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Quasi Convex Integrands and Lower Semicontinuity in BV

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We prove a lower semicontinuity theorem, in BV setting, for multiple integrals of the calculus of variations with quasi convex integrands. The key result is a deep analysis on the behaviour of an L_1–convergent sequence in BV. More precisely, we links up a local mean-value convergence of the gradients with the local oscillation of the surfaces and a suitable localization of a sequential Jensen's-type inequality. The present result extends to BV setting the lower semicontinuity theorem due to Fonseca-Müller [26] and improves our previous result given in [7] for convex integrands.

1. Introduction

We discuss here the lower semicontinuity of multiple integrals of the calculus of variations

$$\int_\Omega F(x, u(x), \mathcal{D}u(x))\, dx \tag{1}$$

with respect to L_1–convergence in BV setting, for quasi convex integrands. Here $\mathcal{D}x$ denotes the "essential gradient" of the BV function u, i.e. the density of the absolutely continuous part of the distributional derivative with respect to Lebesgue measure.

For a survey on the lower semicontinuity of quasi-convex integrands in Sobolev's spaces we refer to Dacorogna [23], where also a wide list of references can be found.

More recently, integral functional with quasi-convex integrands was studied (in various settings), among the others, by Ambrosio - Dal Maso [4], Fonseca - Müller [26, 27], Ambrosio [3], Malý [29] and Fonseca - Leoni [25].

The approach we propose in the present paper is based on two main results.

The first deals with the behaviour of the gradients of an L_1–convergent sequence (Lemma 2 in [19], see also Proposition 3.7 in [7]):

a sequence $u_k : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^\nu$, $k \in \mathbb{N}$, *in* $W^{1,1}$ *which* L_1*–converges to a* BV *function* u_0 *satisfies the following mean-value condition*

$$\lim_{h\to 0}\lim_{k\to\infty} \fint_{B(x_0,h)} \mathcal{D}\, u_k(x)\, dx = \mathcal{D}\, u_0(x_0) \qquad a.e. \text{ in } \Omega \tag{mv}$$

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where

$$\fint_{B(x_0,h)} u(x)\,dx = [\text{meas}(B(x_0,h))]^{-1} \int_{B(x_0,h)} u(x)\,dx.$$

We wish to recall that *mean-value* condition (mv) revealed a key property in order to deal with lower semicontinuity in BV setting.

The second important result is our characterization of the lower semicontinuity of a sequence of integrals $\int_\Omega f_k(x)\,dx$, $k \in \mathbb{N}$ expressed by means of the following local condition (called *lower mean-value*):

$$\liminf_{h\to 0} \liminf_{k\to\infty} \fint_{B(x_0,h)} f_k(x)dx \ge f_0(x_0) \qquad \text{a.e. in } \Omega. \tag{lmv}$$

By virtue of the (mv)-condition of the gradients, a specific characterization for integrals of type (1) can be deduced from this general result, in terms of a suitable localization of a sequential Jensen's-type inequality (see Theorems 4.3, 4.4).

$$\liminf_{h\to 0} \liminf_{k\to +\infty} \left\{ \fint_{B(x_0,h)} F(x, u_0(x_0), \mathcal{D}u_k(x))\,dx - F\left(x_0, u_0(x_0), \fint_{B(x_0,h)} \mathcal{D}u_k(x)\,dx\right)\right\} \ge 0. \tag{Js}$$

In the present research we analyze the behaviour of a sequence in BV thoroughly. Precisely, we get the following result which links up (mv)–condition on the gradients with the local oscillation of the surfaces and (Js)–inequality under mild assumptions on the integrand (see Lemma 5.2).

Lemma 1.1. *Assume that $(u_k)_k$ is a sequence in $W^{1,\infty}$ which has equibounded variation and L_1–converges to a BV function u_0. Then for a.e. $x_0 \in \Omega$ there exists a sequence $(U_{h,k})_{k\in\mathbb{N}}$ of subsets in $B(x_0,h)$ such that*

(1) $\displaystyle\lim_{h\to 0} \lim_{k\to\infty} \frac{\text{meas}\,(U_{h,k})}{\text{meas}\,(B(x_0,h))} = 1;$

(2) $\displaystyle\lim_{h\to 0} \lim_{k\to\infty} \sup_{x\in U_{h,k}} |u_k(x) - u_0(x_0)| = 0;$

(mv) $\displaystyle\lim_{h\to 0} \lim_{k\to\infty} \fint_{U_{h,k}} \mathcal{D}u_k(x)\,dx = \mathcal{D}u_0(x_0).$

Moreover if we assume that $F : \mathbb{R}^{\nu n} \to \mathbb{R}$ is quasi convex and $0 \le F(v) \le C(1 + |v|)$, $v \in \mathbb{R}^{\nu n}$, then the following Jensen's-type inequality holds

$$\liminf_{h\to 0} \liminf_{k\to\infty} \left[\fint_{U_{h,k}} F\left(\mathcal{D}u_k(x)\right)\,dx - F\left(\fint_{U_{h,k}} \mathcal{D}u_k(x)\,dx\right)\right] \ge 0. \tag{Js}$$

As an application of this lemma, we prove the following lower semicontinuity theorem.

Theorem 1.2 (Main result). *Let Ω be a bounded open set and let $A \subset \mathbb{R}^{\nu+n}$ be closed. Assume that $(u_k)_{k\in\mathbb{N}}$ is a sequence in $W^{1,1}(\Omega,\mathbb{R}^n)$ such that*

(i) $u_k(x) \in A$ *a.e.*, $k \in \mathbb{N}$;

(ii) $(u_k)_{k\in\mathbb{N}}$ *has equibounded variation and* L_1*–converges to some* u_0 *that belongs to* $BV(\Omega,\mathbb{R}^n)$.

Let $F:\Omega\times A\times\mathbb{R}^{\nu n}\to\mathbb{R}$ *be a Carathèodory function such that for a.e.* $x_0\in\Omega$ *the following conditions are satisfied*

(iii) $F(\cdot,\cdot,v)/1+|v|$ *is lower semicontinuous in* $(x_0,u_0(x_0))$*, uniformly with respect to* v;

(iv) $F(x_0,u_0(x_0),\cdot)$ *is quasi convex;*

(v) $0\le F(x_0,u_0(x_0),v)\le C(1+|v|),\quad v\in\mathbb{R}^{\nu n}$.

Then $u_0(x)\in A$, *a.e. and*

$$\liminf_{k\to\infty}\int_\Omega F(x,u_k(x),\mathcal{D}u_k(x))\,dx\ge\int_\Omega F(x,u_0(x),\mathcal{D}u_0(x))\,dx.$$

For the sake of comparison with the literature on the subject, we wish to mention that the present result can be considered as an extension to BV-setting of the lower semicontinuity theorem by Fonseca-Müller [26] with an improvement of the assumptions on $F(\cdot,\cdot,v)$.

