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Dynamics of Positive Multiconvex Relations*

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A notion of *multiconvex relation* as a union of a finite number of convex relations is introduced. For a particular case of *multiconvex process*, that is, a union of a finite set of convex processes, we define the notions of the joint and the generalized spectral radius in the same manner as for matrices. We prove the equivalence of these two values if all component processes are positive, bounded, and closed.

Keywords: star-shaped set, convex relation, convex process, rate of growth, generalized spectral radius

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1. Introduction

The dynamics of economic growth is often modeled by conic set-valued maps with convex graphs, that is, by *convex processes*, see for example [1, 2, 7, 9, 12, 14]. These processes are usually assumed to be defined on the entire cone $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\}$ and increasing with respect to this cone. In mathematical theory of economic dynamics these special convex processes are called *superlinear mappings* [7].

In many situations, however, a choice exists between several alternative schemes of economic development; each one is modeled by its own convex process (or superlinear mapping) S_j. Detailed economic motivation is given in Section 7 below.

Since the graphs of all processes contain the origin, the graph of the process $S = \cup_{j=1,\dots,k} S_j$ is a *star-shaped with respect to zero* set in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. If the components S_j are general set-valued mappings with convex graphs, S will be called a *multiconvex mapping* or *multiconvex relation*. The dynamics of the model is described by the sequence of iterations S^m of the relation S.

The dynamical behavior of the sequence $\{S^m\}$ is characterized by the totality of its trajectories $\{x_0, x_1, \dots\}$, $(x_i, x_{i+1}) \in S$ for $i = 0, 1, \dots$. We study the maximal rate of growth of trajectories in terms of two kinds of spectral radii. We extend the notions of the joint and the generalized spectral radius from systems of matrices (multilinear relations

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on the whole space), [3, 4, 5, 6], to multiconvex relations, and prove that they are equal to each other for a wide class of positive processes.

The paper is a direct continuation of [15], where main attention has been paid to the limit behavior of *compact* relations, star-shaped with respect to zero.

2. Star-shaped and multiconvex relations

We consider set-valued mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, also called relations. Any relation S is identified with its graph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in S(x)\}$, so in the sequel we will use the same notation S for the relation and its graph and, for example, speak about unions of relations meaning unions of their graphs. The domain of S is the set $\mathrm{dom}(S) = \{x \in \mathbb{R}^n : S(x) \neq \emptyset\}$.

A *trajectory* (finite or infinite) of the relation S is a sequence of points $\{\dots x_i, x_{i+1}, \dots\}$ in $\mathbb{R}^n$ such that

$$x_{i+1} \in S(x_i)$$

for all admissible indices i.

Definition 2.1. A *composition* of two relations P and S in $\mathbb{R}^n \times \mathbb{R}^n$ is a relation PS defined as

$$PS(x) = P(S(x)) = \{z \in \mathbb{R}^n : \exists y \in S(x) \text{ such that } z \in P(y)\}.$$

We will study iterations S^m of a single relation or of a sequence of different relations $S_{j_m} \dots S_{j_1}$.

The notion of star-shapedness is a generalization of convexity. We will consider a class of zero-centered star-shaped relations and its important subclass of *conic* relations.

Definition 2.2. A set S in a linear space L is called star-shaped with respect to zero (respectively, conic) if $\alpha S \subseteq S$ for any $0 \leq \alpha \leq 1$ (for any $0 \leq \alpha < \infty$).

A relation P is called star-shaped with respect to zero (conic) if its graph is star-shaped with respect to zero (conic) in $\mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.3. A relation $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is *convex* if its graph is convex. A convex conic relation is called a convex process [11].

As is known, the composition of two convex (conic) relations is again a convex (conic) relation. Now we introduce the central notion of this paper.

Definition 2.4. A *multiconvex relation* S in $\mathbb{R}^n$ is a multivalued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose graph is a union of graphs of a finite number of convex relations S_j, $j = 1, \dots, k$ (we will write $S = \cup_{j=1,\dots,k} S_j$).

