$$

for every bounded and continuous $\varphi : [0,1]\times\Gamma\times\mathbb{R}^m \to \mathbb{R}$.

To this end, let us consider the measure D_μ on $[0,1]\times\Gamma$ defined in (5) by means of the chattering parameter μ. By applying Lusin's theorem (see Federer [19], Theorem 2.3.5 and Bertsekas and Shreve [7], Proposition 7.20) to the measurable mapping

$$\begin{aligned} \nu :\ [0,1]\times\Gamma &\to \mathcal{P}(\mathbb{R}^m) \\ (t,\gamma) &\mapsto \nu(t,\gamma), \end{aligned}$$

for any $\varepsilon \in (0,1)$ there exists a compact set $H_\varepsilon \subseteq [0,1]\times\Gamma$ such that the restriction $\nu_{|H_\varepsilon}$ is continuous and $D_\mu(H_\varepsilon) \ge 1-\varepsilon$. Moreover we can, of course, assume that $\varepsilon_1 < \varepsilon_2$ imply $H_{\varepsilon_2} \subseteq H_{\varepsilon_1}$. By Theorem 5.1, for every $\varepsilon \in (0,1)$ there exists a relaxed control ν^ε which is continuous on $[0,1]\times\Gamma$ and such that $\nu^\varepsilon(t,\gamma) = \nu(t,\gamma)$ for every $(t,\gamma) \in H_\varepsilon$ and $\operatorname{supp}\nu^\varepsilon(t,\gamma) \subseteq K$ for every $(t,\gamma) \in [0,1]\times\Gamma$. Let

$$H := \bigcup_{\varepsilon\in(0,1)} H_\varepsilon.$$

As $(H_\varepsilon)_\varepsilon$ is a monotone family of measurable sets, then H is measurable and $D_\mu(H) = 1$. If $(t,\gamma) \in H$ then there exists $\varepsilon_0 \in (0,1)$ such that $\nu^\varepsilon(t,\gamma) = \nu(t,\gamma)$ for every $\varepsilon \in (0,\varepsilon_0)$ and this implies that as ε goes to 0 the functions

$$(t,\gamma) \mapsto \int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\,\nu^\varepsilon(t,\gamma)(d\lambda) \tag{16}$$

converge D_μ-almost everywhere to the function

$$(t,\gamma) \mapsto \int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\,\nu(t,\gamma)(d\lambda).$$

As moreover

$$\Big|\int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\,\nu^\varepsilon(t,\gamma)(d\lambda)\Big| \le \max_{\lambda\in K} |\varphi(t,\gamma,\lambda)| =: \psi(t,\gamma),$$

and ψ is a bounded and continuous function, then the desired convergence of (14) to (15) can be obtained by applying the Lebesgue's dominated convergence theorem to the family of functions (16).

As a second step, let us show that as $h \to \infty$

$$D_{\rho_h}(\nu^\varepsilon) \to D_\mu(\nu^\varepsilon) \text{ weakly } \forall \varepsilon \in (0,1), \tag{17}$$

that is, for every bounded and continuous function $\varphi : [0,1]\times\Gamma\times\mathbb{R}^m \to \mathbb{R}$ and for every $\varepsilon \in (0,1)$ the sequence

$$\int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\,\nu^\varepsilon(t,\gamma)(d\lambda)\Big]\delta_{\rho_h(t)}(d\gamma)\Big]\,dt$$

converges, as $h \to \infty$, to

$$\int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\,\nu^\varepsilon(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\Big]\,dt.$$

As $\delta_{\rho_h} \to \mu$ weakly, to prove the convergence above is enough to check that the function defined on $[0,1]\times\Gamma$ by

$$(t,\gamma) \mapsto \int_{\mathbb{R}^m} \varphi(t,\gamma,\lambda)\,\nu^\varepsilon(t,\gamma)(d\lambda)$$

is bounded and continuous. Boundedness is trivial, while to prove continuity is easy. Indeed, as the relaxed controls ν^ε are equi-supported in K then this compact set can replace the integration domain $\mathbb{R}^m$ and, if (t_n,γ_n) is a sequence converging to (t_0,γ_0) in $[0,1]\times\Gamma$ then the sequence of functions defined on K by $\varphi(t_n,\gamma_n,\cdot)$ converges uniformly to $\varphi(t_0,\gamma_0,\cdot)$. Finally, this fact, together with the continuity of ν^ε in (t_0,γ_0) implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} \varphi(t_n,\gamma_n,\lambda)\,\nu^\varepsilon(t_n,\gamma_n)(d\lambda) = \int_{\mathbb{R}^m} \varphi(t_0,\gamma_0,\lambda)\,\nu^\varepsilon(t_0,\gamma_0)(d\lambda)$$

which proves the desired continuity.

By putting together (13) and (17) and using a suitable diagonal argument we eventually get that there exists a sequence (ε_h) such that, as $h \to \infty$

$$D_{\rho_h}(\nu^{\varepsilon_h}) \to D_\mu(\nu) \text{ weakly}$$

and the result is achieved by taking, of course, $\nu_h = \nu^{\varepsilon_h}$.

As the diagonal process mentioned above deviates from the standard one, we spend a couple of words to illustrate it. Let's denote by d the metric of the weak convergence of probability measures. As

$$D_{\rho_h}(\nu^{1/2}) \to D_\mu(\nu^{1/2}) \text{ weakly as } h \to \infty$$

then there exists $n_1 \in \mathbb{N}$ such that

$$d\big(D_{\rho_h}(\nu^{1/2}), D_\mu(\nu^{1/2})\big) \le \frac{1}{2} \text{ for every } h \ge n_1.$$

Let's define then $\varepsilon_{n_1} = 1/2$. In the same way, as

$$D_{\rho_h}(\nu^{1/4}) \to D_\mu(\nu^{1/4}) \text{ weakly as } h \to \infty$$

then there exists $n_2 > n_1 \in \mathbb{N}$ such that

$$d\big(D_{\rho_h}(\nu^{1/4}), D_\mu(\nu^{1/4})\big) \le \frac{1}{4} \quad \text{for every } h \ge n_2$$

and define then $\varepsilon_n = 1/2$ if $n_1 < n < n_2$ and $\varepsilon_{n_2} = 1/4$. By induction we can construct an increasing sequence of integers $(n_k)_k$ such that

$$d\big(D_{\rho_h}(\nu^{1/2^k}), D_\mu(\nu^{1/2^k})\big) \le \frac{1}{2^k} \quad \text{for every } h \ge n_k$$

and define then $\varepsilon_n = 1/2^{k-1}$ if $n_{k-1} < n < n_k$ and $\varepsilon_{n_k} = 1/2^k$. Then we have that for every $k \in \mathbb{N}$ there exists n_k such that, for every $h \ge n_k$

$$d\big(D_{\rho_h}(\nu^{\varepsilon_h}), D_\mu(\nu)\big) \le d\big(D_{\rho_h}(\nu^{\varepsilon_h}), D_\mu(\nu^{\varepsilon_h})\big) + d\big(D_\mu(\nu^{\varepsilon_h}), D_\mu(\nu)\big) \le \frac{1}{2^k} + d\big(D_\mu(\nu^{\varepsilon_h}), D_\mu(\nu)\big)$$

and the conclusion follows from the fact that ε_h goes to 0 when k goes to ∞. □

Proof of Theorem 6.2. First of all we observe that $\nu(t,\gamma) = \delta_{u(t)} \in \mathcal{B}(u)$, hence $\mathcal{B}(u) \ne \emptyset$ $\forall u \in U$. Let us begin by showing that for all $(u,y) \in U \times Y$ and $\nu \in \mathcal{B}(u)$ it is

$$\Gamma_{\text{seq}}(U^-, Y^-) \limsup_{h\to\infty} F_h(u,y) \le \widetilde{F}(\mu,\nu,y).$$

By Remark 4.1 it is equivalent to prove that

$$\Gamma_{\text{seq}}(U^-, Y^-) \limsup_{h\to\infty} \overline{F}_h(u,y) \le \widetilde{F}(\mu,\nu,y)$$

where $\overline{F}_h$ denotes the lower semicontinuous envelope of F_h in the topology of $U \times Y$.