Moreover the interest of the present research remains even in the particular case of a convex integrand (see Section 6). In fact, for functional of type (1), we can here remove the Lipschitz-type condition we had assumed on $F(x,\cdot,v)$ in [7].

The results of this paper were extended by Comparato [21] to integral functionals

$$\int_\Omega F(x,(\mathcal{U}\,u)(x),(\mathcal{L}\,u)(x))\,dx \tag{2}$$

where $\mathcal{U}$ and $\mathcal{L}$ are continuous operators.

These integrals were already studied in [7] for convex integrands, in BV-setting.

Finally, we wish to mention that the present research finds applications to closure theorems and existence results for optimal control problems ([10, 11]) which are connected with the study of a variational model for the plastic deformation of beams and plates under loads of different types [5, 6, 18, 20, 31, 32, 33].

2. Preliminaries

We denote by $\mathbb{N}$ the set of all integers $k\ge 1$, and by $\mathbb{R}^+$, $\mathbb{R}_0^+$ the set of positive or non-negative real numbers respectively.

Let ν, n and m be given integers. Let $\Omega\subset\mathbb{R}^\nu$ be a bounded open set.

According to standard notations, we denote by $L_1(\Omega,\mathbb{R}^m)$ the space of summable functions $x:\Omega\to\mathbb{R}^m$, by $W^{1,1}(\Omega,\mathbb{R}^m)$ the Sobolev space of functions $x\in L_1(\Omega,\mathbb{R}^m)$ whose distributional derivatives are summable functions, and by $BV(\Omega,\mathbb{R}^m)$ the space of functions $x\in L_1(\Omega,\mathbb{R}^m)$ which are of bounded variation in the sense of Cesari [6a]. Moreover, let $W^{1,\infty}(\Omega,\mathbb{R}^m)$ be the space of functions which are essentially bounded together with their distributional derivatives, let $C_0^\infty(\Omega,\mathbb{R}^m)$ be the space of C^∞ functions with compact support and let $W_0^{1,\infty}(\Omega,\mathbb{R}^m)$ denote the closure of $C_0^\infty(\Omega,\mathbb{R}^m)$ in $W^{1,\infty}(\Omega,\mathbb{R}^m)$.

Let $\mathbb{M}$ denote the space of the measurable functions $f : \Omega \to \mathbb{R}$ whose negative part f^- is summable.

Given a BV function u, we denote by $\mathcal{D}u = \left(\frac{\partial u^i}{\partial x_j},\ i = 1, ..., m,\ j = 1, ..., \nu\right)$ the "essential gradient" i.e. the density of the absolutely continuous part of the distributional derivative with respect to the Lebesgue measure and call $\mathcal{D}u$ the gradient of u.

Given a point $x_0 \in \Omega$ and a constant $h > 0$, we put

$$B_h(x_0) = \{x \in \mathbb{R}^\nu :\ x_{0j} - h \leq x_{0j} \leq x_{0j} + h,\ j = 1, ..., \nu\}.$$

In the case the point x_0 is clearly determined, we briefly write $B_h(x_0) = B_h$.

We will adopt the following notation, given a function $z : \mathbb{R}^+ \times \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^n$ and a point $t \in \mathbb{R}^+$ such that

$$\lim_{h\to 0} \liminf_{k\to\infty} \liminf_{s\to t} z^i(h,k,s) = \lim_{h\to 0} \limsup_{k\to\infty} \limsup_{s\to t} z^i_k(h,k,s) = z^i_0 \qquad i = 1, \ldots, n$$

we briefly put

$$\lim_{h\to 0} \widetilde{\lim_{k\to\infty}} \widetilde{\lim_{s\to t}} z(h,k,s) = z_0.$$

For $\zeta : \mathbb{R}^+ \times \mathbb{N} \to \mathbb{R}^n$, we put

$$\lim_{h\to 0} \widetilde{\lim_{k\to\infty}} \zeta(h,k) = \zeta_0.$$

when similar equalities as above hold.

3. The mean-value and lower mean-value conditions

We recall the definition of mean-value and lower mean-value conditions we introduced in [8, 9] respectively (see also [7]).

Definition 3.1. We say that a sequence $(v_k)_{k\geq 0}$ in $L_1(\Omega, \mathbb{R}^m)$ satisfies the *mean value* (mv) *condition* at a point $x_0 \in \Omega$ provided

$$\operatorname{ess} \lim_{h\to 0} \widetilde{\lim_{k\to\infty}} \fint_{B_h} v_k(x)dx = v_0(x_0). \tag{mv}$$

We say that $(v_k)_{k\geq 0}$ satisfies (mv) *condition on* Ω if (mv) holds at a.e. point $x_0 \in \Omega$.

Definition 3.2. We say that a sequence $(f_k)_{k\geq 0}$ in $\mathbb{M}$ satisfies the *lower mean value* (lmv) *condition* at a point $x_0 \in \Omega$ provided

$$\operatorname{ess\ liminf}_{h\to 0} \liminf_{k\to\infty} \fint_{B_h} f_k(x)dx \geq f_0(x_0). \tag{lmv}$$

We say that $(f_k)_{k\geq 0}$ satisfies (lmv) *condition on* Ω if (lmv) holds at a.e. point $x_0 \in \Omega$.

Of course (mv) implies (lmv), moreover we recall some important results on (mv) conditions that will be used in what follows (for the detail and other results see [8]).

Proposition 3.3. *If* $v_k \rightharpoonup v_0$ *weakly in* $L_1(\Omega, \mathbb{R}^m)$, *then the sequence* $(v_k)_{k\geq 0}$ *satisfies* (mv) *on* Ω.

The converse is not true in general (see [7, Remark 3.5]).

Proposition 3.4. *Let* $(u_k)_{k\in\mathbb{N}}$ *be a sequence in* $W^{1,1}(\Omega, \mathbb{R}^n)$ *which* L_1*–converges to a function* $u_0 \in BV(\Omega, \mathbb{R}^n)$.

Then there exists a subsequence of the gradients $(\mathcal{D}u_{s_k})_{k\geq 0}$ *which satisfies* (mv) *on* Ω.

Following the proof of Theorem 5.1 in [7], the following result can be proved.