Note that, for any trajectory $X = \{\dots, x_i, x_{i+1}, \dots\}$ of a multiconvex relation, there exists an appropriate product of component relations $\dots S_{j_{i+1}} S_{j_i} \dots$ such that X is a trajectory of this product, that is,

$$x_{i+1} \in S_{j_{i+1}}(x_i), \quad \text{for all } i.$$

In particular, any periodic trajectory $\{x_1, \dots, x_m, x_1, \dots\}$ is a trajectory of some periodic sequence of component relations $S_{j_1}, \dots, S_{j_m}$. In this case, the point x_1 is a fixed point of the convex relation $S_{j_m} \dots S_{j_1}$. Any multiconvex relation $S = \cup_{j=1,\dots,k} S_j$ satisfying $0 \in S_j(0)$ for $j = 1, \dots, k$, is star-shaped with respect to zero.

Definition 2.5. A multiconvex process in $\mathbb{R}^n \times \mathbb{R}^n$ is a set-valued map whose graph is a union of graphs of a finite number of convex processes.

Definition 2.6. A multiconvex process is called *multilinear* if all S_i are single-valued linear maps, each one restricted to a convex cone $\mathrm{dom}(S_i) = K_i \subseteq \mathbb{R}^n$.

3. Norms and spectral radii

For conic relations, a notion of norm is defined:

Definition 3.1. Given a norm $\|\cdot\|$ in $\mathbb{R}^n$, the corresponding norm of a conic relation $S \in \mathbb{R}^n \times \mathbb{R}^n$ is defined as

$$\|S\| = \sup_{\|x\|=1,\ y \in S(x)} \|y\|$$

if $\mathrm{dom}(S) \neq \{0\}$ and $\|S\| = +\infty$ if $\mathrm{dom}(S) = \{0\}$ and $S(0) \neq \{0\}$. If $\mathrm{dom}(S) = \{0\}$ and $S(0) = \{0\}$, we set $\|S\| = 0$.

Obviously,

$$\|SP\| \leq \|S\| \cdot \|P\| \tag{1}$$

for any pair S, P of conic relations. A conic relation S is *bounded* if $\|S\| < \infty$.

In the sequel we will only consider bounded relations. The following assertion is well known.

Proposition 3.2. *A convex process S with closed graph is bounded if and only if $S(0) = \{0\}$ and if and only if it is compact-valued, that is, if $S(x)$ is a compact set for each $x \in \mathrm{dom}(S)$.*

Proof. Each set $S(x)$ is closed, hence, if it is not compact, then it is unbounded and $\|S\| = +\infty$.

Vice-versa, if $\|S\| = +\infty$, there exist sequences $\{x_i\}$ and $\{y_i\}$ in $\mathbb{R}^n$ such that $x_i \to 0$ as $i \to \infty$, $\|y_i\| = 1$, $i = 1, 2, \dots$, and $y_i \in S(x_i)$, $i = 1, 2, \dots$. Choosing a subsequence, we conclude that $S(0) \neq \{0\}$ and, hence, $S(0)$ is unbounded. □

In the same manner as for matrices, let us define the notion of spectral radius for conic relations:

$$\rho(S) = \limsup_{m \to \infty} \left(\|S^m\|\right)^{\frac{1}{m}}. \tag{2}$$

In other words, $\rho(S)$ is the maximal average growth rate of long trajectories of S in the following sense.

Proposition 3.3. *Let S be a conic relation and let λ be a real number, $\lambda \geq 0$. Then $\rho(S) \leq \lambda$, if and only if, for any $\lambda_1 > \lambda$, there exists an N such that, for any $m \geq N$ and for any trajectory $x_0, \dots, x_m$ of S, the relation*

$$\|x_m\| \leq \lambda_1^m \|x_0\|$$

holds.

Note also that the value of $\rho(S)$ does not depend on a particular norm $\|\cdot\|$ in $\mathbb{R}^n$.

Proposition 3.4. *For any bounded relation S,*

$$\rho(S) = \lim_{m\to\infty} (\|S^m\|)^{\frac{1}{m}} = \inf_{m=1,2,\ldots} (\|S^m\|)^{\frac{1}{m}}.$$

