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Vector Variational Principles; ε-Efficiency and Scalar Stationarity

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The aim of this paper is to present several versions of vector variational principles related to some type of metrically consistent ε- efficiency and to the approximate necessary first order efficiency condition.

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1. Introduction

For a scalar convex function a point which is "almost minimizing" is not necessarily an "almost stationary" one, as we can see considering the convex smooth function (see [11]) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = \exp(x_1^2 - x_2)$ and the sequence $n \mapsto x^n = (n, n^2 + \ln\sqrt{n})$. We have $f(x^n) = 1/\sqrt{n} \to 0 = \inf f$ but $\nabla f(x^n) = (2\sqrt{n}, -1/\sqrt{n}) \not\to (0,0)$.

Relations between stationary and minimizing sequences have been considered in [11, 20] for constrained scalar mathematical programming problems, where it is also shown the following, which is immediate from Ekeland variational principle.

Theorem 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then for any minimizing sequence (x_k), there exists a nearby sequence (y_k) such that*

$$\text{(i)}\ \lim_{k\to+\infty}(x_k - y_k) = 0, \quad \text{(ii)}\ \lim_{k\to+\infty} f(y_k) = \inf_{x\in\mathbb{R}^n} f(x), \quad \text{(iii)}\ \lim_{k\to+\infty}\nabla f(y_k) = 0.$$

The aim of this paper is to present some generalizations of the above result to nonsmooth vector optimization problems in infinite dimension spaces.

For a scalar minimization problem the optimal value $\inf f(\mathbb{R}^n)$ is a singleton (eventually $-\infty$). A different situation occurs in a vector minimization problem where the set of "optimal" values (which is the "infimal" set) may be infinite and sometimes unbounded. There are several way to define an approximate efficient solution. We introduce some types of ε-efficiency (with ε a positive scalar) which are metrically consistent in the sense that each $\varepsilon-$efficient solution is located in an ε-neighborhood of the infimal set. Approaching the infimal set in an asymptotic manner we consider the *asymptotically weakly Pareto optimizing (*called also *asymptotically weakly efficient) sequences.* On the other hand the

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approximate necessary first order weakly-efficiency condition leads us to define the notion of *weakly scalarly stationary (or weakly Kuhn-Tucker) sequence.*

2. Preliminaries

2.1. The infimal set

Let $\mathcal{Y}$ be a topological space endowed with a partial order relation denoted $\leq_C$ (C is a subset of $\mathcal{Y}$ related to the order relation).

Let $T \subset \mathcal{Y}$. Denote

$$\min(T|C) = \{a \in T| \; a \leq_C x, \;\; \forall x \in T\}$$

$$\mathrm{MIN}(T|C) = \{a \in T| \; \forall x \in T, \; x \leq_C a \Longrightarrow x = a\}.$$

Changing $\leq_C$ in $\geq_C$ we obtain the sets $\max(T|C)$ and $\mathrm{MAX}(T|C)$. Also

$$\inf(T|C) = \max(\{x \in \mathcal{Y}| \; x \leq_C y, \; \forall y \in T\}|C),$$

and in the same way $\sup(T|C)$ is the least upper bound of T.

Note that the above subsets may be empty, and $\min(T|C)$, $\inf(T|C)$, $\max(T|C)$, $\sup(T|C)$ contain no more than one element. Remark also that their definition does not use the topological structure of $\mathcal{Y}$.

The minimal set $\mathrm{MIN}(T|C)$ (resp. the maximal set $\mathrm{MAX}(T|C)$) is also called Pareto or efficient set when dealing with a minimization (resp. maximization) problem. The efficient set may be infinite or even unbounded.

The chronic fault of the sets $\inf(T|C)$ and $\sup(T|C)$ is that they may be "far" from T (e.g. if $\mathcal{Y} = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $y \leq_C y' \Longleftrightarrow y' - y \in C$ and T is the unit (closed or open) disk, then $\inf T = \{(-1, -1)\}$ and $\sup T = \{(1, 1)\}$ which are located at the Euclidean distance of $\sqrt{2} - 1 > 0$ from T). That is why we will consider the *infimal* set (see also [18] or [24]) as follows:

$$\mathrm{INF}(T|C) = \{a \in \bar{T}| \; \forall x \in T, \; x \leq_C a \Longrightarrow x = a\},$$

where $\bar{T}$ stands for the closure of the set T (and reversing the order we define the supremal set $\mathrm{SUP}(T|C)$). This set coincides with $\mathrm{MIN}(T|C)$ (resp. $\mathrm{MAX}(T|C)$) if T is closed. Moreover

$$\min(T|C) \subset \mathrm{MIN}(T|C); \;\; \mathrm{MIN}(\bar{T}|C) \subset \mathrm{INF}(T|C); \;\; \mathrm{MIN}(T|C) \subset \mathrm{INF}(T|C)$$

and it is possible to have strictly inclusions. If $\min(T|C)$ is not empty (hence is a singleton), then we have $\min(T|C) = \inf(T|C) = \mathrm{MIN}(T|C)$. Note also that we may have

$$\inf(T|C) \not\subset \mathrm{INF}(T|C).$$

Example 2.1. Let $\mathcal{Y} = \mathbb{R}^2$ with the norm topology, and $C = \mathbb{R}^2_+$, $y \leq_C y' \Longleftrightarrow y' - y \in C$.

(1) For $T = [0, 1[\times]0, 1[$ we have

$$\min(T|C) = \mathrm{MIN}(T|C) = \emptyset, \;\; \mathrm{MIN}(\bar{T}|C) = \inf(T|C) = \{(0, 0)\}$$

and $\mathrm{INF}(T|C) = [0, 1] \times \{0\}$.