Lemma 3.5. *Let* $(v_k)_{k\in\mathbb{N}}$ *be a bounded sequence in* $L_1(\Omega, \mathbb{R}^m)$. *Then, for a.e.* $x_0 \in \Omega$

$$\operatorname*{ess\,lim}_{h\to 0} \limsup_{k\to\infty} \fint_{B_h} |v_k(x)|\, dx < +\infty.$$

Proof. Let $I = [a, b]^\nu = \Pi_{i=1}^\nu [a^i, b^i] \supset \Omega$ be a given interval. Let us consider the sequence $\phi_k : I \to \mathbb{R}$, $k \in \mathbb{N}$ defined by

$$\phi_k(x) = \int_{[a,x]^\nu} |v_k(\xi)|\, d\xi.$$

Note that the functions $(\phi_k)_{k\in\mathbb{N}}$ are absolutely continuous in the sense of Vitali and have equi-bounded Vitali variation. Thus, by Helly's theorem, there exists a function $\phi_0 : I \to \mathbb{R}$ which has bounded Vitali variation and such that (for a suitable subsequence)

$$\phi_k \longrightarrow \phi_0 \qquad \text{pointwise.}$$

By virtue of Proposition 3.8 in [7], we get that the superficial derivatives $(D^*\phi_k)_{k\geq 0}$ satisfy (mv) condition in $[a, b]^\nu$. Since $D^*\phi_k(x) = |v_k(x)|$ a.e. in Ω, the lemma follows immediately. □

4. Characterizations of lower semicontinuity

The main property of (lmv) condition is the following general characterization of lower semicontinuity (see Theorem 11 in [9]).

Theorem 4.1. *Let* $(f_k)_{k\in\mathbb{N}}$ *be a sequence in* $\mathbb{M}$ *and assume that there exists a function* $\lambda \in L_1$ *such that* $f_k(x) \geq \lambda(x)$, *a.e. in* Ω.

Then the following conditions are equivalent

(i) $(f_k)_{k\geq 0}$ *satisfies* (lmv) *on* Ω;

(ii) *for every measurable set* $E \subset \Omega$, *which has nonempty interior and boundary with null measure, the lower semicontinuity condition holds*

$$\liminf_{k\to\infty} \int_E f_k(x)\, dx \geq \int_E f_0(x)\, dx.$$

Let us introduce the following generalization of Jensen's inequality.

Definition 4.2. We shall say that a function $f : \mathbb{R}^m \to \mathbb{R}$ satisfies the *sequential localized Jensen's inequality* at the point $x_0 \in \Omega$ with respect to a sequence $(v_k)_{k\in\mathbb{N}}$ in $L^1(\Omega, \mathbb{R}^m)$ provided

$$\operatorname*{ess\,liminf}_{h\to 0} \liminf_{k\to\infty} \left\{ \fint_{B_h} f(v_k(x))\, dx - f\left(\fint_{B_h} v_k(x)\, dx \right) \right\} \geq 0. \tag{Js}$$

We shall say that f satisfies (Js) on Ω provided (Js) holds at a.e. point $x_0 \in \Omega$.

The following result is an easy consequence of Theorem 4.1.

Theorem 4.3. *Assume that* $(v_k)_{k\in\mathbb{N}}$ *in* $L^1(\Omega, \mathbb{R}^m)$ *satisfies* (mv) *on* Ω *and* $f : \mathbb{R}^m \to \mathbb{R}$ *is continuous in* $v_0(x_0)$.

Then the following conditions are equivalent

(i) *the sequence* $f_k = f(v_k(\cdot))$, $k \geq 0$, *satisfies* (lmv) *in* $x_0 \in \Omega$;
(ii) f *satisfies* (Js) *in* $x_0 \in \Omega$ *with respect to the sequence* $(v_k)_{k\in\mathbb{N}}$.

Proof. Note that (mv) condition and the continuity of f ensure that for a.e. $x_0 \in \Omega$

$$\operatorname*{ess\,lim}_{h\to 0} \widetilde{\lim_{k\to\infty}} f\left(\fint_{B_h} v_k(x)\, dx \right) = f(v_0(x_0)).$$

□

Condition (Js) is trivially satisfied by convex integrands. For quasi convex integrands (see [23]) the following result holds.

Theorem 4.4. *Assume that* $f : \mathbb{R}^{\nu n} \to \mathbb{R}$ *satisfies the assumptions*

(i) *it is quasi convex;*
(ii) $f(v) \leq C(1 + |v|), \quad v \in \mathbb{R}^{\nu n}$.

Let $(u_k)_{k\in\mathbb{N}}$ *be a sequence in* $W^{1,1}(\Omega, \mathbb{R}^n)$ *and let* $u_0 \in BV(\Omega, \mathbb{R}^n)$ *be such that*

(iii) $(u_k)_{k\in\mathbb{N}}$ *has equibounded variation and* L_1*−converges to* u_0.

Then f *satisfies* (Js) *in* Ω *with respect to the sequence* $(\mathcal{D}u_k)_{k\in\mathbb{N}}$.

We omit the proof since this result can also be considered as a corollary of main Theorem 5.3, by virtue of Theorems 4.1 and 4.3.

5. The main lower semicontinuity result

Before stating the main lower semicontinuity result, let us prove two lemmas that will be usefull in what follows.

Lemma 5.1. *Let* $B = B(x_0, r) \subset \Omega$ *be a given ball, let* $(u_k)_{k\in\mathbb{N}}$ *be a sequence in* $W^{1,\infty}(B, \mathbb{R}^n)$ *and let* $u_0 \in BV(B, \mathbb{R}^n)$ *be such that*

(i) $\mathcal{D}u_0(x_0)$ *exists and* $\lim_{h\to 0} \fint_{B_h} \frac{|u_0(x)|}{h}\, dx = 0$;

(ii) $\sup_{k\in\mathbb{N}} \int_B |\mathcal{D}u_k(x)|\,dx = W < +\infty.$

Then there are three functions $\alpha,\ \beta :]0,\overline{h}[\to \mathbb{R}^+$ *and* $t :]0,\overline{h}[\times\mathbb{N} \to \mathbb{R}^+,$ *with* $0 < \overline{h} \le r\nu^{-\frac{1}{2}}$ *such that*

(1) $\alpha(h) \le t(h,k) \le \beta(h)$ *for every* $(h,k) \in]0,\overline{h}[\times\mathbb{N}$;

(2) $\lim_{h\to 0} \fint_{B_h} \frac{|u_0(x)|}{\alpha(h)}\,dx = 0 \qquad \lim_{h\to 0} \frac{\beta(h)}{h} = 0 \qquad \lim_{h\to 0} \frac{\beta(h)}{\alpha(h)} = +\infty.$