The case where $(\mu,\nu,y) \notin \widetilde{\Lambda}$ is trivial because the right hand side is $+\infty$. Suppose then $(\mu,\nu,y) \in \widetilde{\Lambda}$ and $\nu \in \mathcal{B}(u)$. We have to show that there exists a sequence (u_h, y_h) such that

$$u_h \to u \quad \text{weakly* in } L^\infty(0,1;\operatorname{co}(K)), \tag{18}$$

$$y_h \to y \quad \text{uniformly}, \tag{19}$$

$$\limsup_{h\to\infty} \overline{F}_h(u_h,y_h) \le \widetilde{J}(\mu,\nu,y). \tag{20}$$

Let $(\nu_h)_h$ be a sequence of relaxed controls equi-supported in K and such that

$$D_{\rho_h}(\nu_h) \to D_\mu(\nu) \text{ weakly as } h \to \infty; \tag{21}$$

the existence of such a sequence is ensured by Lemma 6.5. Then define for every $h \in \mathbb{N}$

$$u_h(t) = \int_{\mathbb{R}^m} \lambda\, \nu_h\big(t, \rho_h(t)\big)(d\lambda), \tag{22}$$

and y_h as the solution to the Cauchy problem

$$\begin{cases} y'(t) = \int_{\mathbb{R}^m} g\big(t, y(t), \lambda, \rho_h(t)\big)\, \nu_h\big(t, \rho_h(t)\big)(d\lambda) \\ y(0) = y_0 \end{cases} \tag{23}$$

that is

$$y_h(t) = y_0 + \int_0^t \Big[\int_{\mathbb{R}^m} g\big(\tau, y_h(\tau), \lambda, \rho_h(\tau)\big)\, \nu_h\big(\tau, \rho_h(\tau)\big)(d\lambda)\Big]\, d\tau.$$

As the supports of the measures $\nu_h(t,\gamma)$ are all contained in K, then $u_h \in L^\infty(0,1;K)$ and (18) follows by (21) and the fact that $\nu \in \mathcal{B}(u)$. To prove (19) we note that thanks to the assumptions (d) and (e) and the fact that all the relaxed controls ν_h have support in K, the functions y_h are equi-bounded and, by (e), they are also equi-continuous. Thus by the Ascoli-Arzelà's theorem there exist $\overline{y} \in C[0,1]$ and a subsequence (y_{h_k}) such that

$$y_{h_k} \to \overline{y} \text{ uniformly.}$$

By (e) follows easily that $\overline{y} = y$ and the whole sequence converges.

It remains to prove (20). The value of the functional $\overline{F}_h$ at the point (u_h, y_h) has been characterized by Mascolo and Migliaccio in [23] to be the minimum of the functional

$$\widetilde{F}(\rho_h, \sigma, y_h) = \int_0^1 \Big[\int_{\mathbb{R}^m} f\big(t, y_h(t), \lambda, \rho_h(t)\big)\sigma(t)(d\lambda)\Big]\, dt$$

taken on the measurable mappings $\sigma : [0,1] \to \mathcal{P}(K)$ such that

$$\int_{\mathbb{R}^m} \lambda\, \sigma(t)(d\lambda) = u_h(t) \tag{24}$$

and that

$$y_h'(t) = \int_{\mathbb{R}^m} g\big(t, y_h(t), \lambda, \rho_h(t)\big)\, \sigma(t)(d\lambda),\ y(0) = y_0 \tag{25}$$

Here, as usual, $\mathcal{P}(K)$ is the space of probability measures over K and measurability is taken in the sense stated in Section 2. Since (22) and (23), $\sigma_h(t) = \nu_h(t, \rho_h(t))$ is among the admissible relaxed controls satisfying (24) and (25), and therefore we have

$$\begin{aligned} \overline{F}_h(u_h, y_h) &\le \widetilde{F}(\rho_h, \sigma_h, y_h) \\ &= \int_0^1 \Big[\int_{\mathbb{R}^m} f\big(t, y_h(t), \lambda, \rho_h(t)\big)\, \nu_h(t, \rho_h(t))(d\lambda)\Big]\, dt \\ &= \int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} f\big(t, y_h(t), \lambda, \gamma\big)\, \nu_h(t,\gamma)(d\lambda)\Big]\delta_{\rho_h(t)}(d\gamma)\Big]\, dt \end{aligned} \tag{26}$$

and since $D_{\rho_h}(\nu_h) \to D_\mu(\nu)$ weakly, by Theorem 3.6 (see also Remark 3.7) the right hand side of (26) converges to $\widetilde{J}(\mu, \nu, y)$ and (20) holds (the f-tightness condition required by Theorem 3.6 is trivially satisfied here because ν is equi-supported in K together with each element of the sequence ν_h).

To complete the proof it remains to show that also the inequality

$$\inf\{\widetilde{F}(\mu,\nu,y)\ :\ \nu\in\mathcal{B}(u)\}\le\Gamma_{\rm seq}(U^-,Y^-)\liminf_{h\to\infty}F_h(u,y)$$

holds true. Let $(u,y)\in U\times Y$ be such that

$$\Gamma_{\rm seq}(U^-,Y^-)\liminf_{h\to\infty}F_h(u,y)<+\infty.$$

Then there exists $(u_h,y_h)\in\Lambda_h$ such that, as $h\to\infty$

$$\begin{aligned}&u_h\to u\ \text{ weakly* in } L^\infty(0,1;\mathrm{co}(K))\\&y_h\to y\ \text{ uniformly}\\&\lim_{h\to\infty}J_h(u_h,y_h)=\Gamma_{\rm seq}(U^-,Y^-)\liminf_{h\to\infty}F_h(u,y).\end{aligned}$$

We have to show that there exists $\nu\in\mathcal{B}(u)$ such that $(\mu,\nu,y)\in\widetilde{\Lambda}$ and

$$\widetilde{J}(\mu,\nu,y)\le\lim_{h\to\infty}J_h(u_h,y_h).\tag{27}$$

Since

$$\int_\Gamma\Big[\int_{\mathbb{R}^m}|\lambda|\,\delta_{u_h(t)}(d\lambda)\Big]\delta_{\rho_h(t)}(d\gamma)=|u_h(t)|$$

and because $(u_h)_h$ is equi-bounded then the sequence $\big(\mu_h,\nu_h\big)$, with $\mu_h(t)=\delta_{\rho_h(t)}$ and $\nu_h(t,\gamma)=\delta_{u_h(t)}$, satisfies condition $i)$ of the Definition 3.4 and, by Proposition 3.5, the sequence of measures $\big(D_{\rho_h}(u_h)\big)_h$ associated to $(\delta_{\rho_h},\delta_{u_h})$ is relatively compact with respect to the weak convergence of probability measures on $[0,1]\times\Gamma\times\mathbb{R}^m$. Therefore a subsequence, let us say $D_{\rho_{h_j}}(u_{h_j})$, converges weakly to a probability measure σ on $[0,1]\times\Gamma\times\mathbb{R}^m$. Then, denoted by $\Pi_{[0,1]\times\Gamma}D_{\rho_{h_j}}(u_{h_j})$ and $\Pi_{[0,1]\times\Gamma}\sigma$ the projections of such measures on $[0,1]\times\Gamma$, we have that

$$\Pi_{[0,1]\times\Gamma}D_{\rho_{h_j}}(u_{h_j})\to\Pi_{[0,1]\times\Gamma}\sigma\ \text{ weakly as } j\to\infty$$

and, of course, $\Pi_{[0,1]\times\Gamma}D_{\rho_{h_j}}(u_{h_j})=D_{\rho_{h_j}}$, where $D_{\rho_{h_j}}$ is the measure on $[0,1]\times\Gamma$ defined in (5). But, on the other hand, assumption (11) implies that $D_{\rho_{h_j}}\to D_\mu$ weakly, so that, by uniqueness of the limit we obtain that

$$\Pi_{[0,1]\times\Gamma}\sigma=D_\mu$$

that is

$$\sigma(E\times G\times\mathbb{R}^m)=\int_E\mu(t)(G)\ dt$$

for every pair of measurable sets E and G. By the disintegration theorem (see for instance Valadier [27], Theorem at page 182 and the following Remark 1) the measure σ can be disintegrated with respect to its projection D_μ, that is, there exists a measurable family $\big(\nu(t,\gamma)\big)_{(t,\gamma)\in[0,1]\times\Gamma}$ of probabilities such that

$$\sigma(E\times G\times B) = \int_E \Big[\int_G \Big[\int_B \nu(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\Big]\,dt$$

for every $E\times G\times B \in \mathcal{L}(0,1)\otimes\mathcal{B}(\Gamma)\otimes\mathcal{B}(\mathbb{R}^m)$, turning then out to be equal to the measure $D_\mu(\nu)$ associated to (μ,ν) . Summarizing, we have seen that there exists a suitable relaxed control ν such that

$$D_{\rho_{h_j}}(u_{h_j}) \to D_\mu(\nu) \text{ weakly as } j\to\infty. \tag{28}$$