(2) For $T = \{(\cos t, \sin t) |\ t \in [0,\ \pi[\}$ we have

$$\min(T|C) = \emptyset,\ \ \mathrm{MIN}(T|C) = \{(1,\ 0)\},$$

$$\min(\bar{T}|C) = \mathrm{MIN}(\bar{T}|C) = \inf(T|C) = \{(-1,\ 0)\},\ \mathrm{INF}(T|C) = \{(1,\ 0),\ (-1, 0)\}.$$

(3) For $T = \{(x_1, x_2) |\ x_1^2 + x_2^2 < 1\}$ we have

$$\min(T|C) = \mathrm{MIN}(T|C) = \emptyset,\ \ \inf(T|C) = \{(-1, -1)\},$$

$$\mathrm{INF}(T|C) = \{(\cos t, \sin t) |\ t \in [\pi,\ \frac{3\pi}{2}]\} = \mathrm{MIN}(\bar{T}|C).$$

2.2. The weakly infimal set in the policy space

Throughout this paper Y denotes a real Banach space (the "policy space") endowed with a closed convex pointed[1] cone C (i.e.,$C = \bar{C}$, $\mathbb{R}_+ C + \mathbb{R}_+ C \subset C$ and $C \cap -C = \{0\}$). This cone defines a (partial) order relation in Y given by : $x \leq_C y \iff y - x \in C$ compatible with vector addition and positive scalar multiplication. Thus Y is a (partially) ordered Banach space. We will assume that the interior $\mathrm{int}\, C$ is not empty. The cone $C_0 = \mathrm{int}\, C \cup \{0\}$ defines another partial order relation $x \leq_{C_0} y \iff y - x \in C_0$, and when $x \leq_{C_0} y$, $x \neq y$ we put $x <_C y$, i.e.

$$x <_C y \iff y - x \in \mathrm{int}\, C.$$

We will also denote by B the closed unit ball in any normed vector space. If $S \subset Y$, and $y \in Y$, we denote $\mathrm{d}(y,\ S) = \inf_{z \in S} \|y - z\|$, with the convention $\mathrm{d}(y,\ \emptyset) = +\infty$.

Let $T \subset Y$. Since we are interested by vector minimization problems, in the sequel we will study only the (weakly) minimal or infimal sets, but obviously reversing the inequality (or replacing C by $-C$) we may obtain analogous properties for the (weakly) maximal or supremal sets.

We will define the *weakly efficient* (or *minimal*) *set* by

$$\begin{aligned}\mathrm{MIN}_w(T|C) = \mathrm{MIN}(T|C_0) &= \{y \in T |\ \ \nexists z \in T :\ z <_C y\}\\ &= \{y \in T |\ -y + T \subset Y \setminus (-\mathrm{int}\, C)\}.\end{aligned}$$

It is easy to see the following.

Remark 2.2. $\mathrm{MIN}_w(T|C)$ contains $\mathrm{MIN}(T|C)$, and if T is closed, then $\mathrm{MIN}_w(T|C)$ is closed.

Also, we define the *weakly infimal* set by

$$\begin{aligned}\mathrm{INF}_w(T|C) = \mathrm{INF}(T|C_0) &= \{y \in \bar{T} |\ \ \nexists z \in T :\ z <_C y\}\\ &= \{y \in \bar{T} |\ -y + T \subset Y \setminus (-\mathrm{int}\, C)\}.\end{aligned}$$

It is obvious that

$$\mathrm{INF}(T|C) \subset \mathrm{INF}_w(T|C).$$

[1]Some results of this paper hold even when C is not pointed, but to simplify the presentation we make this hypothesis.

Proposition 2.3.

$$\mathrm{INF}_w(T|C) = \mathrm{MIN}_w(\bar{T}|C),$$

(hence $\mathrm{INF}_w(T|C)$ *is closed) and*

$$\mathrm{MIN}_w(T|C) \subset \mathrm{INF}_w(T|C) \subset \mathrm{bd}\, T,$$

where $\mathrm{bd}\, T$ *is the boundary of* T*.*

Proof. The inclusion $\mathrm{MIN}_w(\bar{T}|C) \subset \mathrm{INF}_w(T|C)$ being obvious, let us prove the converse. Since $Y \setminus \mathrm{int}\, C$ is closed, for each $y \in \mathrm{INF}_w(T|C)$ we have $y - \bar{T} = \overline{y - T} \subset Y \setminus \mathrm{int}\, C$, hence $y \in \mathrm{MIN}_w(\bar{T}|C)$.

The inclusion $\mathrm{MIN}_w(T|C) \subset \mathrm{INF}_w(T|C)$ is obvious. Let $y \in \mathrm{INF}_w(T|C)$. If $y \notin \mathrm{bd}\, T$ it follows that $y \in \mathrm{int}\, T$. Hence for sufficiently small $\varepsilon > 0$ we have $y + \varepsilon B \subset T$. Take $v \in \mathrm{int}\, C \cap B$. It follows that $y - \varepsilon v <_C y$ with $y - \varepsilon v \in T$ contradicting $y \in \mathrm{INF}_w(T|C)$. □

It is easy to adapt some existence results known for the efficient set (see e.g. [19, Chapter 2, section 3]), in the case of the weakly infimal set. Thus, we can state the following.

Proposition 2.4. *Let* T *be such that, for some* $y \in T$*, the set* $T_y = (y - C) \cap \bar{T}$ *is* C-complete *i.e.,* T_y *contains no decreasing net* $(y_\alpha)_{\alpha \in I}$ *such that* $T_y \subset \bigcup_{\alpha \in I} (T \setminus (y_\alpha - C))$*.*

Then $\mathrm{INF}_w(T|C)$ *is not empty.*

Remark 2.5. If the set T_y is compact then it is C-complete.