Proof. As follows from (1), the sequence $p_m = \ln\|S^m\|$ satisfies the inequality $p_{a+b} \le p_a + p_b$. Suppose $\gamma > \inf_m \frac{p_m}{m}$ and choose m such that $\gamma = \frac{p_m}{m} + \varepsilon$, $\varepsilon > 0$. We get, for any integer $K > 0$, $p_{mK} \le mK(\gamma - \varepsilon)$. Taking into account $p_{mK+l} \le p_{mK} + p_l$, $0 \le l < m$, we get

$$p_N \le N(\gamma - \varepsilon) + C$$

for $C = \max\{p_0, \ldots, p_{m-1}\}$ and for all integer $N > 0$. Thus

$$\limsup_{N>0} \frac{p_N}{N} \le \gamma$$

and the required assertion follows. □

If S is a multiconvex process, $\rho(S)$ will be also called the *joint spectral radius* of the system $\{S_j\}$ of component convex processes, compare [4, 13]. The definition of the joint spectral radius does not depend on the representation of S as a union of convex processes S_j, while the following one, generally, does.

Definition 3.5. For a multiconvex process $S = \cup_{j=1\ldots k} S_j$, its *generalized spectral radius* is defined as

$$\hat{\rho}(S) = \sup_{m=1,2\ldots} \sup_{j_1,\ldots,j_m} \left(\rho(S_{j_m}\ldots S_{j_1})\right)^{\frac{1}{m}}. \tag{3}$$

For any finite sequence $\{j_1, \ldots, j_m\}$, we have

$$\rho(S_{j_m}\ldots S_{j_1}) \le \rho(S^m)$$

because $S_{j_m}\ldots S_{j_1} \subseteq S^m$ (in the sense of graphs). Hence,

$$\left(\rho(S_{j_m}\ldots S_{j_1})\right)^{\frac{1}{m}} \le \rho(S),$$

which, in turn, implies

$$\hat{\rho}(S) \le \rho(S). \tag{4}$$

In other words, the maximal growth rate for all periodic products of convex processes of the form $\ldots(S_{j_m}\ldots S_{j_1})(S_{j_m}\ldots S_{j_1})$ is less or equal to the overall growth rate $\rho(S)$.

An important theorem on the equivalence of the joint and the generalized spectral radii for multilinear relations defined on the whole $\mathbb{R}^n$ has been formulated as a conjecture in [4] and proved in [3], see also [5, 6].

Theorem 3.6. *Let S be a multilinear relation*

$$S(x) = \{M_1x, \ldots, M_kx\} \tag{5}$$

defined on the whole $\mathbb{R}^n$ by a finite family of $n\times n$-matrices M_j. Then

$$\rho(S) = \hat{\rho}(S).$$

Suppose now that the matrices M_j are *positive*, that is, all their elements are nonnegative. Consider the restriction S_+ of the relation (5) to $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n\}$.

Proposition 3.7. *Let S be a multilinear relation* (5) *with positive matrices M_i. If $\|x\|$ in $\mathbb{R}^n$ is chosen as*

$$\|x\| = \sum_{i=1}^{n} |x_i|,$$

then $\|S_+\| = \|S\|$.

Proof. First, let $k = 1$. For any $x \in \mathbb{R}^n$, let us reverse the sign of its negative components. This operation will not change the norm of x; the norm of $M_1 x$ cannot decrease because M_1 is positive. In the case of a finite number of matrices, the required statement follows now from $\|S\| = \max_{j=1,\ldots,k} \|M_j\|$. □

As a consequence, we get the following result.

Theorem 3.8. *Let $\{M_1, \ldots, M_k\}$ be a finite family of positive $n\times n$-matrices, and let S_+ be the corresponding multilinear relation on $\mathbb{R}^n_+$. Then*

$$\rho(S_+) = \hat{\rho}(S_+).$$

Proof. Owing to Theorem 3.6, it suffices to prove that

$$\rho(S_+) = \rho(S), \quad \hat{\rho}(S_+) = \hat{\rho}(S), \tag{6}$$

where S is the relation (5).

Since M_j are positive matrices, we have $S_j(\mathbb{R}^n_+) \subseteq \mathbb{R}^n_+$ and, hence,

$$(S_{j_m})_+ \ldots (S_{j_1})_+ = (S_{j_m} \ldots S_{j_1})_+ \tag{7}$$

for any finite product $S_{j_m} \ldots S_{j_1}$. Proposition 3.7 and (7) imply

$$\|(S_{j_m})_+ \ldots (S_{j_1})_+\| = \|S_{j_m} \ldots S_{j_1}\|.$$

Now, equations (6) and, hence, the statement of the theorem follow directly from the definitions of the joint and the generalized spectral radii. □

Theorem 3.6 cannot be automatically transferred to general convex processes, or even to general multilinear relations (not necessarily defined on the whole $\mathbb{R}^n$), as the following example shows.