By well-known properties of the weak convergence of probability measures follows immediately that the supports of $\nu(t,\gamma)$ are contained all in K so that it is trivial to check that the family $(\delta_{\rho_h},\delta_{u_h})_h \cup (\mu,\nu)$ is f-tight and by Theorem 3.6 we get

$$\lim_{j\to\infty} J_{h_j}(u_{h_j},y_{h_j}) = \lim_{j\to\infty} \widetilde{J}(\mu_{h_j},\nu_{h_j},y_{h_j}) = \widetilde{J}(\mu,\nu,y)$$

that is (27) holds. It remains to check that $\nu\in\mathcal{B}(u)$ and that $(\mu,\nu,y)\in\widetilde{\Lambda}$. We already remarked that ν is equi-supported in K. Equality

$$\int_\Gamma \Big[\int_{\mathbb{R}^m} \lambda\,\nu(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma) = u(t) \quad \text{a.e. } t\in[0,1]$$

follows by passing to the limit as $j\to\infty$ each side of

$$\int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} \lambda\varphi(t)\,\delta_{u_{h_j}(t)}(d\lambda)\Big]\delta_{\rho_{h_j}(t)}(d\gamma)\Big]\,dt = \int_0^1 u_{h_j}(t)\varphi(t)\,dt$$

where $\varphi:[0,1]\to\mathbb{R}^m$ is any continuous function. Indeed, by (28) and (8) the left hand side converges to

$$\int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} \lambda\,\varphi(t)\,\nu(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\Big]\,dt$$

and since $u_h\to u$ weakly* in $L^\infty(0,1;\mathrm{co}(K))$ then the right one tends to

$$\int_0^1 u(t)\varphi(t)dt.$$

To conclude we have to show that $(\mu,\nu,y)\in\widetilde{\Lambda}$. We observe that the fact that $(u_h,y_h)\in\Lambda_h$ can be written, using the previous notation, as

$$y_h(t) = y_0 + \int_0^t \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} g\big(\tau,y_h(\tau),\lambda,\gamma\big)\,\delta_{u_h(\tau)}(d\lambda)\Big]\delta_{\rho_h(\tau)}(d\gamma)\Big]\,d\tau.$$

Since $y_h\to y$ uniformly, then by hypothesis (e) we have

$$\lim_{h\to\infty}\int_0^t \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} \Big(g\big(\tau,y(\tau),\lambda,\gamma\big) - g\big(\tau,y_h(\tau),\lambda,\gamma\big)\Big)\delta_{u_h(\tau)}(d\lambda)\Big]\delta_{\rho_h(\tau)}(d\gamma)\Big]\,d\tau = 0$$

and being $D_{\rho_{h_j}}(u_{h_j}) \to D_\mu(\nu)$ weakly we obtain

$$\lim_{j\to\infty} \int_0^t \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} g\big(\tau, y(\tau), \lambda, \gamma\big) \delta_{u_{h_j}(\tau)}(d\lambda)\Big] \delta_{\rho_{h_j}(\tau)}(d\gamma)\Big] d\tau =$$

$$= \int_0^t \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} g\big(\tau, y(\tau), \lambda, \gamma\big)\, \nu(\tau,\gamma)(d\lambda)\Big] \mu(\tau)(d\gamma)\Big] d\tau.$$

Summarizing, we have

$$\begin{aligned}\lim_{j\to\infty} y_{h_j}(t) &= y_0 + \lim_{j\to\infty} \int_0^t \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} g\big(\tau, y_{h_j}(\tau), \lambda, \gamma\big)\, \delta_{u_{h_j}(\tau)}(d\lambda)\Big] \delta_{\rho_{h_j}(\tau)}(d\gamma)\Big] d\tau \\ &= y_0 + \int_0^t \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} g\big(\tau, y(\tau), \lambda, \gamma\big)\, \nu(\tau,\gamma)(d\lambda)\Big] \mu(\tau)\Big](d\gamma) d\tau\end{aligned}$$

and, since $y_h(t) \to y(t)$, we get

$$y(t) = y_0 + \int_0^t \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} g\big(\tau, y(\tau), \lambda, \gamma\big)\, \nu(\tau,\gamma)(d\lambda)\Big] \mu(\tau)(d\gamma)\Big] d\tau$$

that is $(\mu, \nu, y) \in \widetilde{\Lambda}$. □

7. A representation theorem

When the space of parameters is also compact then formula (12), which identifies the Γ-limit using parametrized measures, can be made more explicit. For the case of a constant sequence a representation theorem has been obtained by Mascolo and Migliaccio ([22], Theorem 4.4). We extend that result to the general case of the variational convergence. Let be given a sequence of optimal control problems

$$\min\left\{ \int_0^1 f\big(t, y(t), u(t), \rho_h(t)\big)\, dt \;:\; \begin{array}{l} y'(t) = g\big(t, y(t), u(t), \rho_h(t)\big), \\ y(0) = y_0, \\ u(t) \in K \text{ a.e. } t \in [0,1] \end{array} \right\}$$

where K is a compact subset of $\mathbb{R}^m$, Γ is a compact metric space, $\rho_h : [0,1] \to \Gamma$ is a sequence of parameter functions. We have already observed in Section 4 that the variational convergence problem for such a sequence consists in finding the Γ-limit

$$\Gamma_{\text{seq}}(U^-, Y^-) \lim_{h\to\infty} F_h(u,y) \tag{29}$$

where $U = L^\infty(0,1;\text{co}(K))$ with the weak* topology, $Y = W^{1,1}(0,1;\mathbb{R}^n)$ with the metric of uniform convergence, and

$$F_h(u,y) = \int_0^1 f\big(t, y(t), u(t), \rho_h(t)\big)\, dt + \chi_{\begin{cases} y'(t) = g\big(t, y(t), u(t), \rho_h(t)\big), u(t) \in K \\ y(0) = y_0. \end{cases}}$$

The main result of this section is given by the following theorem which solves the problem of the explicit characterization of the Γ-limit (29).

Theorem 7.1. *Let us assume* (a), (b), (c), (d), (e) *of* Section 3. *If the sequence of parameter functions* (ρ_h) *converges weakly to a chattering parameter* μ, *that is*

$$\delta_{\rho_h(t)} \to \mu \quad \textit{weakly,}$$

and if there exists a finite positive measure $\bar\mu$ *such that* $\mu(t)$ *is absolutely continuous with respect to* $\bar\mu$ *for every* $t \in [0,1]$, *then for every* $u \in U$ *and* $y \in Y$ *we have*

$$\Gamma_{\mathrm{seq}}(U^-, Y^-) \lim_{h\to\infty} F_h(u,y) = \int_0^1 \check f\big(t, y(t), u(t), y'(t)\big)dt + \chi_{y(0) = y_0}$$

where the integrand $\check f$ *is obtained in the following way: setting for every* $(t, y, \lambda, \gamma, v) \in [0,1]\times\mathbb{R}^n\times\mathbb{R}^m\times\Gamma\times\mathbb{R}^n$

$$\varphi(t,y,\lambda,\gamma,v) = f(t,y,\lambda,\gamma) + \chi_{\{v = g(t,y,\lambda,\gamma),\ \lambda\in K\}}$$

and denoting by

$$\varphi^*(t,y,\lambda^*,\gamma,v^*) = \sup\{\lambda^*\lambda + v^*v - \varphi(t,y,\lambda,\gamma,v) \ : \ \lambda\in\mathbb{R}^m,\ v\in\mathbb{R}^n\}$$

the Fenchel duality transform of the function φ *with respect to the variables* λ *and* v, *the others being frozen, then*

$$\check f(t,y,\lambda,v) = \sup\{\lambda^*\lambda + v^*v - \int_\Gamma \varphi^*(t,y,\lambda^*,\gamma,v^*)\,\mu(t)(d\gamma) \ : \ \lambda^*\in\mathbb{R}^m,\ v^*\in\mathbb{R}^n\}. \quad (30)$$

This expression for the integrand $\check f$ is summarized in formula (2). The whole section is devoted to prove the theorem. First of all we recall that if $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$ is a proper function (i.e. such that $\varphi(x) < +\infty$ for at least one point $x \in \mathbb{R}^d$) then the conjugate of φ is the function defined on $\mathbb{R}^d$ by

$$\varphi^*(x^*) = \sup\big\{x^*x - \varphi(x) \ : \ x\in\mathbb{R}^d\big\}.$$

If φ is a proper function defined on a proper subset of $\mathbb{R}^d$ then we denote by φ^* the conjugate of the extension of φ to $\mathbb{R}^d$ by $+\infty$. We recall that φ^* is convex and lower semicontinuous and that $\varphi = \varphi^{**}$ for every proper, convex, lower semicontinuous function φ. We denote moreover by

$$\Delta_d = \{(\alpha_0,\dots,\alpha_d)\in\mathbb{R}^{d+1} :\ \alpha_j \geq 0 \text{ and } \sum_{j=0}^d \alpha_j = 1\}$$

the d-simplex of $\mathbb{R}^d$, and recall that if M is a subset of $\mathbb{R}^d$ then

$$\mathrm{co}(M) = \{\sum_{j=0}^d \alpha_j x_j \ : \ (\alpha_0,\dots,\alpha_d)\in\Delta_d,\ x_0,\dots,x_d\in M\}$$

is the convex hull of M. If φ is the indicator function of M then φ^{**} is the indicator of the closed convex hull, $\overline{\mathrm{co}}(M)$, of M.