Moreover, for every $(h,k) \in]0,\overline{h}[\times\mathbb{N}$ *and every* $\alpha(h) \le s < t = t(h,k)$ *there exists a function*

$$w^k_{s,t} : B \to \mathbb{R}^n \quad \textit{in} \quad W^{1,\infty}(B,\mathbb{R}^n) \quad \textit{such that}$$

$$w^k_{s,t}(x) = u_k(x) \quad \textit{in} \quad \{x:\ |u_k(x)| \le s\} \qquad w^k_{s,t}(x) = 0 \quad \textit{in} \quad \{x:\ |u_k(x)| \ge t\}$$

and with the property that

(3) $\operatorname{ess\,sup}_{x\in B} |w^k_{s,t}(x)| \le t$ *for every* $0 < h < \overline{h},\quad k \in \mathbb{N}$;

(4) $\operatorname{ess\,lim}_{h\to 0}\ \widetilde{\lim}_{k\to+\infty}\ \widetilde{\lim}_{s\to t} \fint_{B_h} \mathcal{D}w^k_{s,t}(x)\,dx = 0$;

(5) $\operatorname{ess\,lim}_{h\to 0}\ \widetilde{\lim}_{k\to+\infty}\ \widetilde{\lim}_{s\to t^-} [(\operatorname{meas}(B_h))]^{-1} \int_{B_h\cap\{x:\ s\le|u_k(x)|\le t\}} |\mathcal{D}w^k_{s,t}(x)|\,dx = 0.$

Proof. Let $0 < \overline{h} \le r\nu^{-\frac{1}{2}}$ be fixed and put $\mathcal{D}_0 = \mathcal{D}u_0(x_0)$.
Denote by $\Theta :]0,\overline{h}[\to \mathbb{R}^+$ the function

$$\Theta(h) = h^2 + \fint_{B_h} |u_0(x)|\,dx$$

and consider the functions $\alpha, \beta :]0,\overline{h}[\to \mathbb{R}^+$ defined by

$$\alpha(h) = \sqrt{h\,\Theta(h)} \qquad\qquad \beta(h) = \sqrt[3]{h^2\,\Theta(h)}.$$

By virtue of assumption (i) we have

$$\lim_{h\to 0} \frac{\beta(h)}{h} = 0 \tag{5.1}$$

$$\lim_{h\to 0} \frac{\Theta(h)}{\alpha(h)} = 0 \tag{5.1'}$$

$$\lim_{h\to 0} \frac{\beta(h)}{\alpha(h)} = +\infty \tag{5.1''}$$

hence condition (2) holds moreover, it is not restrictive to assume that

$$0 < \alpha(h) < \beta(h) < h \qquad\qquad \text{for every } h \in]0,\overline{h}]. \tag{5.2}$$

Let $k \in \mathbb{N}$ and $h \in]0,\overline{h}]$ be fixed.

For every $t \in [\alpha(h), \beta(h)]$ and every $0 < s < t$, let $\phi_{s,t} \in \mathcal{C}_0^\infty([0,1], \mathbb{R}_0^+)$ be a cut off function such that

$$\phi_{s,t}(\zeta) = 1 \qquad \text{if} \quad 0 \le \zeta \le s \qquad\qquad \phi_{s,t}(\zeta) = 0 \qquad \text{if} \quad \zeta \ge t$$

$$\operatorname*{ess\,sup}_{\zeta \in [0.1]} |\phi'_{s,t}(\zeta)| \le \frac{C}{t-s},$$

where C is a constant. Let $w_{s,t}^k : B \to \mathbb{R}^n$ be the function defined by

$$w_{s,t}^k(x) = \phi_{s,t}(|u_k(x)|) \cdot u_k(x).$$

Note that $w_{s,t}^k \in W^{1,\infty}(B, \mathbb{R}^n)$ and a.e. in B we have

$$|w_{s,t}^k(x)| \le t \tag{5.3}$$

$$|\mathcal{D} w_{s,t}^k(x)| \le \frac{C}{t-s} \, |\mathcal{D}|u_k(x)|| \cdot |u_k(x)| + |\mathcal{D} u_k(x)| \tag{5.4}$$

$$\mathcal{D} w_{s,t}^k(x) = \mathcal{D} u_k(x) \qquad \text{if } |u_k(x)| < s, \qquad\qquad \mathcal{D} w_{s,t}^k(x) = 0 \qquad \text{if } |u_k(x)| > t. \tag{5.5}$$

Since $w_{s,t}^k \in W^{1,1}(B, \mathbb{R}^n)$, for a.e. $0 < h' < h$ and $i = 1, \dots, n$, $j = 1, \dots, \nu$

$$\int_{B_{h'}} \mathcal{D} w_{s,t}^{k,i}(x)\, dx = \int_{B_{h'}^j} \left[w_{s,t}^{k,i}(x_0^j - h', \xi) - w_{s,t}^{k,i}(x_0^j + h', \xi) \right] d\xi$$

where $B_{h'}^j = \prod_{l=1,\dots,\nu,\ l \ne j} [x_0^l - h', x_0^l + h']$ and taking (5.3) into account of we get

$$\left| \fint_{B_{h'}} \mathcal{D} w_{s,t}^{k,i}(x)\, dx \right| \le \frac{\operatorname{meas}(B_{h'}^j)}{\operatorname{meas}(B_{h'})}\, 2t \le \frac{t}{h'} \le \frac{\beta(h')}{h'}.$$

Thus by the arbitrariness of h' we have

$$\left| \fint_{B_h} \mathcal{D} w_{s,t}^{k,i}(x)\, dx \right| \le \frac{\beta(h)}{h} \tag{5.6}$$

which gives (4) by virtue of (5.1).

Let us prove now that , for a.e. $t \in [\alpha(h), \beta(h)]$, we have

$$\lim_{s \to t} \int_{B_h \cap \{x:\ s \le |u_k(x)| \le t\}} |\mathcal{D} u_k(x)|\, dx = 0 \tag{5.7}$$

$$\begin{aligned} \limsup_{s \to t} \frac{1}{t-s} \int_{B_h \cap \{x:\ s \le |u_k(x)| \le t\}} & |\mathcal{D} u_k(x)| \cdot |u_k(x)|\, dx \le \\ & \le t H_{\nu-1}\left(\{x \in B_h :\ |u_k(x)| = t\}\right) \end{aligned} \tag{5.8}$$

where $H_{\nu-1}$ denotes the Hausdorff measure.