Let us consider the set

$$C^+ = \{\lambda \in Y^* |\ \langle \lambda, y \rangle \geq 0 \ \ \forall y \in C\},$$

where Y^* is the topological dual of the space Y and $\langle \cdot, \cdot \rangle$ stands for the duality product.

Theorem 2.6. *Assume* $\mathrm{int}\, C^+ \neq \emptyset$ *and let* $T \subset Y$ *be such that there exists* $\lambda \in \mathrm{int}\, C^+$*, bounded from below on* T*. Then, for every* $y \in T \setminus \mathrm{INF}_w(T|C)$*, there exists* $z \in \mathrm{INF}_w(T|C)$ *such that* $z <_C y$*.*

Proof. We need first the following.

Lemma 2.7. *Suppose* $\mathrm{int}\, C^+$ *not empty and let* $\lambda \in \mathrm{int}\, C^+$*. Then*

$$\inf_{y \in C,\ \|y\|=1} \langle \lambda, y \rangle \geq \mathrm{d}(\lambda,\ Y^* \setminus C^+).$$

Proof. For each positive number $r < \alpha = \mathrm{d}(\lambda,\ Y^* \setminus C^+)$ we have $\lambda + rB \subset C^+$. Let $y \in C,\ \|y\| = 1$. According to the Hahn-Banach theorem, there exists $e \in Y^*$, $\|e\| = 1$, such that $\langle e, y \rangle = 1$. Thus $0 \leq \langle \lambda - re, y \rangle$, hence $r \leq \langle \lambda, y \rangle$. Letting $r \to \alpha_-$ we obtain the result. □

Now, we go back to the proof of the theorem. This can be done using “Phelps’ extemization principle” (see [1, Theorem 2.5.]), but thanks to one of the referees, we can give a direct proof.

Let λ and y as stated in the theorem. There exists $y' \in T$ such that $y' <_C y$. Let us show that the section $T_{y'} = (y' - C) \cap \bar{T}$ is C-complete. Indeed, if (y_α) is a decreasing net in $T_{y'}$, then $(\langle \lambda, y_\alpha \rangle)$ is a decreasing, bounded from below net in $\mathbb{R}$, hence it is fundamental. According to the previous Lemma, for every $e \in C \setminus \{0\}$, we have

$$\langle \lambda, e \rangle \geq \|e\| \cdot \mathrm{d}(\lambda,\, Y^* \setminus C^+).$$

This implies that (y_α) is fundamental, hence it has a limit, say $\bar{y} \in T_{y'}$. This shows that the family $(T_{y'} \setminus (y_\alpha - C))$ cannot cover $T_{y'}$, hence $T_{y'}$ is C-complete. Thus $\mathrm{INF}_w(T_{y'}|C) \neq \emptyset$, and since $\mathrm{INF}_w(T_{y'}|C) \subset \mathrm{INF}_w(T|C)$, any $z \in \mathrm{INF}_w(T_{y'}|C)$ satisfies the conclusion of the theorem. $\square$

2.3. The extended space $\overline{Y}$

We will briefly recall some results presented in [8].

In the real convex analysis, we allow the values $+\infty$, $-\infty$ in order to handle convex functions defined on the whole space X, and for other practical reasons. For the same reasons we will extend the space Y to a topological partially ordered space $\overline{Y} = Y \cup \{+\infty_C, -\infty_C\}$ where $+\infty_C$ and $-\infty_C$ are two distinct elements not belonging to Y (some other results about the extended space can be found in [9, 22]).

A *neighborhood* of $+\infty_C$ ($-\infty_C$) in $\overline{Y}$ is a set N, ($-N$ resp.) such that $\{+\infty_C\} \cup (y + C) \subset N$ for some $y \in Y$. Then the topology $\bar{\tau}$ of $\overline{Y}$ is defined by

$$\bar{\tau} = \tau \cup \{\mathcal{O} \subset \overline{Y} |\ \mathcal{O} \text{ is a neighborhood of } +\infty_C \text{ or of } -\infty_C \\ \text{and } \mathcal{O} \setminus \{+\infty_C,\, -\infty_C\} \in \tau\},$$

where τ is the topology of Y. Thus, the imbedding $Y \subset \overline{Y}$ is continuous, and Y is dense in $\overline{Y}$.

The algebraic operations are extended in the same way as in the convex analysis (see [8]). Note that $+\infty_C - \infty_C = +\infty_C$; $0 \cdot \infty_C = 0_Y$.

We will extend the relations $\leq_C$, $\leq_{C_0}$ and $<_C$ to $\overline{Y}$ by

$$\forall y \in Y,\ \ -\infty_C \leq_C y \leq +\infty_C,\ \ -\infty_C \leq_{C_0} y \leq +\infty_C,\ \ -\infty_C <_C y <_C +\infty_C.$$

Note that, despite the analogy with the extended real line $\bar{\mathbb{R}}$, the space $\overline{Y}$ is in general not compact (even in the case $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$).

Remark 2.8. Following our definition, a sequence $(y_n)_n \subset \overline{Y}$ tends to $+\infty_C$ in $\overline{Y}$ iff for every $y \in Y$ there exists some n_0 such that, for all $n \geq n_0$ we have $y \leq_C y_n$. Also $y_n \to -\infty_C$ iff $-y_n \to +\infty_C$.

Remark 2.9. We have also that $y_n \to +\infty_C$ (or $y_n \to -\infty_C$) and $y_n \in Y$ implies $\|y_n\| \to +\infty$. Of course, the converse is false.