Assume, from now on, the hypotheses (a), (b), (c), (d), (e) of Section 3. Let $\mu(t)$ be the weak limit of the chattering parameter functions $\delta_{\rho_h(t)}$. For every $(t,y) \in [0,1] \times \mathbb{R}^n$ we set

$$E(t,y) = \{(\lambda, v, \gamma) \in \mathbb{R}^m \times \mathbb{R}^n \times \Gamma \ : \ v = g(t,y,\lambda,\gamma),\ \lambda \in K\}$$

and consider the functional defined on $\mathbb{R}^m \times \mathbb{R}^n$ by

$$\begin{aligned} q_{t,y}(\lambda, v) \ &= \left(\int_\Gamma \chi^*_{E(t,y)}(\lambda^*, v^*, \gamma)\mu(t)(d\gamma) \right)^* (\lambda, v) \\ &= \sup_{(\lambda^*,v^*)\in\mathbb{R}^m\times\mathbb{R}^n} \Big\{ \lambda^*\lambda + v^*v - \int_\Gamma \max_{\lambda\in K}\{\lambda^*\lambda + v^*g(t,y,\lambda,\gamma)\}\mu(t)(d\gamma) \Big\}. \end{aligned} \tag{31}$$

Let now set $Q(t,y) = \{(\lambda, v) \ : \ q_{t,y}(\lambda, v) = 0\}$. Since $q_{t,y} \geq 0$, then

$$Q(t,y) = \{(\lambda, v) \ : \ q_{t,y}(\lambda, v) \leq 0\}. \tag{32}$$

Set finally

$$\overline{\Lambda} = \big\{(u,y) \in U \times Y \ : \ \big(u(t), y'(t)\big) \in Q(t, y(t)) \text{ a.e. } t \in [0,1],\ y(0) = y_0\big\}. \tag{33}$$

Proposition 7.2. *Let F_h be as in* (9)*. Then the following proposition holds*

$$\Gamma_{\text{seq}}(U^-, Y^-) \lim_{h\to\infty} F_h(u,y) < +\infty \quad \Rightarrow \quad (u,y) \in \overline{\Lambda}.$$

Proof. By formula (12), the Γ-limit is finite if and only if

$$\min\{\widetilde{F}(\mu, \nu, y) \ : \ \nu \in \mathcal{B}(u)\} < +\infty$$

and by definition of $\widetilde{F}$ this implies that there exists $\nu \in \mathcal{B}(u)$ such that $(\mu, \nu, y) \in \widetilde{\Lambda}$. Then for almost every $t \in [0,1]$

$$\begin{aligned} y'(t) \ &= \int_\Gamma \Big[\int_K g\big(t, y(t), \lambda, \gamma\big)\, \nu(t,\gamma)(d\lambda) \Big] \mu(t)(d\gamma) \\ u(t) \ &= \int_\Gamma \Big[\int_K \lambda\, \nu(t,\gamma)(d\lambda) \Big] \mu(t)(d\gamma) \end{aligned}$$

and substituting into the expression (31) of $q_{t,y(t)}\big(u(t), y'(t)\big)$ we obtain for a.e. $t \in [0,1]$

$$\begin{aligned} q_{t,y(t)}\big(u(t), y'(t)\big) = \sup_{\lambda^*,\, v^*} \Big\{ \int_\Gamma \Big[\int_K \Big(\lambda^*\lambda + v^* g\big(t, y(t), \lambda, \gamma\big)\Big)\, \nu(t,\gamma)(d\lambda) + \\ - \max_{\lambda\in K}\{\lambda^*\lambda + v^* g\big(t, y(t), \lambda, \gamma\big)\}\Big]\, \mu(t)(d\gamma) \Big\} \leq 0 \end{aligned}$$

that is $(u,y) \in \overline{\Lambda}$. □

In the sequel we shall omit, for simplicity when not essential, the indication of the variables t and y. In this way the construction of the set Q becomes

$$Q = \{(\lambda, v) \ : \ q(\lambda, v) \leq 0\}$$

where

$$q(\lambda, v) = \left(\int_\Gamma \chi^*_E(\lambda^*, v^*, \gamma)\, \mu(d\gamma) \right)^* (\lambda, v)$$

and

$$E = \{(\lambda, v, \gamma) \in \mathbb{R}^m \times \mathbb{R}^n \times \Gamma \ : \ v = g(\lambda, \gamma),\ \lambda \in K\}.$$

In the same way we set

$$E' = \{(\lambda, \gamma, v, z) \in \mathbb{R}^m \times \Gamma \times \mathbb{R}^n \times \mathbb{R} \ : \ v = g(\lambda, \gamma),\ z = f(\lambda, \gamma),\ \lambda \in K\}$$

$$q'(\lambda, v, z) = \left(\int_\Gamma \chi^*_{E'}(\lambda^*, v^*, z^*, \gamma)\, \mu(d\gamma) \right)^* (\lambda, v, z) \tag{34}$$

$$Q' = \{(\lambda, v, z) \ : \ q'(\lambda, v, z) \leq 0\}.$$

For $(\lambda, v) \in Q$ let $Q'_{\lambda,v} = \{z \in \mathbb{R} \ : \ (\lambda, v, z) \in Q'\}$ that is the (λ, v)-section of the set Q'.

Remark 7.3. $Q'_{\lambda,v} \neq \emptyset$ for every $\lambda, v \in Q$. Indeed, a straightforward computation shows that whenever $\lambda, v \in Q$ then Q' contains at least one among $\left(\lambda, v, \int_\Gamma \min_{\lambda \in K} f(\lambda, \gamma)\, \mu(d\gamma)\right)$ and $\left(\lambda, v, \int_\Gamma \max_{\lambda \in K} f(\lambda, \gamma)\, \mu(d\gamma)\right)$.

Lemma 7.4. *Let $M'(t, y)$ be the set defined by*

$$\begin{aligned} M' = \Big\{ & (\lambda, v, z) \in K \times \mathbb{R}^m \times \mathbb{R} \ : \ \textit{there exists } \lambda(\cdot) \in \mathfrak{M}(\Gamma; K) \textit{ such that} \\ & \lambda = \int_\Gamma \lambda(\gamma)\, \mu(d\gamma),\ v = \int_\Gamma g\big(\lambda(\gamma), \gamma\big)\, \mu(d\gamma),\ z = \int_\Gamma f\big(\lambda(\gamma), \gamma\big)\, \mu(d\gamma) \Big\}. \end{aligned} \tag{35}$$

The following equalities hold

$$Q'(t, y) = \operatorname{co} M'(t, y) = \overline{\operatorname{co}}\, M'(t, y).$$

Proof. First of all, let us prove that for every $\lambda^* \in \mathbb{R}^m$, $v^* \in \mathbb{R}^n$, $z^* \in \mathbb{R}$ and for every fixed $y \in \mathbb{R}^n$ and every $t \in [0, 1]$ the following equality holds

$$\begin{aligned} \sup_{\lambda \in \mathfrak{M}(\Gamma; K)} & \left\{ \int_\Gamma \Big(\lambda^* \lambda(\gamma) + v^* g\big(t, y, \lambda(\gamma), \gamma\big) + z^* f\big(t, y, \lambda(\gamma), \gamma\big) \Big) \mu(t)(d\gamma) \right\} = \\ & = \int_\Gamma \max_{\lambda \in K} \{\lambda^* \lambda + v^* g(t, y, \lambda, \gamma) + z^* f(t, y, \lambda, \gamma)\} \mu(t)(d\gamma). \end{aligned} \tag{36}$$

The inequality $\leq$ is trivial. To prove the opposite, let us consider the marginal function

$$V(\gamma) = \max_{\lambda \in K} \{\lambda^* \lambda + v^* g(t, y, \lambda, \gamma) + z^* f(t, y, \lambda, \gamma)\}$$

and the marginal set-valued map

$$S(\gamma) = \{\lambda \in K \ : \ V(\gamma) = \lambda^* \lambda + v^* g(t, y, \lambda, \gamma) + z^* f(t, y, \lambda, \gamma)\}.$$