Note that, since $\mathcal{D}u_k$ is bounded and u_k is summable in B_h, then we get respectively

$$\int_{B_h \cap \{x:\ s \le |u_k(x)| \le t\}} |\mathcal{D}u_k(x)|\, dx \le C_k \cdot \text{meas}\,(B_h \cap \{x:\ s \le |u_k(x)| \le t\})$$

where C_k is a constant depending on k, and

$$\lim_{s\to t} \text{meas}\,(B_h \cap \{x:\ s \le |u_k(x)| \le t\}) = 0 \qquad \text{for a.e. t}$$

which gives (5.7); moreover, the coarea formula (see [34]) ensures

$$\int_s^t H_{\nu-1}\left(\{x \in B_h:\ |u_k(x)| = \tau\}\right) d\tau = \int_{B_h \cap \{x:\ s \le |u_k(x)| \le t\}} |\,\mathcal{D}|u_k(x)|\,|\, dx$$

hence we deduce

$$\frac{1}{t-s}\int_{B_h \cap \{x:\ s \le |u_k(x)| \le t\}} |\,\mathcal{D}|u_k(x)|\,| \cdot |u_k(x)|\, dx \le \fint_s^t t\, H_{\nu-1}\left(\{x \in B_h:\ |u_k(x)| = \tau\}\right) d\tau$$

and (5.8) follows by Lebesgue density theorem.

Note that still from the coarea formula, putting

$$\lambda = \underset{t\in[\alpha(h),\beta(h)]}{\text{ess inf}}\ tH_{\nu-1}\left(\{x \in B_h:\ |u_k(x)| = t\}\right),$$

we have

$$\int_{B_h} |\,\mathcal{D}|u_k(x)|\,|\, dx = \int_{\mathbb{R}_0^+} H_{\nu-1}\left(\{x \in B_h:\ |u_k(x)| = \tau\}\right) d\tau \ge$$

$$\ge \int_{\alpha(h)}^{\beta(h)} \frac{\lambda}{\tau}\, d\tau = \lambda\, \log \frac{\beta(h)}{\alpha(h)}. \quad (5.9)$$

Now, let $t = t(h,k) \in [\alpha(h), \beta(h)]$ be choosen in such a way that (5.7), (5.8) hold and (see (5.9)) in such a way that

$$\lambda \le tH_{\nu-1}\left(\{x \in B_h:\ |u_k(x)| = t\}\right) \le \int_{B_h} |\,\mathcal{D}|u_k(x)|\,|\, dx \,/\, \log \frac{\beta(h)}{\alpha(h)}.$$

Passing to the limits for $h \to 0$ and $k \to +\infty$ and taking Lemma 3.5 and (5.1") into account, we obtain

$$\underset{h\to 0}{\text{ess lim}} \limsup_{k\to+\infty} [\text{meas}(B_h)]^{-1}\ t\ H_{\nu-1}\ \left(\{x \in B_h:\ |u_k(x)| = t\}\right) \le$$

$$\le \underset{h\to 0}{\text{ess lim}} \limsup_{k\to+\infty} \left[\fint_{B_h} |\,\mathcal{D}|u_k(x)|\,|\, dx \,/\, \log \frac{\beta(h)}{\alpha(h)}\right] = 0. \quad (5.10)$$

From (5.8) and (5.10) we deduce that

$$\underset{h\to 0}{\text{ess lim}}\ \widetilde{\lim_{k\to+\infty}}\ \widetilde{\lim_{s\to t}}\ \frac{[\text{meas}(B_h)]^{-1}}{t-s} \int_{B_h \cap \{x:\ s \le |u_k(x)| \le t\}} |\mathcal{D}|u_k(x)|\,| \cdot |u_k(x)|\, dx = 0. \quad (5.11)$$

Moreover, by (5.4) it follows

$$\int_{B_h\cap\{x:\ s\le|u_k(x)|\le t\}} \left|\mathcal{D}w^k_{s,t}(x)\right|\,dx \le$$
$$\le \frac{C}{t-s}\int_{B_h\cap\{x:\ s\le|u_k(x)|\le t\}} |\,\mathcal{D}|u_k(x)|\,|\cdot|u_k(x)|\,dx + \int_{B_h\cap\{x:\ s\le|u_k(x)|\le t\}} |\mathcal{D}u_k(x)|\,dx$$

and by virtue of (5.11) and (5.7), we have (5). □

Lemma 5.2. *Let* $(u_k)_{k\in\mathbb{N}}$ *be a sequence in* $W^{1,\infty}(\Omega,\mathbb{R}^n)$ *and let* $u_0\in BV(\Omega,\mathbb{R}^n)$ *be such that*

(i) $(u_k)_{k\in\mathbb{N}}$ *converges to* u_0 *in* $L_1(\Omega,\mathbb{R}^n)$

(ii) $\sup_{k\in\mathbb{N}}\int_\Omega|\mathcal{D}u_k(x)|\,dx = W<+\infty.$

Then for a.e. $x_0\in\Omega$ *and for*

$$\pi(x)=u_0(x_0)+\langle\mathcal{D}u_0(x_0),x-x_0\rangle,\quad x\in\Omega \qquad and \qquad \tilde u_k=u_k-\pi,\quad k\ge 0,$$

the following results hold.

There exists a function $t:]0,1[\times\mathbb{N}\to\mathbb{R}^+$ *such that* $\lim_{h\to 0}t(h,k)=0$ *for every* $k\in\mathbb{N}$, *and*

(1) $\lim_{h\to 0}\liminf_{k\to\infty}\dfrac{\operatorname{meas}(B_h\cap\{x:\ |\tilde u_k(x)|\le t\})}{\operatorname{meas}(B_h)}=1;$

(2) $\operatorname{ess}\lim_{h\to 0}\widetilde{\lim_{k\to\infty}}\ \fint_{B_h\cap\{x:\ |\tilde u_k(x)|\le t\}}\mathcal{D}u_k(x)\,dx=\mathcal{D}u_0(x_0)$

(3) $\sup_{B_h\cap\{x:\ |\tilde u_k(x)|\le t\}}|u_k(x)-u_0(x_0)|\le t+|\mathcal{D}u_0(x_0)|\cdot h$

where $t=t(h,k)$.