For $T \subset \overline{Y}$, $\bar{T}$ stands for the closure in the space $\overline{Y}$, and to simplify the notation, we put $\mathrm{MIN}\, T = \mathrm{MIN}(T|C)$, $\mathrm{MIN}_w T = \mathrm{MIN}_w(T|C)$, $\mathrm{INF}_w T = \mathrm{INF}_w(T|C)$ etc. Note that, if $T \neq \{+\infty_C\}$, then

$$\mathrm{INF}_w T = \{y \in \bar{T} |\ -y + T \subset \overline{Y} \setminus (-\operatorname{int} C \cup \{-\infty_C\})\}.$$

All the results about the (weakly) minimal or infimal set presented in the previous section in the space Y, hold for the space $\overline{Y}$ too, excepting the relation $\mathrm{INF}_w(T|C) \subset \mathrm{bd}\, T$. Moreover, according to the definition, we obtain easily the following.

Proposition 2.10. *The following statements are equivalent:*

(i) $-\infty_C \in \mathrm{INF}_w(T)$.
(ii) $\mathrm{INF}_w(T) = \{-\infty_C\}$.
(iii) $\exists (y_n) \subset T \;\; y_n \to -\infty_C$.
(iv) $-\infty_C \in \bar{T}$.

Proposition 2.11. *Let $T \subset Y \cup \{+\infty_C\}$ such that $-\infty_C \notin \bar{T}$ and $T \neq \{+\infty_C\}$. Then*

$$\mathrm{INF}_w T = \mathrm{INF}_w(T \setminus \{+\infty_C\}),$$

and these sets coincide with the weakly infimal set associated to $T \setminus \{+\infty_C\}$ in the space Y.

Proposition 2.12. *Let (y_n) be a sequence in Y. Then*

$$\lambda \in C^+ \setminus \{0\},\; y_n \to +\infty_C \Longrightarrow \langle \lambda, y_n \rangle \to +\infty. \tag{1}$$

Proof. Let $e \in \mathrm{int}\, C$. Then for any $\lambda \in C^+ \setminus \{0\}$ we have $\langle \lambda, e\rangle > 0$ (otherwise, let $r > 0$ such that $e + rB \subset C$; we have $\lambda(e + rB) = r\lambda(B) \subset \mathbb{R}_+$, $\lambda(-B) = \lambda(B)$ wich implies $\lambda = 0$). For any $k \in \mathbb{R}_+$ there exists some $n_0 \in \mathbb{N}$ such that $ke \le y_n$ for all $n > n_0$. Thus $k\langle \lambda, e\rangle \le_C \langle \lambda, y_n\rangle$. □

We will extend each element $\lambda \in C^+ \setminus \{0\}$ to a function (denoted λ too) $\lambda : \overline{Y} \to \bar{\mathbb{R}}$ putting

$$\langle \lambda, +\infty_C \rangle = +\infty; \quad \langle \lambda, -\infty_C \rangle = -\infty.$$

We will denote

$$\Lambda = \{\lambda : \overline{Y} \to \bar{\mathbb{R}} |\; \lambda_{|Y} \in C^+;\; \|\lambda_{|Y}\| = 1;\; \lambda(-\infty_C) = -\infty,\; \lambda(+\infty_C) = +\infty\}.$$

Notice also the following converse of Proposition 2.12.

Proposition 2.13. *Let C be a polyhedral convex pointed cone with non empty interior (thus Y must be a finite dimensional space!). Then, for any sequence $(y_n) \subset Y$ such that*

$$\forall \lambda \in \Lambda, \quad \langle \lambda, y_n \rangle \to +\infty,$$

we have

$$y_n \to +\infty_C.$$

Proof. Since C^+ is a polyhedral cone (see [23]), there exists $\{\lambda_1, \ldots, \lambda_p\} \subset \Lambda$ such that

$$C^+ = \{\sum_{i=1}^{p} \alpha_i \lambda_i |\; (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}_+^p\}.$$

Let $e \in \mathrm{int}\, C$. For each $i \in \{1, \ldots, p\}$ there exists $n_i \in \mathbb{N}$ such that

$$\langle \lambda_i, y_n \rangle \ge \langle \lambda_i, e \rangle, \quad \forall n > n_i.$$

Let $n_0 = \max(n_1, \dots, n_p)$. Then

$$\forall \lambda \in C^+,\ n > n_0, \quad \langle \lambda, y_n \rangle \geq \langle \lambda, e \rangle.$$

This implies

$$\forall n > n_0, \quad e \leq_C y_n,$$

hence $y_n \to +\infty_C$. □

2.4. Extended values vector optimization problem

Unless otherwise stated, X stands for a reflexive Banach space. Let us consider a map $F : X \to \overline{Y}$. We will denote by $\mathrm{dom}(F)$ the effective domain of F, i.e.

$$\mathrm{dom}(F) = \{x \in X |\ F(x) \neq +\infty_C\}.$$

We say that F is *proper* if $\mathrm{dom}(F) \neq \emptyset$ and $F(x) \neq -\infty_C$, $\forall x \in \mathrm{dom}(F)$.

For the vector "minimization" problem:

$$(VMP) \qquad \text{"minimize" } F(x),\ \ x \in X$$

we say that $a \in X$ is a *weakly Pareto solution* if

$$F(a) \in \mathrm{MIN}_w\, F(X),$$

i.e., there is no $x \in X$ such that $F(x) <_C F(a)$. The weakly Pareto (or efficient) set will be denoted by $E_w = F^{-1}(\mathrm{MIN}_w\, F(X))$. Note that, if $F : X \to \overline{Y}$ is continuous, then E_w is closed.

Note also that, if $-\infty_C \in F(X)$, then $E_w = F^{-1}(\{-\infty_C\})$.

Let $F : X \to \overline{Y}$ be a map.