By Theorem 6 of Aubin and Cellina [5], Sec. 2, Ch. 1, V is a continuous real function on Γ, and S is upper semicontinuous. By the continuity of f and g with respect to λ, for every γ the sets $S(\gamma)$ are closed and non empty. By Proposition 2 of [5], Sec. 1, Ch. 1, the map S has a closed graph and therefore it admits a measurable selection (see Castaing and Valadier [14], Theorem III.30), that is there exists a measurable function $\overline{\lambda} : \Gamma \to K$ such that

$$V(\gamma) = \lambda^* \overline{\lambda}(\gamma) + v^* g(t, y, \overline{\lambda}(\gamma), \gamma) + z^* f(t, y, \overline{\lambda}(\gamma), \gamma) \quad \text{for every } \gamma \in \Gamma.$$

Then we have

$$\begin{aligned}\int_\Gamma V(\gamma)\mu(t)(d\gamma) &= \int_\Gamma \Big(\lambda^* \overline{\lambda}(\gamma) + v^* g\big(t, y, \overline{\lambda}(\gamma), \gamma\big) + z^* f\big(t, y, \overline{\lambda}(\gamma), \gamma\big)\Big)\mu(t)(d\gamma) \\ &\le \sup_{\lambda \in \mathfrak{M}(\Gamma;K)} \Big\{ \int_\Gamma \Big(\lambda^* \lambda(\gamma) + v^* g\big(t, y, \lambda(\gamma), \gamma\big) + z^* f\big(t, y, \lambda(\gamma), \gamma\big)\Big)\mu(t)(d\gamma)\Big\}\end{aligned}$$

that is the desired inequality. By (36), for every $(\lambda, v, z) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$, we get

$$\begin{aligned}&\chi_{\overline{\mathrm{co}}\, M'(t,y)}(\lambda, v, z) = \\ &= \sup_{\lambda^*, v^*, z^*} \Big\{ \lambda^* \lambda + v^* v + z^* z + \\ &\qquad - \sup_{\lambda \in \mathfrak{M}(\Gamma;K)} \Big\{ \int_\Gamma \Big(\lambda^* \lambda(\gamma) + v^* g\big(t, y, \lambda(\gamma), \gamma\big) + z^* f\big(t, y, \lambda(\gamma), \gamma\big)\Big)\mu(t)(d\gamma)\Big\}\Big\} \\ &= \sup_{\lambda^*, v^*, z^*} \Big\{ \lambda^* \lambda + v^* v + z^* z + \\ &\qquad - \int_\Gamma \max_{\lambda \in K} \big\{ \lambda^* \lambda + v^* g\big(t, y, \lambda, \gamma\big) + z^* f\big(t, y, \lambda, \gamma\big)\big\}\mu(t)(d\gamma)\Big\} = q_{t,y}(\lambda, v).\end{aligned}$$

By the properties of f and g it follows that $M'(t, y)$ is a compact subset of $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$, so that $\overline{\mathrm{co}}\, M'(t, y) = \mathrm{co}\, M'(t, y)$. □

Let's define now

$$\begin{aligned}\check{f}(\lambda, v) &= \Big(\int_\Gamma \Big(f(\lambda^*, \gamma) + \chi_{\{v^* = g(\lambda^*, \gamma),\ \lambda^* \in K\}}\Big)^* \mu(d\gamma)\Big)^*(\lambda, v) = \\ &= \sup_{(\lambda^*, v^*) \in \mathbb{R}^m \times \mathbb{R}^n} \big\{ \lambda^* \lambda + v^* v - \int_\Gamma \max_{\lambda \in K} \{\lambda^* \lambda + v^* g(\lambda, \gamma) - f(\lambda, \gamma)\}\mu(d\gamma)\big\}.\end{aligned} \tag{37}$$

The following lemma, similar to Lemma 7.7 of [22], states the most useful properties of the function $\check{f}$.

Lemma 7.5. *We have*

$$\check{f}(\lambda, v) \le b \quad \textit{for every } (\lambda, v, b) \in Q'; \tag{38}$$

$$\check{f}(\lambda, v) = \inf Q'_{\lambda, v} \quad \textit{for every } (\lambda, v) \in Q; \tag{39}$$

$$\big(\lambda, v, \check{f}(\lambda, v)\big) \in Q' \quad \textit{for every } (\lambda, v) \in Q. \tag{40}$$

Namely, if $v = \int_\Gamma \big[\int_{\mathbb{R}^m} g(\lambda, \gamma)\, \nu(\gamma)(d\lambda)\big]\mu(d\gamma)$ *and* $b = \int_\Gamma \big[\int_{\mathbb{R}^m} f(\lambda, \gamma)\, \nu(\gamma)(d\lambda)\big]\mu(d\gamma)$, *from* (38) *we obtain*

$$\check{f}(\lambda, v) \le \int_\Gamma \big[\int_{\mathbb{R}^m} f(\lambda', \gamma)\, \nu(\gamma)(d\lambda')\big]\mu(d\gamma) \tag{41}$$

for any measurable family $\{\nu(\gamma)\}_{\gamma\in\Gamma}$ *of probability measures over* $\mathbb{R}^m$ *satisfying*

$$\int_\Gamma \Big[\int_{\mathbb{R}^m} \lambda'\,\nu(\gamma)(d\lambda')\Big]\mu(d\gamma) = \lambda.$$

Finally

$$\check f(\lambda, v) = +\infty \quad \text{if and only if} \quad (\lambda, v) \notin Q. \tag{42}$$

Proof. Let $(\lambda, v, b) \in Q'$. This means that

$$0 \ge q'(\lambda, v, b) = \sup_{\lambda^*,v^*,z^*} \Big\{\lambda^*\lambda + v^*v + z^*b - \int_\Gamma \max_{\lambda\in K}\{\lambda^*\lambda + v^*g(\lambda,\gamma) + z^*f(\lambda,\gamma)\}\,\mu(d\gamma)\Big\}$$

which implies $\lambda^*\lambda + v^*v + z^*b \le \int_\Gamma \max_{\lambda\in K}\{\lambda^*\lambda + v^*g(\lambda,\gamma) + z^*f(\lambda,\gamma)\}\,\mu(d\gamma)$ for every $\lambda^* \in \mathbb{R}^m$, $v^* \in \mathbb{R}^n$, $z^* \in \mathbb{R}$, so that, taking $z^* = -1$, we get

$$\lambda^*\lambda + v^*v - b \le \int_\Gamma \max_{\lambda\in K}\{\lambda^*\lambda + v^*g(\lambda,\gamma) - f(\lambda,\gamma)\}\,\mu(d\gamma)$$

for every $\lambda^* \in \mathbb{R}^m$ and $v^* \in \mathbb{R}^n$, and (38) follows from the definition of $\check f$. Let us prove (39). By (38) we have $\check f(\lambda, v) \le \inf Q'_{\lambda,v}$. Assume by contradiction that $\check f(\lambda, v) < \inf Q'_{\lambda,v}$; then $\big(\lambda, v, \check f(\lambda, v)\big) \notin Q'$. By definition of Q' and Q we have $q'\big(\lambda, v, \check f(\lambda, v)\big) > 0$ and $q(\lambda, v) \le 0$ that is

$$\sup_{\lambda^*,v^*,z^*} \{\lambda^*\lambda + v^*v + z^*\check f(\lambda, v) - \int_\Gamma \max_{\lambda\in K}\{\lambda^*\lambda + v^*g(\lambda,\gamma) + z^*f(\lambda,\gamma)\}\,\mu(d\gamma)\} > 0 \tag{43}$$

and

$$\sup_{\lambda^*,v^*} \{\lambda^*\lambda + v^*v - \int_\Gamma \max_{\lambda\in K}\{\lambda^*\lambda + v^*g(\lambda,\gamma)\}\,\mu(d\gamma)\} \le 0. \tag{44}$$

From (43) it follows that there exists $\varepsilon > 0$, $\lambda^* \in \mathbb{R}^m$, $v^* \in \mathbb{R}^n$, $z^* \in \mathbb{R}$ such that

$$\lambda^*\lambda + v^*v + z^*\check f(\lambda, v) \ge \int_\Gamma \max_{\lambda\in K}\{\lambda^*\lambda + v^*g(\lambda, v) + z^*f(\lambda, v)\}\,\mu(d\gamma) + \varepsilon. \tag{45}$$