Example 3.9. Let $K_1 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and $K_2 = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$. Consider a system of two linear relations S_1, S_2 defined on K_1, K_2 respectively and suppose that S_i is a rotation on a fixed angle α_i such that the triplet $\{\alpha_1, \alpha_2, \pi\}$ is linearly independent over the field of rational numbers, that is, the equality

$$r_0\pi + r_1\alpha_1 + r_2\alpha_2 = 0$$

for some rational coefficients r_i implies $r_i = 0$, $i = 1, 2, 3$.

Since $K_1 \cup K_2 = \mathbb{R}^2$, any finite trajectory of $S = S_1 \cup S_2$ can be prolonged indefinitely. Obviously, $\|x_i\| = \|x_0\|$ for any trajectory of S. Hence the joint spectral radius of the system is equal to 1. However, for any fixed finite sequence of relations $S_{j_1}, \ldots, S_{j_m}$, $j_i \in \{1, 2\}$, and for its any nontrivial trajectory $\{x_i\}$, the subsequence $\{x_0, x_m, x_{2m}, \ldots\}$ must be dense on the circle $\{y \in \mathbb{R}^2 : \|y\| = \|x_0\|\}$, which is impossible because $x_{km} \in K_{j_1}$ for all $k = 0, 1, \ldots$. Therefore, the spectral radius of the convex relation $S_{j_m} \ldots S_{j_1}$ equals 0, and hence the same is true for the generalized spectral radius of S itself.

4. Positive multiconvex relations and processes

Definition 4.1. A convex relation S is *positive* if its graph belongs to $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ and $\mathrm{dom}(S) = \mathbb{R}^n_+$. A positive closed convex conic relation is called a superlinear mapping.

Definition 4.2. A relation S is called increasing if for any pair $x, y \in \mathbb{R}^n_+$ such that $x \leqq y$ (that is, $y - x \in \mathbb{R}^n_+$), there exists a vector $h \in \mathbb{R}^n_+$ such that $S(x) + h \subseteq S(y)$. In particular, for any $p \in S(x)$, there exists a $q \in S(y)$ such that $q \geqq p$ (this property will be denoted $S(y) \geq S(x)$).

Lemma 4.3. *Any compact-valued positive convex relation S with closed graph is* increasing.

Proof. The ray $L = \{x + \alpha(y - x) : \alpha \geq 0\}$ belongs to $\mathbb{R}^n_+$, hence $S(x + \alpha(y - x))$ is nonempty for all $\alpha \geq 0$. Choose a point $z_\alpha \in S(x + \alpha(y - x))$ for each α and then let $\alpha \to +\infty$. Since S is convex, we get

$$S(y) \supseteq \frac{(\alpha - 1)S(x) + z_\alpha}{\alpha}. \tag{8}$$

Owing to the compactness of $S(y)$ and $S(x)$, the vector-function z_α/α is bounded and thus has a limit point $h \geqq 0$ as $\alpha \to \infty$. Again, the compactness of $S(y)$ makes it possible to pass to the limit in (8) as $\alpha \to \infty$ and to get the required statement. □

The following assertion is a direct consequence of Lemma 4.3.

Lemma 4.4. *If $\{x_0, x_1 \ldots\}$ is a trajectory of a positive compact-valued multiconvex relation S and $x'_0 \geqq x_0$, then there exists a trajectory $\{x'_0, x'_1 \ldots\}$ of S majorizing $\{x_0, x_1, \ldots\}$, that is, satisfying $x'_i \geqq x_i$, $i = 0, 1, \ldots$.*

Proof. If $x' \geqq x$ and $y \in S(x)$ then there exists a $y' \in S(x')$ such that $y' \geqq y$, thus a required trajectory can be extended indefinitely. □

Definition 4.5. A multiconvex relation $S = \cup_{j=1,\ldots,k} S_j$ is called *positive* (*compact-valued, superlinear*) if all the relations S_j are positive (compact-valued, superlinear).