Definition 2.14. We say that F is *C-convex* if

$$\forall \theta \in [0,\, 1],\ \forall x,\, x' \in X,\ \ F((1-\theta)x + \theta x') \leq_C (1-\theta)F(x) + \theta F(x').$$

Definition 2.15. We say that F is *C-semicontinuous* if the *level sets*

$$L(F; \alpha) = \{x \in X |\ F(x) \leq_C \alpha\}$$

are closed for all $\alpha \in Y$.

For each $\lambda \in \Lambda$, we will consider the function $F_\lambda : X \to \bar{\mathbb{R}}$ given by

$$F_\lambda = \lambda \circ F.$$

Definition 2.16. We say that F is *positively lower semicontinuous (continuous)* if for any $\lambda \in \Lambda$, the function F_λ is lower semicontinuous (continuous resp.).

Remark 2.17.

- F is C-convex if and only if F_λ is convex for every $\lambda \in C^+ \setminus \{0\}$.
- If F is C-convex then $L(F; \alpha)$ is a convex set in X for all $\alpha \in Y$.

- If F is positively lower semicontinuous, then F is C-semicontinuous (because $L(F;\alpha) = \cap_{\lambda\in C^+\setminus\{0\}}\{x \in X|\ F_\lambda(x) \le \langle\lambda,\alpha\rangle\}$).
- If F is C-convex the effective domain $\mathrm{dom}(F)$ is a convex set.

Definition 2.18. Let $F : X \to \overline{Y}$ be a C-convex map (Gâteaux differentiable, with $\mathrm{dom}(F)$ open resp.). A point $a \in X$ ($a \in \mathrm{dom}(F)$ resp.) is called *weakly scalarly stationary*, if there exists $\lambda \in \Lambda$ such that

$$0 \in \partial F_\lambda(a),$$

where

$$\partial F_\lambda(a) = \{x^* \in X^*|\ F_\lambda(x) - F_\lambda(a) \ge \langle x^*, x-a\rangle,\ \ \forall x \in X\}$$

is the usual subdifferential of the extended real values convex function F_λ at a (the Gâteaux derivative resp.).

We denote by S_w the set of the weakly scalarly stationary points.

Proposition 2.19.
(i) *If $F : X \to \overline{Y}$ is C-convex, then $E_w = S_w$.*
(ii) *If $F : X \to \overline{Y}$ is Gâteaux differentiable with* $\mathrm{dom}(F)$ *open, then $E_w \subset S_w$.*

Proof. The part (i) has been proved in [8], and (ii) can be found in [6] for Fréchet differentiable functions, and it is easy to generalize it for Gâteaux differentiable functions. □

Let $F : X \to Y \cup \{+\infty_C\}$ be a proper map.

We will denote

$$\begin{aligned}\mathrm{INF}_w\,(F) = \mathrm{INF}_w(F(X)) &= \{y \in \overline{F(X)}|\ F(X) - y \subset \overline{Y} \setminus (-\,\mathrm{int}\,C \cup \{-\infty_C\})\}\\ &= \{y \in \overline{F(X)}|\ F(X) \cap (y - \mathrm{int}\,C) = \emptyset\}.\end{aligned}$$

Remark 2.20. It is easy to see that:

(i) $F(E_w) = F(X) \cap \mathrm{INF}_w(F) \subset \mathrm{INF}_w(F)$.
(ii) $E_w = F^{-1}(\mathrm{INF}_w(F))$.
(iii) E_w may be empty while $\mathrm{INF}_w(F)$ is not empty (see [6]).

Definition 2.21. The point $a \in X$ is $\varepsilon-$ *weakly efficient* if there exists $e \in B$, such that $F(a) - \varepsilon e \in \mathrm{INF}_w(F)$, where $\varepsilon > 0$.

Remark 2.22. Note that, for $F(a) \in Y$, we have that a is $\varepsilon-$ weakly efficient iff $\mathrm{d}(F(a); \mathrm{INF}_w(F)) \le \varepsilon$.

Remark 2.23. In the literature we can find a different definition for the approximate weakly efficiency. Thus (see [18, 16]), given a *vector* $\vec{\varepsilon} \in C$ (or $\vec{\varepsilon} \in Y$), a point $x \in X$ is an $\vec{\varepsilon}-$ weakly efficient solution if there is no $x' \in X$ such that $F(x') <_C F(x) - \vec{\varepsilon}$. The disadvantage of this definition is that the value $F(x)$ *may be far from the set* $\mathrm{INF}_w(F)$.

Example 2.24. Consider $X = Y = \mathbb{R}^2$ with the euclidean structure, $C = \mathbb{R}^2_+$, and $F : X \to Y$ such that $\forall (x_1, x_2) \in \mathbb{R}^2,\ F(x_1, x_2) = (x_1^2 + x_2^2,\ 10^8 x_2^2)$. Thus $F(X) = \{(y_1,\ y_2) \in \mathbb{R}^2_+ |\ y_2 \le 10^8 y_1\}$. Hence $\mathrm{INF}_w(F)$ is the half-axis $\mathbb{R}_+ \cdot (1,\ 0)$. For the point $a = (0;\ 10^{-2})$, we have $F(0;\ 10^{-2}) = (10^{-4},\ 10^4)$, hence a is an $\vec{\varepsilon}-$ weakly efficient solution with $\vec{\varepsilon} = (10^{-4},\ 0)$ but $\mathrm{d}(F(a);\ \mathrm{INF}_w(F)) = 10^4$.

Definition 2.25. The point $a \in X$ is $\varepsilon-$ *correct weakly efficient* if there exists $e \in C \cap B$, such that $F(a) - \varepsilon e \in \mathrm{INF}_w(F)$, where $\varepsilon > 0$.

Remark 2.26. It is obvious that every $\varepsilon-$ correct weakly efficient point is also an $\varepsilon-$ weakly efficient point. Also, a point $a \in X$ is weakly efficient $\Longleftrightarrow$ a is $\varepsilon-$ correct weakly efficient for all $\varepsilon > 0 \Longleftrightarrow a$ is $\varepsilon-$ weakly efficient for all $\varepsilon > 0$.