From (44)) and (45) it follows immediately that $z^* \ne 0$. Let us show that in fact $z^* < 0$. By Remark 7.3, there exists $b \in \mathbb{R}$ such that $(\lambda, v, b) \in Q'$; then

$$\lambda^*\lambda + v^*v + z^*b \le \int_\Gamma \max_{\lambda\in K}\{\lambda^*\lambda + v^*g(\lambda,\gamma) + z^*f(\lambda,\gamma)\}\,\mu(d\gamma)$$

and combining with (45) we get $z^*(\check f(\lambda, v) - b) > 0$ which, by (38), implies $z^* < 0$. Dividing by $|z^*|$ in (45) we have

$$\frac{\lambda^*\lambda}{|z^*|} + \frac{v^*v}{|z^*|} - \check f(\lambda, v) \ge \int_\Gamma \max_{\lambda\in K}\{\frac{\lambda^*\lambda}{|z^*|} + \frac{v^*g(\lambda,\gamma)}{|z^*|} - f(\lambda,\gamma)\}\,\mu(d\gamma) + \frac{\varepsilon}{|z^*|}.$$

Setting $s^* = \lambda^*/|z^*|$, $w^* = v^*/|z^*|$, we have

$$\check f(\lambda, v) + \frac{\varepsilon}{|z^*|} \ge s^*\lambda + w^*v - \int_\Gamma \max_{\lambda\in K}\{s^*\lambda + w^*g(\lambda,\gamma) - f(\lambda,\gamma)\}\,\mu(d\gamma)$$

which contradicts the definition of $\check{f}(\lambda, v)$ and therefore proves (39) and (40). The proof of (41) is straightforward. It remains to prove (42). If $(\lambda, v) \in Q$, by Remark 7.3 there exists $b \in \mathbb{R}$ such that $(\lambda, v, b) \in Q'$, then, by (38), $\check{f}(\lambda, v) \leq b < +\infty$. Conversely, suppose that $(\lambda, v) \notin Q$, that is $q(\lambda, v) > 0$. Then, by the definition of q, there exist $\lambda^* \in \mathbb{R}^m$, $v^* \in \mathbb{R}^n$, such that

$$\lambda^*\lambda + v^*v - \int_\Gamma \max_{\lambda \in K}\{\lambda^*\lambda + v^*g(\lambda, \gamma)\}\, \mu(d\gamma) > 0.$$

Then, by setting $a = \lambda^*\lambda + v^*v$ and $b = \int_\Gamma \max_{\lambda \in K}\{\lambda^*\lambda + v^*g(\lambda, \gamma)\}\mu(d\gamma)$, we have $b < a$. If $\lambda_n^* = n\lambda^*$ and $v_n^* = nv^*$, we have, by definition of $\check{f}$

$$\begin{aligned}
\check{f}(\lambda, v) &\geq s\lambda_n^* + wv_n^* - \int_\Gamma \max_{\lambda \in K}\{\lambda_n^*\lambda + v_n^*g(\lambda, \gamma) - f(\lambda, \gamma)\}\, \mu(d\gamma) \\
&\geq na - \int_\Gamma \max_{\lambda \in K}\{\lambda_n^*\lambda + v_n^*g(\lambda, \gamma)\}\, \mu(d\gamma) + \int_\Gamma \min_{\lambda \in K} f(\lambda, \gamma)\, \mu(d\gamma) \\
&\geq n(a - b) + \int_\Gamma \min_{\lambda \in K} f(\lambda, \gamma)\, \mu(d\gamma).
\end{aligned}$$

and passing to the limit as $n \to +\infty$ we get $\check{f}(\lambda, v) = +\infty$. □

Consider the following functionals defined on $U \times Y$:

$$\check{J}(u, y) = \int_0^1 \check{f}\big(t, y(t), u(t), y'(t)\big)dt, \qquad \check{F}(u, y) = \check{J}(u, y) + \chi_{\overline{\Lambda}}(u, y) \tag{46}$$

where $\overline{\Lambda}$ is defined as in (33).

Theorem 7.6. *Let F_h and $\check{F}$ be as in* (9) *and in* (46)*. Then for each $(u, y) \in U \times Y$*

$$\Gamma_{\text{seq}}(U^-, Y^-) \lim_{h \to \infty} F_h(u, y) = \check{F}(u, y).$$

To prove the theorem we make use of the following measurable selection lemma (see Warga [28], Theorem I.7.10).

Lemma 7.7. *Let $\mathcal{K}$ be a compact metric space, and $\Phi : [0, 1] \times \mathcal{K} \to \mathbb{R}^n$ a measurable function, continuous with respect to the second variable for almost every $t \in [0, 1]$. Let $g : [0, 1] \to \mathbb{R}^n$ be measurable and such that*

$$g(t) \in \Phi(t, \mathcal{K}) = \{y \in \mathbb{R}^n \ :\ y = \Phi(t, \lambda),\ \lambda \in \mathcal{K}\} \ \text{ for a.e. } t \in [0, 1].$$

Then there exists a measurable function $v : [0, 1] \to \mathcal{K}$ such that

$$g(t) = \Phi(t, v(t)) \ \text{ for a.e. } t \in [0, 1].$$

Proof of Theorem 7.6. Let $(u, y) \notin \overline{\Lambda}$; by Theorem 7.2 and by definition of $\check{F}$ we have

$$\Gamma_{\text{seq}}(U^-, Y^-) \lim_{h \to \infty} F_h(u, y) = \check{F}(u, y) = +\infty.$$

Let now $(u, y) \in \overline{\Lambda}$; by (41) we have

$$\check{F}(u, y) \leq \int_0^1 \Big[\int_\Gamma \Big[\int_{\mathbb{R}^m} f\big(t, y, \lambda, \gamma\big)\, \nu(t, \gamma)(d\lambda)\Big] \mu(t)(d\gamma)\Big]\, dt$$

for every $\nu \in \mathcal{B}(u)$, so that inequality

$$\check{F}(u, y) \leq \Gamma_{\text{seq}}(U^-, Y^-) \lim_{h\to\infty} F_h(u, y)$$

follows directly by (12). Let us prove the opposite inequality. By definition of $\overline{\Lambda}$ (see (33)) and by (40), we have

$$\big(u(t), y'(t), \check{f}(t, y(t), u(t), y'(t))\big) \in Q'(t, y(t)) \quad \text{a.e. } t \in [0, 1]. \tag{47}$$

By Lemma 7.4 we have

$$Q'(t, y) = \overline{\text{co}}\, M'(t, y) \quad \text{for every } (t, y) \in [0, 1] \times \mathbb{R}^n \tag{48}$$

where $M'(t, y)$ is defined as in (35). Then, by (47) and (48) we have

$$\big(u(t), y'(t), \check{f}(t, y(t), u(t), y'(t))\big) \in \overline{\text{co}}\, M'(t, y(t)) \quad \text{a.e. } t \in [0, 1]. \tag{49}$$

Consider now the function

$$\Phi(t, v) = \sum_{j=0}^{m+n+1} \alpha_j \Big(\Phi_b(t, \nu_j), \Phi_g(t, \nu_j), \Phi_f(t, \nu_j)\Big)$$

where

$$v = \big(\alpha_0, \cdots, \alpha_{m+n}, \nu_0(\cdot), \cdots, \nu_{m+n}(\cdot)\big) \in \Delta_{m+n} \times \mathfrak{M}(\Gamma; \mathcal{P}(K))^{m+n+1}$$

and

$$\begin{aligned}
\Phi_b(t, \nu_j) &= \int_\Gamma \Big[\int_K \lambda\, \nu_j(\gamma)(d\lambda)\Big]\, \mu(t)(d\gamma), \\
\Phi_g(t, \nu_j) &= \int_\Gamma \Big[\int_K g\big(t, y(t), \lambda, \gamma\big)\, \nu_j(\gamma)(d\lambda)\Big]\, \mu(t)(d\gamma), \\
\Phi_f(t, \nu_j) &= \int_\Gamma \Big[\int_K f\big(t, y(t), \lambda, \gamma\big)\, \nu_j(\gamma)(d\lambda)\Big]\, \mu(t)(d\gamma).
\end{aligned} \tag{50}$$

In view of the application of Lemma 7.7 we have to look at v as an element of a space $\mathcal{K}$ with a compact metric which ensures continuity of Φ with respect to v. To this aim, let us consider the linear mapping

$$\begin{aligned}
D_{\bar{\mu}} : \mathfrak{M}(\Gamma; \mathcal{P}(K)) &\to \mathcal{P}([0, 1] \times \Gamma \times K) \\
\nu \quad &\mapsto D_{\bar{\mu}}(\nu)
\end{aligned}$$

defined in Section 3. Substituting $\mu(t)(d\gamma)$ with $\dfrac{d\mu(t)}{d\bar{\mu}}(\gamma)\bar{\mu}(d\gamma)$ in (50) and using (8), we can observe that $\Phi(t, v)$ depends only on the $D_{\bar{\mu}}$-equivalence class of the controls ν_j. Therefore v can be eventually considered as an element of the set

$$\mathcal{K} = \Delta_{m+n} \times \big(\mathfrak{M}(\Gamma; \mathcal{P}(K))/\ker D_{\bar{\mu}}\big)^{m+n+1}.$$