Definition 4.6. A positive convex relation S is *normal* (see [7, 14]) if, for any $x \in \mathbb{R}^n_+$,

$$(S(x) - \mathbb{R}^n_+) \cap \mathbb{R}^n = S(x).$$

Definition 4.7. The *normal hull* of a convex relation S is defined as

$$\mathrm{Nh}(S)(x) = (S(x) - \mathbb{R}^n_+) \cap \mathbb{R}^n_+.$$

For a multiconvex relation, the normal hull is defined as the union of normal hulls of its components.

The following assertion is an immediate consequence of the definitions.

Proposition 4.8. *The operation of taking the normal hull conserves positivity, boundedness, and closedness of multiconvex relations.*

Note also that, for any pair of increasing positive relations P, S we have

$$\mathrm{Nh}(PS) = \mathrm{Nh}(P)\,\mathrm{Nh}(S).$$

Proposition 4.9. *For a bounded positive multiconvex process S, the operation of taking the normal hull conserves the values of both the joint and the generalized spectral radius.*

Proof. As follows from Lemma 4.4, any monotone norm (say, the norm $\|x\| = \sum_{i=1}^{n} |x_i|$) is conserved by the operation of taking the normal hull. Since both values of spectral radii are independent on the particular norm used in $\mathbb{R}^n$, the required statement follows. $\square$

We will also need the following assertion.

Proposition 4.10. *Any closed bounded positive multiconvex relation $S(x)$ is Hausdorff continuous on $\mathbb{R}^n_+$.*

Proof. In the case of a single normal convex relation this result is well known, we give a proof for completeness.

The upper semicontinuity of S at $x \in \mathbb{R}^n_+$ follows from the closedness of S. Suppose now that there exists a point $y \in S(x)$ and a sequence $x_i \to x$ as $i \to \infty$ such that $d(y, S(x_i)) > \varepsilon > 0$, $i = 1, 2, \ldots$, where

$$d(x, A) = \inf_{y \in A} \|x - y\|.$$

Because of upper semicontinuity of S, there exists a $p \in \mathbb{R}^n$, $\|p\| = 1$, such that

$$\max_{z \in S(x)} \langle p, z \rangle > \limsup_{i \to \infty} \max_{z \in S(x_i)} \langle p, z \rangle.$$

This means that the support function $s_p(y) = \max_{z \in S(y)} \langle p, z \rangle$ is not upper semicontinuous at x on the set $\mathbb{R}^n_+$. This is, however, impossible, because $s_p(y)$ is a concave function on a polyhedral set $\mathbb{R}^n_+$ in a finite-dimensional space.

Finally, if each S_j is Hausdorff continuous, then so is the union $S = \cup_{j=1,\ldots,k} S_j$. $\square$

5. Equivalence of spectral radii

Our main goal in this paper is to prove the following result.

Theorem 5.1. *Let $S = \cup_{j=1,\ldots,k} S_j$ be a multivalued process, where all S_j are closed superlinear mappings with $S_j(0) = \{0\}$. Then*

$$\rho(S) = \hat{\rho}(S).$$

Proof. Suppose the contrary, that is,

$$\rho(S) > \hat{\rho}(S).$$

Without loss of generality we can assume that $\rho(S) > 1$ and $\hat{\rho}(S) < 1$; this can be achieved by considering the process αS instead of S for an appropriate $\alpha > 0$. Owing to Proposition 4.9, we can also assume S to be normal.

Now, let us consider the set $\mathcal{T}$ of all finite trajectories $X = \{x_1, \ldots, x_m\}$ of S such that $\|x_1\| = 1$. By assumption, for any $N > 0$, there exists a trajectory X in $\mathcal{T}$ such that $\|x_m\| \geq N$. Since $\mathbb{R}^n_+$ has only a finite number of faces, we can assume that all $x_1 \in \mathrm{ri}(F)$, where F is a non-zero face of $\mathbb{R}^n$. Because of normality of S, we can choose $F = \mathbb{R}^n_+$; however, this will not be the case at the subsequent steps of the iteration procedure, see below.

For $x, y \in \mathbb{R}^n_+$, let us say that x *dominates* y (denote $x \succ y$) if and only if y belongs to the face of $\mathbb{R}^n_+$ generated by x (see, for instance, [7]). Equivalently, $x \succ y$ if and only if $\alpha x \geqq y$ for some $\alpha > 0$.