Remark 2.27. If we assume that F_λ is bounded from below for some $\lambda \in \Lambda \cap \operatorname{int} C^+$, then using Theorem 2.6 (with $T = F(X) \backslash \{+\infty_C\}$) and Proposition 2.11, we have that for each $a \in \mathrm{dom}(F)$, there exists $y \in \operatorname{int} C \cup \{0\}$, hence $y \in C$ such that $F(a) - y \in \mathrm{INF}_w(F)$. However, when $\mathrm{d}(F(a); \mathrm{INF}_w(F)) < \varepsilon$, we cannot be sure that $\|y\| \le \varepsilon$. Of course, there exists $z \in \mathrm{INF}_w(F)$ such that $\|F(a) - z\| < \varepsilon$, but not necessarily $z \in C$.

Example 2.28. Consider $X = Y = \mathbb{R}^2$ with the euclidean structure, $C = \mathbb{R}^2_+$, and $F : X \to \overline{Y}$ such that $F(x) = x$ for all $x \in T = \{(x_1,\ x_2) \in \mathbb{R}_- \times \mathbb{R} |\ x_2 \ge -10^3;\ \ (x_1 + x_2) \cdot (x_2 - 10^6 x_1) = 0\}$, and $F(x) = +\infty_C$ elsewhere. Thus $\mathrm{INF}_w(F) = \mathrm{INF}_w(T) = \{(t,\ -t) |\ t < -10^{-3}\} \cup \{(-10^{-3},\ -10^3)\}$ and the point $(0;\ 0)$ is $10^{-3}\sqrt{2}$-weakly efficient but it is not $10^{-3}\sqrt{2}$-correct weakly efficient (it is $\varepsilon-$correct weakly efficient for $\varepsilon \ge 10^3 \sqrt{1 + 10^{-12}}$).

Definition 2.29. A sequence (x_n) in $\mathrm{dom}(F)$ will be called:

(1) *asymptotically weakly Pareto optimizing* (a.w.p.) if

$$\mathrm{d}(F(x_n),\ \mathrm{INF}_w\ (F)) \to 0 \quad \text{or} \quad F(x_n) \to -\infty_C$$

(2) *correct asymptotically weakly Pareto optimizing (c.a.w.p.)* if there exists a sequence $(\varepsilon_n) \subset \mathbb{R}^*_+$ such that

$$(i)\ \lim_{n\to\infty} \varepsilon_n = 0;\ \ (ii)\ \forall n,\ x_n \text{ is } \varepsilon_n- \text{ correct weakly efficient.}$$

(3) *weakly scalarly stationary* (w.s.s.) if F is C-convex or F is Gâteaux differentiable with $\mathrm{dom}(F)$ open, and there exists a sequence $(\lambda_n) \subset \Lambda$ verifying the following:

$$\forall n \in \mathbb{N},\ \ \exists \xi_n \in \partial F_{\lambda_n}(x_n) :\ \ \|\xi_n\|_{X^*} \to 0, \tag{2}$$

which is equivalent to saying that

$$\forall n \in \mathbb{N},\ \ \partial F_{\lambda_n}(x_n) \ne \emptyset,\ \ \mathrm{d}(0; \partial F_{\lambda_n}(x_n)) \to 0.$$

Remark 2.30. If (x_n) is an a.w.p. sequence, then $\mathrm{INF}_w(F) \ne \emptyset$ and we have two possibilities:

(i) $-\infty_C \notin \mathrm{INF}_w(F)$, and in this case $\mathrm{d}(F(x_n), \mathrm{INF}_w(F)) \to 0$.
(ii) $\mathrm{INF}_w(F) = \{-\infty_C\}$, and in this case $F(x_n) \to -\infty_C$.

Note that (i) is equivalent to the following:

(i') *there exists a sequence* (ε_n) *of positive real numbers tending to* 0 *such that, for all* $n \in \mathbb{N}$, x_n *is* ε_n- *weakly efficient.*

3. An a.w.p. sequence is close to a w.s.s. one

3.1. A vector variational principle

Here we consider that X is a finite dimensional euclidean space identified with its dual.

Theorem 3.1. *Let $F : X \to Y \cup \{+\infty_C\}$ be a proper map, positively lower semicontinuous (l.s.c.) such that $\emptyset \neq \mathrm{INF}_w(F) \subset Y$.*

Then, for any $\varepsilon > 0$, $\gamma > 0$ and $a \in X$ such that $\mathrm{d}(F(a);\ \mathrm{INF}_w(F)) < \varepsilon$ there exists $a' \in \operatorname{dom} F$ and $e \in B \subset Y$ such that:

(i) $\|a - a'\| \leq \gamma$,

(ii) *a' is a weakly efficient point for the map $G : X \to Y \cup \{+\infty_C\}$, given by*

$$x \mapsto G(x) = F(x) - \varepsilon\varphi(1 - \frac{\|x - a\|^2}{\gamma^2})e, \tag{3}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing, convex differentiable function such that

$$\varphi(\mathbb{R}_-) = \{0\}, \quad \varphi'(1) \leq 2, \quad \varphi(1) = 1,$$

(for example $\varphi(t) = 0$, if $t < 0$; $\varphi(t) = t^2$ if $t \geq 0$.)