As Γ is compact then the space $\mathcal{P}([0,1]\times\Gamma\times K)$ with the topology d of weak convergence of probability measures is metric and compact. Moreover, with an argument similar to that used in the proof of (27), one can prove that $D_{\bar\mu}$ has a closed range. Then the inverse image topology on $\mathfrak{M}(\Gamma;\mathcal{P}(K))$ is compact but, as $D_{\bar\mu}$ may be not injective, it may be not metrizable. Nevertheless, on $\mathfrak{M}(\Gamma;\mathcal{P}(K))/\ker D_{\bar\mu}$ the quotient topology is compact and metrizable, a metric d_μ being given by

$$d_\mu([\nu_1],[\nu_2]) := d(D_{\bar\mu}(\nu_1), D_{\bar\mu}(\nu_2)).$$

With this topology on $\mathfrak{M}(\Gamma;\mathcal{P}(K))/\ker D_{\bar\mu}$ and the norm topology of $\mathbb{R}^{m+n+1}$ on Δ_{m+n} the function Φ is continuous with respect to variable v (see Remark 3.2). Hence the selection Lemma 7.7 applies and there exist measurable functions $\alpha_j(t)$ and $\nu_j(t,\cdot)$ with $\big(\alpha_0(t),\dots,\alpha_{n+m+1}(t)\big)\in\Delta_{m+n+1}$ and $\nu_j\in\mathfrak{M}([0,1]\times\Gamma,\mathcal{P}(K))$, $j=0,\dots,m+n+1$ such that for almost every $t\in[0,1]$

$$\begin{aligned}
u(t) &= \sum_{j=0}^{m+n+1}\alpha_j(t)\int_\Gamma\Big[\int_K \lambda\,\nu_j(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\\
y'(t) &= \sum_{j=0}^{m+n+1}\alpha_j(t)\int_\Gamma\Big[\int_K g\big(t,y(t),\lambda,\gamma\big)\,\nu_j(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\\
\check f\big(t,y(t),u(t),y'(t)\big) &= \sum_{j=0}^{m+n+1}\alpha_j(t)\int_\Gamma\Big[\int_K f\big(t,y(t),\lambda,\gamma\big)\,\nu_j(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma).
\end{aligned}$$

Setting

$$\nu(t,\gamma) = \sum_{i=0}^{m+n+1}\alpha_j(t)\nu_j(t,\gamma)$$

we have $\nu\in\mathcal{B}(u)$, $(\mu,\nu,y)\in\widetilde\Lambda$, and

$$\begin{aligned}
\check F(u,y) &= \int_0^1 \check f\big(t,y,u,y'\big)\,dt\\
&= \int_0^1\Big[\int_\Gamma\Big[\int_{\mathbb{R}^m} f\big(t,y,\lambda,\gamma\big)\,\nu(t,\gamma)(d\lambda)\Big]\mu(t)(d\gamma)\Big]dt\\
&\geq \min\big\{\widetilde F(\mu,\nu,y)\ :\ \nu\in\mathcal{B}(u)\big\}.
\end{aligned}$$

□

Proof of Theorem 7.1. Thanks to Theorem 7.6, the proof is reduced to show that

$$\check F(u,y) = \check J(u,y) + \chi_{y(0)=y_0} \quad \text{for each } (u,y)\in U\times Y. \tag{51}$$

The indication of the variables (t,y), which was previously omitted, becomes now essential. We shall consider the set $Q(t,y)$ defined in (32) and the set $Q'(t,y)$ defined in (34) but with the suitable indication of the variables t and y. By applying Lemma 7.5 to these sets we obtain in particular

$$\check f(t,y,\lambda,v) = +\infty \quad\Longleftrightarrow\quad (\lambda,v)\notin Q(t,y)$$

i.e., by definition of the set $\overline{\Lambda}$,

$$\check{f}\big(t, y(t), u(t), y'(t)\big) = +\infty \text{ a.e. } t \in [0,1],\ y(0) = y_0 \quad \Longleftrightarrow \quad (u, y) \notin \overline{\Lambda}.$$

Hence we get (51). □

8. Remarks and examples

In [10] Buttazzo and Cavazzuti studied the variational convergence of sequences of optimal control problems by considering state equations where the state and the control variable was separated, in the sense that the function g was taken to be of the form

$$g(t, y, \lambda, \gamma) = a(t, y, \gamma) + B(t, y, \gamma) \cdot b(t, \lambda, \gamma) \tag{52}$$

where

$$\begin{aligned} a :&\ \ [0,1]\times\mathbb{R}^n\times\Gamma \to \mathbb{R}^n \\ B :&\ \ [0,1]\times\mathbb{R}^n\times\Gamma \to \mathbb{R}^{nk} \\ b :&\ \ [0,1]\times\mathbb{R}^m\times\Gamma \to \mathbb{R}^k \end{aligned}$$

was measurable functions measurable in t and continuous in y, λ and γ satisfying conditions of type (d), (e) (Section 3). Let us recall here a slightly simplified version of some results of [10] which will be compared with those obtained in the previous sections. Let

$$f : [0,1]\times\mathbb{R}^n\times K\times\Gamma \to [0, +\infty[$$

be a function satisfying conditions of type (a), (b), (c) and let g be as in (52) where the functions a, B and b are assumed to be continuous in y, λ and γ. Let

$$\widetilde{f}(t, y, \lambda, z, \gamma) = f(t, y, \lambda, \gamma) + \chi_{z = b(t, \lambda, \gamma)}, \ \lambda \in K. \tag{53}$$

Let ρ_h be a sequence of parameter functions taking values in the compact metric space Γ and such that, as $h \to \infty$

$$\begin{aligned} &\widetilde{f}^*(\cdot, y, \lambda^*, z^*, \rho_h(\cdot)) \to \widetilde{\varphi}(\cdot, y, \lambda^*, z^*) \ \text{ weakly in } L^1(0,1) \\ &a(\cdot, y, \rho_h(\cdot)) \to a(\cdot, y) \ \text{ weakly* in } L^\infty(0,1;\mathbb{R}^n) \\ &B(\cdot, y, \rho_h(\cdot)) \to B(\cdot, y) \ \text{ strongly in } L^1(0,1;\mathbb{R}^n) \end{aligned} \tag{54}$$

for every $y \in \mathbb{R}^n$, $\lambda^* \in \mathbb{R}^m$, $z^* \in \mathbb{R}^k$.

Under such assumptions Buttazzo and Cavazzuti proved that

$$\begin{aligned} &\Gamma_{\text{seq}}(U^-, Y^-) \lim_{h\to\infty} F_h(u, y) = \\ &\quad = \inf\Big\{ \int_0^1 \widetilde{\varphi}^*\big(t, y(t), u(t), z\big)\, dt \ : \ y'(t) = a(t, y) + B(t, y)z,\ y(0) = y_0 \Big\} \\ &\quad = \int_0^1 \overline{f}\big(t, y(t), u(t), y'(t)\big)\, dt + \chi_{y(0) = y_0} \end{aligned}$$

where the function $\overline{f}$ is defined by

$$\overline{f}(t, y, \lambda, v) = \inf_{z\in\mathbb{R}^k} \big\{ \widetilde{\varphi}^*(t, y, \lambda, z) \ : \ v = a(t, y) + B(t, y)z \big\}. \tag{55}$$

Under the additional assumptions that $B(t, y, \gamma) = B(t, y)$, that is B does not depend on γ we shall prove that, when (ρ_h) weakly converge to a limit parametrized measure $\mu(t)$, then

$$\overline{f}\big(t, y, \lambda, v\big) = \check{f}\big(t, y, \lambda, v\big) \text{ for a.e. } t \in [0,1] \text{ and every } y \in \mathbb{R}^n,\ \lambda \in K, v \in \mathbb{R}^n \qquad (56)$$

where $\check{f}$ is defined as in (30). This means that $\overline{f}$ and $\check{f}$ provide the same integral representation of the Γ-limit.

The following lemma will be useful.