Consider the set F' of all $x \in \mathbb{R}^n$ such that some trajectory from x has a node y dominating all $z \in \mathrm{ri}(F)$. More precisely, $x \in F'$ if and only if there exists an $m \geq 0$ and a $y \in S^m(x)$ such that

$$y \succ z \quad \text{for each } z \in \mathrm{ri}(F). \tag{9}$$

Here we define $S^0(x) = \{x\}$. It follows directly from the definition that $x \notin F'$ implies $S(x) \cap F' = \emptyset$.

Let us demonstrate that F' is relatively open in $\mathbb{R}^n_+$. Indeed, suppose $x \in F'$, that is, (9) holds for some $m \geq 0$. The map $S(\cdot)$ is Hausdorff continuous on $\mathbb{R}^n_+$ (Proposition 4.10) and, hence, $S^m(\cdot)$ is also Hausdorff continuous on $\mathbb{R}^n_+$. Therefore, relation (9) holds for all x' in some neighborhood of x.

Denote $G = \mathbb{R}^n_+ \backslash F'$. The set G is closed and invariant for S, that is, $S(x) \subset G$ for any $x \in G$. Moreover, G is a union of some faces of $\mathbb{R}^n_+$. Indeed, due to monotonicity of S, we have $y \in G$ whenever $x \in G$ and $x \succ y$.

Let us now prove that, for any $\varepsilon > 0$, there exists an $N_\varepsilon > 0$ such that $d(\frac{x_m}{\|x_m\|}, G) < \varepsilon$ whenever x_m is the terminal point of a trajectory $X \in \mathcal{T}$ and $\|x_m\| > N_\varepsilon$.

Indeed, let us suppose the contrary. Then, for some $\varepsilon > 0$, there exists a sequence $x^i_{m_i}$ of terminal points of trajectories from $\mathcal{T}$ such that $\lim_{i\to\infty} \|x^i_{m_i}\| = +\infty$ and

$$d\left(\frac{x^i_{m_i}}{\|x^i_{m_i}\|}, G\right) > \varepsilon.$$

Choosing a subsequence, if necessary, we get

$$\lim_{i\to\infty} \frac{x^i_{m_i}}{\|x^i_{m_i}\|} = x \in F'.$$

Since (9) holds, there exists an $\alpha > 0$ such that

$$\alpha y \geq w, \quad \text{for all} \quad w \in F,\ \|w\| = 1.$$

Thus, for some neighborhood V of x and for any $x' \in V$, there exists a $y' \in S^m(x')$ such that

$$2\alpha y' \geq w, \quad \text{for all} \quad w \in F, \ \|w\| = 1.$$

Finally, if we choose $x^i_{m_i}$ such that $\|x^i_{m_i}\| > 2\alpha$ and $\frac{x^i_{m_i}}{\|x^i_{m_i}\|} \in V$, then there exists a trajectory $x^i_1, \ldots, x^i_{m_i}, y_1, \ldots, y_m$ of S such that $y_m \geq x^i_1$. Since S is assumed to be normal, we can replace y_m by x^i_1 and, thus, construct a nontrivial periodic trajectory of S, which is a contradiction to the assumption $\hat{\rho}(S) < 1$.

Let us prove that $\rho(S) \geq 1$, where

$$\rho_G(S) = \limsup_{m \to \infty} (\|S^m\|_G)^{\frac{1}{m}}$$

and

$$\|S\|_G = \sup_{x \in G, \ \|x\|=1, \ y \in S(x)} \|y\|.$$

Indeed, otherwise there exists an $m' > 0$ such that

$$\|y\| < \frac{1}{2}\|x\| \quad \text{for all} \quad x \in G, \ y \in S^{m'}(x). \tag{10}$$

Since the mapping S^m is Hausdorff continuous and the set $G_1 = \{x \in G : \|x\| = 1\}$ is compact, there exists an $\varepsilon > 0$ such that inequality (10) holds also for all x satisfying $d(\frac{x}{\|x\|}, G) < \varepsilon$. Let $\{x_1, \ldots, x_m\}$ be a trajectory from $\mathcal{T}$ and let x_p be the first node of this trajectory such that $\|x_p\| \geq N_\varepsilon\|$ (if it exists). We have $\|x_p\| \leq \|S\|N_\varepsilon$ and

$$d(\frac{x_p}{\|x_p\|}, G) < \varepsilon.$$

Then, from (10) we derive the inequality

$$\|x_{m'}\| \leq \frac{1}{2}\|S\|^{m+1}N_\varepsilon$$

for any trajectory in $\mathcal{T}$. This is, however, a contradiction with the assumption $\rho(S) > 1$.