(iii) *for any $x \in \operatorname{dom} F$:*

$$\|F(x) - G(x)\| \leq \varepsilon. \tag{4}$$

If in addition F is Gâteaux differentiable[2] *with $\operatorname{dom} F$ open set, then*

$$\exists \lambda \in \Lambda, \quad \mathrm{d}(0; \partial F_\lambda(a')) \leq 4\frac{\varepsilon}{\gamma}. \tag{5}$$

Proof. Let $y \in \mathrm{INF}_w(F)$ such that $\|F(a) - y\| < \varepsilon$. Denote by e the vector verifying

$$F(a) - \varepsilon e = y.$$

If $F(a) \in \mathrm{INF}_w(F)$ take $e = 0$, $a' = a$, and (i), (ii), (iii) follow immediately. So we will suppose that $F(a) \notin \mathrm{INF}_w(F)$, hence $e \in B \setminus \{0\}$. Obviously the function $x \mapsto -\varepsilon\varphi(1 - \frac{\|x-a\|^2}{\gamma^2})$ is continuous on X. Thus the function G defined by (3) is positively l.s.c., hence C-semicontinuous, which implies that the set $L(G; y)$ is closed. Note also that $a \in L(G; y)$ because $G(a) = y$. Consider $\hat{\lambda} \in \Lambda$. The function $G_{\hat{\lambda}}$ is l.s.c. as a sum of such functions, hence it attains its minimum on the compact set $(a + \gamma B) \cap L(G; y) \subset \operatorname{dom} G = \operatorname{dom} F$. Thus there exists $a' \in (a + \gamma B) \cap L(G; y)$ such that

$$G_{\hat{\lambda}}(a') = \min_{x \in (a+\gamma B) \cap L(G;y)} G_{\hat{\lambda}}(x).$$

Let $x \in X$ such that $G(x) <_C G(a')$. Since $G(a') \leq_C y$ we must have $x \in L(G; y)$. We have $G_{\hat{\lambda}}(x) < G_{\hat{\lambda}}(a')$ hence $x \notin (a + \gamma B) \cap L(G; y)$. Thus $x \notin a + \gamma B$ which implies that $G(x) = F(x) <_C G(a') \leq_C y$ contradicting the fact that $y \in \mathrm{INF}_w(F)$. Hence there is no $x \in X$ such that $G(x) <_C G(a')$ which means that a' is a weakly efficient point for G. Thus (i) and (ii) have been proved.

[2] we denote the Gâteaux derivative of a function H at a by $\partial H(a)$ or by $H'(a)$

(4) is obvious. Let F be Gâteaux differentiable with $\operatorname{dom} F$ open set. Then G is Gâteaux differentiable at any $x \in \operatorname{dom} F$, and

$$\forall \lambda \in \Lambda, \;\; \partial G_\lambda(x) = \partial F_\lambda(x) - \frac{2\varepsilon\langle \lambda, e\rangle}{\gamma^2}\varphi'(1 - \frac{\|x-a\|^2}{\gamma^2})(x-a).$$

Thus, according to Proposition 2.19, $0 \in \partial G_\lambda(a')$ for some $\lambda \in \Lambda$, and (5) follows immediately. □

The following is obvious.

Corollary 3.2. *Let $F : X \to Y \cup \{+\infty_C\}$ be Gâteaux differentiable, with* $\operatorname{dom} F$ *an open set.*

If (a_k) is an a.w.p. sequence then there exists a sequence (a'_k) such that

(i) $\|a_k - a'_k\| \to 0$,

(ii) *(a'_k) is a w.s.s. sequence.*

(iii) *Moreover, if F is uniformly continuous on* $\operatorname{dom} F$*, then (a'_k) is also an a.w.p. sequence*

3.2. Ekeland vector variational principle for $\varepsilon-$correct weakly efficient points

In this subsection we will see that much more can be said about an $\varepsilon-$correct weakly efficient point. Let X be a reflexive Banach space, and $F : X \to Y \cup \{+\infty\}$.

Theorem 3.3. *Suppose that F is a proper, positively weakly l.s.c. map. Let $\varepsilon > 0$ and let $a \in X$ be any $\varepsilon-$ correct weakly efficient point. Consider some $e \in C \cap B$ such that $F(a) - \varepsilon e \in \mathrm{INF}_w(F)$.*

Then, for any $\gamma > 0$, there exists some $b \in X$ such that :

(i) $\|a - b\| \le \gamma$,

(ii) *b is a weakly efficient point for the perturbed function*

$$x \mapsto G(x) = F(x) + \frac{\varepsilon}{\gamma}\|x-b\|e,$$

(iii) $F(b) \le_C F(a) - \frac{\varepsilon}{\gamma}\|a-b\|e$.

Proof. Consider the function $x \mapsto H(x) = F(x) - \varepsilon(1 - \frac{\|x-a\|}{\gamma})_+ e$, where $t_+ = \max(t, 0)$, $t \in \mathbb{R}$. Put $y = H(a) = F(a) - \varepsilon e$. For each $\lambda \in \Lambda$, the scalar function H_λ is weakly l.s.c. as a sum of weakly l.s.c. functions (since $\langle \lambda, e\rangle \ge 0$ and $x \mapsto -\varepsilon(1 - \frac{\|x-a\|}{\gamma})_+$ is weakly l.s.c., because $x \mapsto \|x-a\|$ is weakly l.s.c. and $t \mapsto t_+$ is continuous and increasing). Thus the level set $L(H; y)$ is weakly closed, hence the set $S = (a + \gamma B) \cap L(H; y)$ is weakly compact. Then, for each $\lambda \in \Lambda$, the set $\arg\min_{x\in S} H_\lambda(x)$ is not empty. So, we can choose $b \in \arg\min_{x\in S} H_\lambda(x)$. Obviously we have (i). To prove (ii), notice first that, for any $x \in S$, we have $H_\lambda(b) \le H_\lambda(x)$ which is equivalent to

$$F_\lambda(b) \le F_\lambda(x) + \frac{\varepsilon\langle \lambda, e\rangle}{\gamma}(\|x-a\| - \|b-a\|).$$