Moreover let $F:\mathbb{R}^{\nu n}\to\mathbb{R}$ *be an integrand which satisfies the conditions*

(iii) *is quasi convex;*

(iv) $0\le F(v)\le C(1+|v|),\quad v\in\mathbb{R}^{\nu n}$

then for a.e. $x_0\in\Omega$, *the following result holds*

(4) $\liminf_{h\to 0}\liminf_{k\to\infty}\left[\fint_{B_h\cap\{x:\ |\tilde u_k(x)|\le t\}}F(\mathcal{D}u_k(x))\,dx-F\left(\fint_{B_h\cap\{x:\ |\tilde u_k(x)|\le t\}}\mathcal{D}u_k(x)\,dx\right)\right]\ge 0.$

Proof. Let $x_0\in\Omega$ be fixed in such a way that the derivative $\mathcal{D}u_0(x_0)$ exists and (see [24])

$$\lim_{h\to 0}\fint_{B_h}\frac{|u_0(x)-u_0(x_0)-\langle\mathcal{D}u_0(x_0),x-x_0\rangle|}{|x-x_0|}\,dx=0. \tag{5.12}$$

Put

$$\mathcal{D}_0=\mathcal{D}u_0(x_0),\quad \pi(x)=u_0(x_0)+\langle\mathcal{D}_0,x-x_0\rangle,\quad x\in\Omega\quad \text{and}\quad \tilde u_k=u_k-\pi,\quad k\ge 0,$$

note that $\tilde{u}_k \in \mathcal{C}^\infty(\mathbb{R}^\nu, \mathbb{R}^n)$, $k \in \mathbb{N}$ and

$$\tilde{u}_k \longrightarrow \tilde{u}_0 \text{ in } L_1 \tag{5.13}$$

$$\sup_{k\in\mathbb{N}} \int_\Omega |\mathcal{D}\tilde{u}_k|\, dx = \tilde{W} = W + \mathcal{D}_0 \text{ meas}(\Omega) \tag{5.13'}$$

Let $B(x_0, r) \subset \Omega$ and let $0 < \overline{h} < r\nu^{-\frac{1}{2}}$ be given.

Let $t :]0, \overline{h}[\times \mathbb{N} \to \mathbb{R}^+$ be the function given by virtue of Lemma 5.1 and, for every $k \in \mathbb{N}$, $0 < h < \overline{h}$ and $0 < s < t = t(h,k)$ we denote by $w^k_{s,t} : B \to \mathbb{R}^n$ the function given in Lemma 5.1, relative to $\tilde{u}_k$ i.e.

$$w^k_{s,t}(x) = \phi_{s,t}(|\tilde{u}_k(x)|) \cdot \tilde{u}_k(x).$$

We recall that $w^k_{s,t} \in W^{1,\infty}(B, \mathbb{R}^n)$ and

$$w^k_{s,t}(x) = \tilde{u}_k(x) \quad \text{if } |\tilde{u}_k(x)| \le s, \qquad w^k_{s,t}(x) = 0 \quad \text{if } |\tilde{u}_k(x)| \ge t \tag{5.14}$$

$$\mathcal{D}w^k_{s,t}(x) = \mathcal{D}u_k(x) - \mathcal{D}_0 \quad \text{if } |\tilde{u}_k(x)| < s, \qquad \mathcal{D}w^k_{s,t}(x) = 0 \quad \text{if } |\tilde{u}_k(x)| > t. \tag{5.14'}$$

Of course we have

$$\frac{\text{meas } (B_h \cap \{x : |\tilde{u}_k(x)| > t\})}{\text{meas}(B_h)} < \frac{1}{\alpha(h)} \fint_{B_h} |\tilde{u}_k(x)|\, dx$$

and hance we deduce from (5.13) that for every $0 < h < \overline{h}$

$$\limsup_{k\to\infty} \frac{\text{meas } (B_h \cap \{x : |\tilde{u}_k(x)| > t\})}{\text{meas}(B_h)} \le \fint_{B_h} \frac{|\tilde{u}_0(x)|}{\alpha(h)}\, dx$$

and by virtue of (2) in Lemma 5.1 we get

$$\lim_{h\to 0} \limsup_{k\to\infty} \frac{\text{meas}\,(B_h \cap \{x : |\tilde{u}_k(x)| > t\})}{\text{meas}(B_h)} = 0$$

which proves (1).

Taking (5.14') and (1) into account, from (4) and (5) in Lemma 5.1, we deduce that

$$\lim_{h\to 0} \widetilde{\lim_{k\to\infty}} \widetilde{\lim_{s\to t}} [\text{meas}(B_h)]^{-1} \int_{B_h \cap \{x:\ |\tilde{u}_k(x)| < s\}} \mathcal{D}u_k(x)\, dx = \mathcal{D}_0. \tag{5.15}$$

Moreover (5.7) in Lemma 5.1 ensures that for every $k \in \mathbb{N}$ and $0 < h < \overline{h}$

$$\lim_{s\to t} \int_{B_h \cap \{x:\ s \le |\tilde{u}_k(x)| \le t\}} |\mathcal{D}u_k(x) - \mathcal{D}_0|\, dx = 0$$

and hence we have

$$\lim_{h\to 0} \widetilde{\lim_{k\to\infty}} \widetilde{\lim_{s\to t}} [\text{meas}(B_h)]^{-1} \int_{B_h \cap \{x:\ s \le |\tilde{u}_k(x)| \le t\}} |\mathcal{D}u_k(x)|\, dx = 0. \tag{5.16}$$

By virtue of (5.15), (5.16) and (1), we deduce (2).

Condition (3) is an immediate consequence of the definition of the function $\tilde{u}_k$.

Now note that

$$\fint_{B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}} F\left(\mathcal{D}u_k(x)\right)\,dx \geq$$
$$\geq \frac{\mathrm{meas}(B_h)}{\mathrm{meas}\left(B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}\right)}\fint_{B_h} F\left(\mathcal{D}w^k_{s,t}(x)+\mathcal{D}_0\right)\,dx\ +$$
$$-\frac{1}{\mathrm{meas}\left(B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}\right)}\int_{B_h\cap\{x:\ |\tilde{u}_k(x)|\geq s\}} F\left(\mathcal{D}w^k_{s,t}(x)+\mathcal{D}_0\right)\,dx \quad (5.17)$$

From (1) above and (1) of Lemma 5.1 we deduce that

$$\operatorname*{ess\,lim}_{h\to 0}\ \liminf_{k\to\infty}\ \liminf_{s\to t}\frac{\mathrm{meas}(B_h)}{\mathrm{meas}\left(B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}\right)}\fint_{B_h} F\left(\mathcal{D}w^k_{s,t}(x)+\mathcal{D}_0\right)\,dx \geq$$
$$\geq F\left(\mathcal{D}_0\right) \quad (5.18)$$