Now, using multiplicative coefficients, if necessary, we conclude that $\rho(S) = \rho_G(S)$. Analogously, $\hat{\rho}_G(S)$ is defined, and, obviously, the inequality $\hat{\rho}_G(S) \leq \hat{\rho}(S)$ holds.

Next, we repeat the argument verbatim for S on G instead of $\mathbb{R}^n_+$ and either come to a contradiction or find an invariant closed set $G' \subset G$, $G' \neq G$, such that, again, G' consists of whole faces of $\mathbb{R}^n_+$ and $\rho_{G'}(S) = \rho_G(S)$. This process must terminate at some step because the number of faces of $\mathbb{R}^n_+$ is finite, and, finally, we come to a contradiction. $\square$

6. Duality

The notion of the *conjugate process* to a convex process can be extended in a natural way to multiconvex processes. Let us recall that, for a given superlinear normal mapping S with $S(0) = \{0\}$ its conjugate S^* is defined by the relation

$$S^* = \{(x^*, y^*) \in \mathbb{R}^*_n \times \mathbb{R}^n_* = |\langle x, x^* \rangle \geq \langle y, y^* \rangle \quad \text{for all} \quad (x, y) \in S\},$$

The conjugate mapping is superlinear and *stable*, that is,

$$S^*(y^*) + \mathbb{R}^n_+ = S^*(y^*) \quad \text{for any} \quad y^* \in \mathbb{R}^n.$$

The inverse mapping $S' = (S^*)^{-1}$ is called *dual* with respect to S.The mapping S' is superlinear and $S'(0) = \{0\}$.

Definition 6.1. The conjugate process to a multiconvex process $S = \cup S_j$, $j = 1, \dots, k$, is the union $S^* = \cup S_j^*$, $j = 1, \dots, k$.

We are going to link together spectral radii of the direct and the conjugate processes. We define both the joint and the generalized spectral radii of the conjugate process using an alternative definition of norm [10, 14]. Namely, for a stable mapping S^*,

$$\|S^*\| = \sup_{x \in \mathbb{R}^n_+, \|x\|=1} \inf_{y \in S^*(x)} \|y\|. \tag{11}$$

Let us recall the formula [10]

$$(AB)^* = B^* A^*. \tag{12}$$

Next, we have to restrict the class of multiconvex processes under consideration in order to use the rich duality theory developed for superlinear mappings ([10, 7].

Definition 6.2. A normal superlinear mapping S with $S(0) = \{0\}$ is *nonsingular* if $S(\mathbb{R}^n_+) \cap \operatorname{int}(\mathbb{R}^n_+) \neq \emptyset$. A multiconvex process $S = \cup S_j$ is nonsingular if all S_j are nonsingular convex processes.

It is clear that a normal superlinear mapping S_j with $S_j(0) = 0$ is singular if and only if there exists a face F of $\mathbb{R}^n_+$, other than $\mathbb{R}^n_+$ itself, such that $S(x) \subseteq F$ for any $x \in \mathbb{R}^n_+$. An obvious assertion follows:

Proposition 6.3. *For a nonsingular multiconvex process $S = \cup S_j$, $j = 1, \dots, k$, all the products $S_{j_m} \dots S_{j_1}$ are nonsingular.*

For nonsingular processes, the equality $\|S\| = \|S^*\|$ holds, and also

$$\overline{\lambda}(S) = \overline{\lambda}(S^*),$$

where

$$\overline{\lambda}(S) = \sup\{\lambda > 0 | \, \lambda x \in S(x) \ \text{ for some nonzero } \ x \geqq 0\}$$

(the von Neumann rate of growth) and

$$\overline{\lambda}(S^*) = \inf\{\lambda > 0 | \, \lambda x \in S^*(x) \text{ for some } x \gg 0\}$$

(here $x \gg 0$ means $x_i > 0$, $i = 1, \dots, k$).