Since $\|x-a\| - \|b-a\| \leq \|x-b\|$, we obtain

$$\forall x \in S, \quad G_\lambda(b) \leq G_\lambda(x). \tag{6}$$

Let now consider $x \in X$ such that

$$G(x) <_C G(b). \tag{7}$$

From (6) we have that $x \notin S$. On the other hand, (7) is equivalent to

$$F(x) <_C F(b) - \frac{\varepsilon}{\gamma}\|x-b\|e. \tag{8}$$

If $x \notin a + \gamma B$, then

$$-\|x-b\| \leq -\|x-a\| + \|a-b\| \leq -\gamma + \|a-b\|$$

hence, from (8),

$$F(x) <_C F(b) + \frac{\varepsilon}{\gamma}(-\gamma + \|a-b\|)e = H(b) \leq_C y$$

contradicting the fact that $y \in \mathrm{INF}_w(F)$.

If $x \in a + \gamma B$, then (8) implies

$$H(x) + \varepsilon(1 - \frac{\|x-a\|}{\gamma})e <_C H(b) + \varepsilon(1 - \frac{\|b-a\| + \|x-b\|}{\gamma})e,$$

and, since $H(b) \leq_C y$, we obtain

$$H(x) <_C y + \frac{\varepsilon}{\gamma}(\|x-a\| - \|b-a\| - \|x-b\|)e \leq_C y.$$

It follows that $x \in L(H; y)$, hence $x \in S$ which is impossible.

In conclusion, there is no $x \in X$ verifying (7), hence b is weakly efficient for G.

(iii) follows immediately from the fact that $H(b) \leq_C y$. □

Corollary 3.4. *With the hypotheses and notations of* Theorem 3.3*, if we have in addition that* $\operatorname{dom} F$ *is an open set and* F *is Gâteaux differentiable on* $\operatorname{dom} F$*, then there exists some* $\lambda \in \Lambda$ *such that*

$$\|(F_\lambda)'(b)\| \leq \frac{\varepsilon}{\gamma}. \tag{9}$$

Proof. The set $C_1 = Y \setminus (-\operatorname{int} C)$ is a closed cone (not convex), i.e. $\mathbb{R}_+ C_1 \subset C_1$. Let $h \in X$. Since b is weakly efficient for G, then for any positive real t sufficiently small such that $b + th \in \operatorname{dom} F$, we have that $G(b+th) - G(b) \in C_1$, hence

$$\frac{F(b+th) - F(b)}{t} + \frac{\varepsilon\|h\|}{\gamma} e \in C_1.$$

Letting $t \to 0^+$, we obtain that

$$F'(b)h + \frac{\varepsilon\|h\|}{\gamma}e \in C_1.$$

Consider the set

$$U = \{F'(b)h + \frac{\varepsilon\|h\|}{\gamma}e |\ h \in X\}.$$

It is easy to see that $U + C \subset C_1$, and $U + C$ is a convex set. Then by the separation theorem, there exists $\lambda \in Y^* \setminus \{0\}$ such that

$$\forall h \in X,\ c \in C, \quad \langle \lambda, F'(b)h + \frac{\varepsilon\|h\|}{\gamma}e\rangle \geq \langle \lambda, -c\rangle.$$

It follows that $\lambda \in C^+$ (and dividing by $\|\lambda\|$ we can suppose that $\lambda \in \Lambda$), and using the fact that $\langle \lambda, F'(b)h\rangle = (F_\lambda)'(b)h = F'_\lambda(b)$, we obtain

$$\forall h \in \operatorname{bd} B, \quad (F_\lambda)'(b)h + \varepsilon\frac{\langle \lambda, e\rangle}{\gamma} \geq 0.$$

Changing h in $-h$ we obtain that

$$\forall h \in \operatorname{bd} B, \quad |(F_\lambda)'(b)h| \leq \varepsilon\frac{\langle \lambda, e\rangle}{\gamma} \leq \frac{\varepsilon}{\gamma},$$

which implies (9). □

Corollary 3.5. *With the hypotheses and notations of* Theorem 3.3, *if we have in addition that F is C-convex, then there exists some $\lambda \in \Lambda$ such that*

$$\mathrm{d}(0,\ \partial F_\lambda(b)) \leq \frac{\varepsilon}{\gamma}. \tag{10}$$

Proof. It is easy to see that the function G is C-convex. Then, according to Proposition 2.19, we have that there exists some $\lambda \in \Lambda$ such that $0 \in \partial G_\lambda(b)$. Thus $0 \in \partial F_\lambda(b) + \frac{\varepsilon\langle\lambda,e\rangle}{\gamma}B$ and (10) follows easily. □

We obtain immediately the following.

Theorem 3.6. *Let F be a proper, positively weakly l.s.c. map. If F is C-convex, or F is Gâteaux differentiable with* $\operatorname{dom} F$ *open, then given any c.a.w.p. sequence (a_k), there exists a sequence (b_k) such that*

(i) $\|a_k - b_k\| \to 0$,
(ii) (b_k) *is w.s.s.,*
(iii) $F(b_k) \leq_C F(a_k)$.

Remark 3.7. Unfortunately, in Theorem 3.6 we cannot be sure that the sequence (b_k) is a.w.p., unless F is C-convex and asymptotically well behaved as defined in [8], or F is uniformly continuous on $\operatorname{dom} F$.

Remark 3.8. It is important to note the differences with the vector variational principle obtained by Loridan in [17], where the initial point (denoted a in our theorem) is an ε solution of some scalarized problem. If the problem is nonconvex then there exists (weakly) efficient points x such that $F(x)$ is far from any ε solution of any scalarized function F_λ, $\lambda \in \Lambda$.

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