Moreover, by assumption (iv) it follows that

$$\frac{1}{\mathrm{meas}\ \left(B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}\right)}\int_{B_h\cap\{x:\ |\tilde{u}_k(x)\geq s\}} F\left(\mathcal{D}w^k_{s,t}+\mathcal{D}_0\right)\,dx \leq$$
$$\leq C(1+\mathcal{D}_0)\ \frac{\mathrm{meas}\ \left(B_h\cap\{x:\ |\tilde{u}_k(x)|\geq s\}\right)}{\mathrm{meas}\ \left(B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}\right)}\ +$$
$$+\ \frac{C\,\mathrm{meas}(B_h)}{\mathrm{meas}\ \left(B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}\right)}\left(\mathrm{meas}(B_h)]^{-1}\int_{B_h\cap\{x:\ |\tilde{u}_k(x)|\geq s\}}\left|\mathcal{D}w^k_{s,t}(x)\right|\,dx\right)$$

and taking (1) above and (5) in Lemma 5.1 into account we get

$$\operatorname*{ess\,lim}_{h\to 0}\ \widetilde{\lim_{k\to\infty}}\ \widetilde{\lim_{s\to t}}\frac{1}{\mathrm{meas}\left(B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}\right)}\fint_{B_h\cap\{x:\ |\tilde{u}_k(x)|\geq s\}} F\left(\mathcal{D}w^k_{s,t}(x)+\mathcal{D}_0\right)\,dx = 0. \quad (5.19)$$

By virtue of the continuity of F, we deduce from (2) that

$$\operatorname*{ess\,lim}_{h\to 0}\ \widetilde{\lim_{k\to\infty}}\ F\left(\fint_{B_h\cap\{x:\ |\tilde{u}_k(x)|\leq t\}}\mathcal{D}u_k(x)\right)\,dx = F\left(\mathcal{D}_0\right). \quad (5.20)$$

Thus (4) follows from (5.17)–(5.20). □

We are ready now to state and prove our main result.

Theorem 5.3 (Lower semicontinuity). *Assume that* $A\subset\mathbb{R}^n$ *is closed. Let* $(u_k)_{k\in\mathbb{N}}$ *be a sequence in* $W^{1,1}(\Omega,\mathbb{R}^n)$ *and let* $u_0\in BV(\Omega,\mathbb{R}^n)$ *be such that*

(i) $u_k(x)\in A$ *a.e.,* $k\in\mathbb{N}$;

(ii) $(u_k)_{k\in\mathbb{N}}$ *has equibounded variation and* L_1*–converges to* u_0.

Let $F : \Omega \times A \times \mathbb{R}^{\nu n} \to \mathbb{R}$ *be a Carathèodory function such that for a.e.* $x_0 \in \Omega$ *the following conditions are satisfied*

(iii) $F(\cdot,\cdot,v)/1+|v|$ *is lower semicontinuous in* $(x_0, u_0(x_0))$, *uniformly with respect to* v;

(iv) $F(x_0, u_0(x_0), \cdot)$ *is quasi convex;*

(v) $0 \le F(x_0, u_0(x_0), v) \le C(1+|v|), \quad v \in \mathbb{R}^{\nu n}$.

Then $u_0(x) \in A$, *a.e. and*

$$\liminf_{k\to\infty} \int_\Omega F(x, u_k(x), \mathcal{D}u_k(x))\, dx \ge \int_\Omega F(x, u_0(x), \mathcal{D}u_0(x))\, dx.$$

Proof. Following the idea adopted by Acerbi - Fusco [1] and successively by Fonseca - Müller [26, 27], it is not restrictive to assume that the sequence $(u_k)_{k\in\mathbb{N}}$ lies in $W^{1,1}(\Omega,\mathbb{R}^n) \cap C^0(\Omega,\mathbb{R}^n)$.

In fact $W^{1,1}(\Omega,\mathbb{R}^n) \cap C^\infty(\Omega,\mathbb{R}^n)$ is dense in $W^{1,1}(\Omega,\mathbb{R}^n)$ (see [2]). Thus, for every $k \in \mathbb{N}$, let $(v_{k,m})_{m\in\mathbb{N}}$ be a sequence in $W^{1,1}(\Omega,\mathbb{R}^n) \cap C^\infty(\Omega,\mathbb{R}^n)$ which $W^{1,1}$–converges to u_k. Of course we may assume that

$$\begin{aligned} &(v_{k,m})_{m\in\mathbb{N}} \text{ converges in } L_1 \text{ and a.e. to } u_k \\ &(\mathcal{D}v_{k,m})_{m\in\mathbb{N}} \text{ converges in } L_1 \text{ and a.e. to } \mathcal{D}u_k. \end{aligned} \tag{5.21}$$

By virtue of the continuity of $F(x,\cdot,\cdot)$ and the assumption (iv), it is easy (using Fatou's Lemma) to prove that

$$\lim_{m\to\infty} \int_\Omega F(x, v_{k,m}(x), \mathcal{D}v_{k,m}(x))\, dx = \int_\Omega F(x, u_k(x), \mathcal{D}u_k(x))\, dx. \tag{5.22}$$

Finally, from (5.21), (5.22), by a standard diagonalization process, we get a sequence $(v_{m_k,k})_{k\in\mathbb{N}}$ in $W^{1,1}(\Omega,\mathbb{R}^n) \cap C^\infty(\Omega,\mathbb{R}^n)$ such that $(v_{m_k,k})_{k\in\mathbb{N}}$ has equibounded variation, L_1–converges to u_0 and

$$\lim_{k\to\infty} \int_\Omega F(x, v_{m_k,k}(x), \mathcal{D}v_{m_k,k}(x))\, dx = \liminf_{k\to\infty} \int_\Omega F(x, u_k(x), \mathcal{D}u_k(x))\, dx.$$

Thus, let us assume that $u_k \in W^{1,1}(\Omega,\mathbb{R}^n) \cap C^0(\Omega,\mathbb{R}^n)$, $k \in \mathbb{N}$.

Let $x_0 \in \Omega$ be fixed in such a way that assertions of Lemmas 3.5 and 5.2 hold.

Since F is non-negative, we have

$$\begin{aligned} &\fint_{B_h} F(x, u_k(x), \mathcal{D}u_k(x))\, dx \ge \\ &\ge \frac{\text{meas}(B_h \cap \{x : |\tilde{u}_k(x)| \le t\})}{\text{meas}(B_h)} \Bigg\{ \fint_{B_h \cap \{x:\ |\tilde{u}_k(x)| \le t\}} [F(x, u_k(x), \mathcal{D}u_k(x)) - F(x_0, u_0(x_0), \mathcal{D}u_k(x))]\, dx + \\ &+ \fint_{B_h \cap \{x:\ |\tilde{u}_k(x)| \le t\}} F(x_0, u_0(x_0), \mathcal{D}u_k(x))\, dx - F\left(x_0, u_0(x_0), \fint_{B_h \cap \{x:\ |\tilde{u}_k(x)| \le t\}} \mathcal{D}u_k(x)\, dx\right) + \\ &\quad + F\left(x_0, u_0(x_0), \fint_{B_h \cap \{x:\ |\tilde{u}_k(x)| \le t\}} \mathcal{D}u_k(x)\, dx\right) - F(x_0, u_0(x_0), \mathcal{D}u_0(x_0)) \Bigg\}. \end{aligned} \tag{5.23}$$