Lemma 8.1. *Let $\widetilde{f}$ and $\widetilde{\varphi}$ be the functions introduced in* (53) *and* (54) *and $\rho_h \to \mu$ weakly in $\mathcal{M}$. Then the following equality holds for every $y \in \mathbb{R}^n$ and almost every $t \in [0,1]$*

$$\widetilde{\varphi}(t, y, \lambda^*, z^*) = \int_\Gamma \widetilde{f}^*(t, y, \lambda^*, z^*, \gamma)\, \mu(t)(d\gamma) \quad \forall \lambda^* \in \mathbb{R}^m,\ \forall z^* \in \mathbb{R}^k. \qquad (57)$$

Proof. For every function $\psi \in L^\infty(0,1)$ we have that

$$\int_0^1 \psi(t)\widetilde{f}^*\big(t, y, \lambda^*, z^*, \rho_h(t)\big)\, dt = \int_0^1 \Big[\int_\Gamma \psi(t)\widetilde{f}^*(t, y, \lambda^*, z^*, \gamma)\, \delta_{\rho_h(t)}(d\gamma)\Big] dt. \qquad (58)$$

Thanks to the continuity assumption with respect to y, λ and γ and the compactness of Γ and K, the function $(t, \gamma) \to \psi(t)\widetilde{f}^*(t, y, \lambda^*, z^*, \gamma)$ belongs to the space $L^1(0,1; C_0(\Gamma))$ for every y, λ^* and z^*. Hence, by Remark 3.2, the sequence in (58) converges to

$$\int_0^1 \psi(t)\Big[\int_\Gamma \widetilde{f}^*(t, y, \lambda^*, z^*, \gamma)\mu(t)(d\gamma)\Big] dt$$

that is

$$\widetilde{f}^*(\cdot, y, \lambda^*, z^*, \rho_h(\cdot)) \to \int_\Gamma \widetilde{f}^*(\cdot, y, \lambda^*, z^*, \gamma)\mu(\cdot)(d\gamma) \text{ weakly in } L^1(0,1)$$

and the claim follows by (54) and the uniqueness of the limit. □

The proof of (56) relies now on an algebraic computation. Indeed, following the rules for constructing the function $\check{f}$ given in the statement of Theorem 7.1

$$\varphi(t, y, \lambda, \gamma, v) = f(t, y, \lambda, \gamma) + \chi_{\{v = a(t,y) + B(t,y)b(t,\lambda,y),\ \lambda \in K\}}$$

so that

$$\begin{aligned}\varphi^*(t, y, \lambda^*, \gamma, v^*) &= v^* a(t,y) + \sup_{\lambda \in K} \big\{\lambda^*\lambda + B(t,y)^* v^* b(t,\lambda,y) - f(t,y,\lambda,\gamma)\big\} \\ &= v^* a(t,y) - \widetilde{f}^*\big(t, y, \lambda^*, B(t,y)^* v^*, \gamma\big)\end{aligned}$$

where $B(t,y)^*$ denotes the transposed of the matrix $B(t,y)$. Then, using Lemma 8.1, we get

$$\begin{aligned}\int_\Gamma \varphi^*(t, y, \lambda^*, \gamma, v^*)\mu(t)(d\gamma) &= v^* a(t,y) + \int_\Gamma \widetilde{f}^*\big(t, y, \lambda^*, B(t,y)^* v^*, \gamma\big)\mu(t)(d\gamma) \\ &= v^* a(t,y) - \widetilde{\varphi}\big(t, y, \lambda^*, B(t,y)^* v^*\big)\end{aligned}$$

so that the expression of $\check{f}$ is, in this case, given by

$$\begin{aligned}\check{f}(t,y,\lambda,v) &= \sup_{(\lambda^*,v^*)\in\mathbb{R}^m\times\mathbb{R}^n} \{\lambda^*\lambda + v^*(v-a(t,y)) - \widetilde{\varphi}\big(t,y,\lambda^*,B(t,y)^*v^*\big)\} \\ &= \sup_{v^*\in\mathbb{R}^n} \big\{v^*(v-a(t,y)) + \sup_{\lambda^*\in\mathbb{R}^m} \{\lambda^*\lambda - \widetilde{\varphi}\big(t,y,\lambda^*,B(t,y)^*v^*\big)\}\big\} \\ &= \inf_{z\in\mathbb{R}^k} \big\{\widetilde{\varphi}^*(t,y,\lambda,z) \ : \ B(t,y)z = v-a(t,y)\big\} \\ &= \overline{f}(t,y,\lambda,v)\end{aligned}$$

where the third equality of the chain comes by the application of Theorem I-22 of Castaing and Valadier [14].

One of the most important features of [3] is that it deals with fully nonlinear constraints in the sense that the state and control variables are not separable in the state equation. Thanks to the work of Artstein, and to the representation Theorem 7.1, we are able to write explicitely the Γ-limit of a sequence of problems with fully nonlinear state equations depending on highly oscillating parameters. This fact is enlightened by the following example which, due to this nonlinearity, cannot be developed into the framework of the previous papers [10], [11], [12], [13], [20].

Example 8.2. Let us consider a sequence of optimal control problems of the form

$$\min\Big\{ \int_0^1 [\psi(t,y)u + \varphi(t,y)]\ dt \ : \ (u,y)\in\Lambda_h\Big\}$$

where the functions $\psi,\varphi : [0,1]\times\mathbb{R} \to [0,+\infty)$ satisfy

$$|\psi(t,x)-\psi(t,y)| \le L(t)|x-y| \text{ for a suitable } L\in L^1(0,1), \tag{59}$$

$$|\varphi(t,y)| \le a(t) + b|y|^p \ \text{ for suitable } a\in L^1(0,1),\ b>0 \text{ and } p\ge 1. \tag{60}$$

and the set of the admissible pairs is given by

$$\begin{aligned}\Lambda_h = \{\, &(u,y)\in L^\infty(0,1)\times W^{1,1}(0,1) \ : \\ &: y' = \rho_h(t)\cos(uy),\ y(0)=0, u(t)\in[-\frac{\pi}{8},\frac{\pi}{8}] \text{ a.e.}\}\end{aligned}$$

where ρ_h is the h-th Rademacher function defined on $[0,1]$ by

$$\rho_h(t) = \begin{cases} 1 & \text{if } t\in[\frac{n}{2^h},\frac{n+1}{2^h}) \text{ with } n \text{ even} \\ -1 & \text{otherwise.}\end{cases}$$

With the same notation of the previous section we can assume

$$U = \big\{u\in L^\infty(0,1) \ : \ u(t)\in[-\frac{\pi}{8},\frac{\pi}{8}] \text{ a.e. } t\in[0,1]\big\}$$

that is $K = [-\frac{\pi}{8},\frac{\pi}{8}]$. In order to give the representation of

$$\Gamma_{\text{seq}}(U^-,Y^-)\lim_{h\to\infty} F_h(u,y)$$

where

$$F_h(u,y) = J(u,y) + \chi_{\Lambda_h}(u,y)$$
$$J(u,y) = \int_0^1 \big[\psi(t,y)u + \varphi(t,y)\big]\,dt$$

we note that hypotheses (59) and (60) ensure continuity to the cost functional $J : U \times Y \to [0,+\infty)$ so that

$$\Gamma_{\mathrm{seq}}(U^-,Y^-)\lim_{h\to\infty} F_h(u,y) = J(u,y) + \Gamma_{\mathrm{seq}}(U^-,Y^-)\lim_{h\to\infty}\chi_{\Lambda_h}(u,y).$$

We can now apply Theorem 7.1 by taking $f = 0$, $\Gamma = [-1,1]$, and

$$g(t,y,\lambda,\gamma) = \gamma\cos(\lambda y).$$

The weak limit of the sequence of parameter functions ρ_h turns out to be the measure (independent of t)

$$\mu(t) = \frac{\delta_{-1}+\delta_1}{2}.$$

and straightforward computations lead to

$$\Big(\int_\Gamma \Big(\chi_{\begin{cases} v^* = g(t,y,\lambda^*,\gamma) \\ \lambda^* \in K\end{cases}}\Big)^* \mu(t)(d\gamma)\Big)^*(\lambda,v) = \chi_{\overline{\Lambda}}(\lambda,v)$$

where

$$\overline{\Lambda} = \{(\lambda,v)\in\mathbb{R}^2 \ : \ |v| \le \beta(\lambda,y)\}$$

and the function β is given by

$$\beta(\lambda,y) = \begin{cases} \sin(|\lambda|y)\sin[(\lambda+\frac{\pi}{8})y] & \text{if } \lambda\in[-\frac{\pi}{8},-\frac{\pi}{16}] \\ \sin^2(\frac{\pi}{16}y) & \text{if } \lambda\in[-\frac{\pi}{16},\frac{\pi}{16}] \\ \sin(|\lambda|y)\sin[(\frac{\pi}{8}-\lambda)y] & \text{if } \lambda\in[\frac{\pi}{16},\frac{\pi}{8}] \end{cases}$$

Summarizing, the limit problem turns out to be

$$\min\{J(u,y) \ : \ |y'| \le \beta(u,y),\ y(0)=0,\ u(t)\in[-\frac{\pi}{8},\frac{\pi}{8}] \text{ a.e. } t\in[0,1]\}.$$

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