Let us recall a classical theorem by R. T. Rockafellar [10]:

Theorem 6.4. *Let S be a normal superlinear mapping with $S(0) = \{0\}$. Then*

$$\overline{\lambda} = \lim_{m \to \infty} \|P^m\|^{\frac{1}{m}} \leqq \|P\|$$

for both $P = S$ and $P = S^*$.

As an obvious consequence, we get

Corollary 6.5. *The generalized spectral radius of a normal compact-valued positive multiconvex process is equal to that of its conjugate. The same is true for the joint spectral radius.*

A stable superlinear mapping Q is called nonsingular if $0 \in Q(x)$ only for $x = 0$. Recall that we use the norm (11) for stable convex and multiconvex processes. Now, the central result of this section follows directly from Theorem 5.1:

Theorem 6.6. *The joint spectral radius $\rho(Q)$ of a nonsingular stable multiconvex process Q coincides with its generalized spectral radius $\hat{\rho}(Q)$.*

Proof. It suffices to notice that any nonsingular stable multiconvex process Q is a conjugate to some nonsingular normal compact-valued positive multiconvex process S; this follows from the same result for stable superlinear mappings [10]. □

Remark 6.7. We consider in this section only some simple results linking multiconvex relations generated by superlinear mappings with conjugate multiconvex relations. Let us mention some more advanced problems, which will be a topic of further research

(1) Much more interesting and complicated is the situation with singular processes and their conjugate. The theory of rates of growth of a normal superlinear mapping S with $S(0) = \{0\}$ can be found in [7]. In particular it follows from results of [7] that the von Neumann rate of growth of a dual mapping S' coincides with the so-called economical rate of growth of the initial mapping S. These results can be extended for multiconvex relations generated by superlinear mappings.

(2) The following definition [14, 8] plays a crucial role in construction of efficient trajectories generated by a normal superlinear mapping S with $S(0) = \{0\}$. A function q defined on $\mathbb{R}^n_+$ is called an efficient function of a normal superlinear mapping S with $S(0) = \{0\}$ if there exists a number α such that $\max\{q(y) : y \in S(x)\} = \alpha q(x)$ for all $x \in \mathbb{R}^n_+$. Efficient functions are closely related to congugate mappings. Their analogues can be defined and studied for multiconvex relations generated by superlinear mappings.

7. Economic interpretation

Positive convex relations are often used for modeling the production activity in mathematical economics. In particular, the well-known von Neumann type model (in other terms, the von Neumann-Gale model; see, for example, [7] and references therein) is described by a superlinear mapping. A pair $(x, y) \in S$, where S is a positive convex relation, is called a technological process with input x and output y. Assume that the producer posesses several positive convex relations $S_1, \ldots, S_m$, which describe different technologies. An input vector x is called infinitely divisible between production mappings $S_1, \ldots, S_m$, if every part of x can be used as an input vector for each of these mappings. If all inputs are infinitely divisible then the production activity of the producer can be described by the convolution S of mappings $S_1, \ldots, S_m$:

$$S(x) = \left\{ \sum_{i=1}^{m} S(x_i) : \sum_{I=1}^{m} x_i = x,\ x_i \geq 0,\ i = 1, \ldots, m \right\}.$$

If $S_1, \dots, S_m$ are superlinear mappings then S is a superlinear mapping as well. A dynamical model generated by the mapping S has been studied in detail (see [8] and references therein).

The alternative to infinite divisibility is a complete indivisibility: each input vector can be used only by one of the mapping $S_1, \dots, S_m$. Let us give the simplest example of a complete indivisibility. Assume that a producer has two technologies, both of them are based on the same unit of equipment and the producer has only one such unit. If this unit is used for one of technologies, it cannot be used for the other. Thus the producer must choose one of these technologies every time. The case of complete indivisibility is studied in this paper.

Models, which are intermediate between models with infinitely divisible and completely indivisible inputs, require a special invistigation. Clearly, this investigation will be based on the the technique, which was developed in the study of both models with infinitely divisible and completely indivisible inputs.

Theorem 5.1 shows that the maximal rate of growth of the economy with completely indivisible inputs can be approximated by growth rates of perodic finite sequences of component processes, that is, it is possible to achieve any suboptimal rate of growth by choosing some periodic rule of changing technologies. All these periodic sequences consist of superlinear mappings and so the theory of von Neumann type production models (see [7, 14, 8] and references therein) can be applied.

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