Now, given $\varepsilon > 0$, from assumption ii), we deduce that a constant $\sigma = \sigma(x_0, u_0, \varepsilon) > 0$ exists such that if $|x - x_0| \leq \sigma$ and $|u - u_0(x_0)| \leq \sigma$, then for every $v \in \mathbb{R}^{\nu n}$ one has

$$F(x, u, v) \geq F(x_0, u_0(x_0), v) - \varepsilon(1 + |v|). \tag{5.24}$$

Note that from (3) of Lemma 5.2, in $B_h \cap \{x : \ |u_k(x) - u_0(x_0)| \leq t\}$, we get

$$|u_k(x) - u_0(x_0)| \leq t + |\mathcal{D}u_0(x_0)|\, h$$

thus, by virtue of (5.24), it is not restrictive to assume that for h sufficiently small

$$\fint_{B_h \cap \{x:\ |\tilde{u}_k(x)| \leq t\}} [F(x, u_k(x), \mathcal{D}u_k(x)) - F(x_0, u_0(x_0), \mathcal{D}u_k(x))]\, dx \geq$$
$$\geq -\varepsilon \left(1 + \fint_{B_h} |\mathcal{D}u_k(x)|\, dx\right)$$

and from Lemma 5.2 we deduce that

$$\liminf_{h \to 0} \liminf_{k \to \infty} \fint_{B_h \cap \{x:\ |\tilde{u}_k(x)| \leq t\}} [F(x, u_k(x), \mathcal{D}u_k(x)) - F(x_0, u_0(x_0), \mathcal{D}u_k(x))]\, dx \geq 0. \tag{5.25}$$

Taking the continuity of $F(x_0, u_0(x_0), \cdot)$ into account, from (2) of Lemma 5.2, we get

$$\liminf_{h \to 0} \liminf_{k \to \infty} \left\{ F\left(x_0, u_0(x_0), \fint_{B_h \cap \{x: |\tilde{u}_k(x)| \leq t\}} \mathcal{D}u_k(x)\, dx\right) - F(x_0, u_0(x_0), \mathcal{D}u_0(x_0)) \right\} \geq 0. \tag{5.26}$$

Finally, from (5.23), (5.25), (5.26) and (1), (4) of Lemma 5.2 we obtain

$$\operatorname*{ess\,lim}_{h \to 0} \widetilde{\lim_{k \to \infty}} \fint_{B_h} F(x, u_k(x), \mathcal{D}u_k(x))\, dx \geq F(x_0, u_0(x_0), \mathcal{D}u_0(x_0)).$$

The assertion is then an immediate consequence of Theorem 4.1. □

6. The convex case

Let us show how our main theorem reduces in the case of a convex integrand.

Following the proof of an analogous result given by Fonseca - Müller [26], the following lemma can be proved.

Lemma 6.1. *Let* $F : \Omega \times A \times \mathbb{R}^{n\nu} \to \mathbb{R}$ *be a continuous function such for a.e.* $x_0 \in \Omega$ *and every* $u_0 \in A$ *it satisfies the conditions*

(i) $F(x_0, u_0, \cdot)$ *is convex;*
(ii) $0 \leq F(x_0, u_0, v) \leq C(1 + |v|), \quad v \in \mathbb{R}^{\nu n}$.

Then for a.e. $x_0 \in \Omega$ *and every* $u_0 \in A$ *the function*

$$\frac{F(\cdot, \cdot, v)}{1 + |v|} \quad \textit{is lower semicontinuous in } (x_0, u_0), \textit{ uniformly with respect to } v.$$

In [22] the following approximation result was proved.

Lemma 6.2. *Let* $F : \Omega \times A \times \mathbb{R}^{n\nu} \to \mathbb{R}_0^+$ *be a continuous function such that for a.e.* $x_0 \in \Omega$ *and every* $u_0 \in A$ *it satisfies the conditions*

(i) $F(x_0, u_0, \cdot)$ *is convex;*

(ii) *for every* $\varepsilon > 0$ *there exists* $\delta = \delta(x_0, u_0, \varepsilon) > 0$ *such that if* $|x - x_0| < \delta$, $|u - u_0| < \delta$ *and* $v \in \mathbb{R}^{\nu n}$ *then*

$$F(x, u, v) \geq (1 - \varepsilon) F(x_0, u_0, v).$$

Then there exists a non decreasing sequence of continuous functions $F_j : \Omega \times \mathbb{R}^n \times \mathbb{R}^{\nu n} \to \mathbb{R}_0^+, j \in \mathbb{N}$, *such that*

(1) $0 \leq F_j(x, u, \cdot)$ *is convex;*

(2) $0 \leq F_j(x, u, v) \leq C_j(1 + |v|)$;

(3) $F(x, u, v) = \sup_{j \in \mathbb{N}} F_j(x, u, v)$.

By virtue of these results, the following result can be deduced from Theorem 5.3.

Corollary 6.3 (Lower semicontinuity). *Assume that* $A \subset \mathbb{R}^n$ *is closed.*

Let $(u_k)_{k \in \mathbb{N}}$ *be a sequence in* $W^{1,1}(\Omega, \mathbb{R}^n)$ *and let* $u_0 \in BV(\Omega, \mathbb{R}^n)$ *be such that*

(i) $u_k(x) \in A$ *a.e.*, $k \in \mathbb{N}$;

(ii) $(u_k)_{k \in \mathbb{N}}$ *has equibounded variation and* L_1*–converges to* u_0.

Let $F : \Omega \times A \times \mathbb{R}^{\nu n} \to \mathbb{R}$ *be a continuous function such that for a.e.* $x_0 \in \Omega$ *the following conditions are satisfied*

(iii) $F(x_0, u_0(x_0), \cdot)$ *is convex;*

(iv) *for every* $\varepsilon > 0$ *there exists* $\delta = \delta(x_0, u_0(x_0), \varepsilon) > 0$ *such that if* $|x - x_0| < \delta$, $|u - u_0(x_0)| < \delta$ *and every* $v \in \mathbb{R}^{\nu n}$ *then*

$$F(x, u, v) \geq (1 - \varepsilon) F(x_0, u_0(x_0), v).$$

Then we have that $u_0(x) \in A$, *a.e. and*

$$\liminf_{k \to \infty} \int_\Omega F(x, u_k(x), \mathcal{D}u_k(x))\, dx \geq \int_\Omega F(x, u_0(x), \mathcal{D}u_0(x))\, dx.$$

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