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Variational Inequalities and Regularity Properties of Closed Sets in Hilbert Spaces*

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Some properties of closed sets which generalize concepts of Convex Analysis are compared and characterized. Some of them have a global character and are concerned with controlling the lack of monotonicity of the Fréchet subdifferential of the indicator function. The connection with the local structure of sets in finite as well as in infinite dimensional spaces is also investigated. Special emphasis is given to a class of sets satisfying an external sphere condition, with locally uniform radius.

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1. Introduction

Let H be a Hilbert space, and let $K \subset H$ be a closed set. Many concepts of tangent and normal vectors to K were defined in the past (see, e. g., [3, 14, 27, 31]), with different – and, possibly, conflicting – purposes. In particular, Clarke's definition privilegiates regularity properties of the cones as multivalued mappings: for example, Clarke normal cone $N_K^c(\cdot)$ at $x \in K$ always admits a (multivalued) upper-hemicontinuous selection. On the other hand, normals in the sense of Bouligand satisfy some geometrical properties that genuine normal vectors should be expected to enjoy. Since a unifying concept does not exist, in order to exclude pathologies it is natural to assume that Clarke and Bouligand cones coincide. Several properties implying the above one are already present in the literature, or can be defined. This paper deals with the comparison and the investigation of some of them, mainly in infinite dimensional spaces. All the known regularity properties are enjoyed by both convex and smooth sets, and are born as generalizations of various characterizations of convexity.

We consider, among others, the following regularity properties: 1) Clarke regularity, i.e. Clarke and (strong) Bouligand normal cones coincide; 2) sleekness, i.e. the (strong)

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Bouligand tangent cone is lower semicontinuous; 3) σ-regularity, i.e. Clarke and (weak) Bouligand cones coincide; 4) φ-convexity, i.e. K satisfies an external sphere condition, with locally uniform radius. All of these properties are shown to be different in infinite dimensional spaces, while 1) $\div$ 3) are equivalent in $\mathbb{R}^d$. Moreover, the chain of implications 4) $\Rightarrow$ 3) $\Rightarrow$ 2) $\Rightarrow$ 1) holds. A fifth property, called property (ω), is also defined and studied. It is intermediate between 3) and 4), and is connected with the closedness of the graph of the boundary of the normal cone and with a kind of local monotonicity. Moreover, property (ω) is closely related with regularity and uniqueness of trajectories of the Moreau process for a class of non-convex sets, which are studied in the paper [16], and it is shown to be equivalent to the $o(1)$-convexity introduced by Shapiro (see [32]). Other properties, such as the equality between strong and weak Bouligand cones or between Clarke and proximal normal cone are also briefly studied. Some of the above concepts are also investigated, sometimes under different names, in the related paper [9], by Bounkhel and Thibault, where a thorough analysis of some regularity conditions for closed subsets of Banach spaces is performed.

The above listed regularity properties can be understood through variational inequalities characterizing them. More precisely, it is well known that an equivalent condition for the convexity of sets is the (monotonicity) relation

$$\langle z - x, y - x \rangle \leq 0\,, \tag{1}$$

valid for all $x, y \in K$ and for all $z \in H$ such that x is the projection of z into K. Most of the properties 1) $\div$ 4), together with the property (ω), can be characterized through a variational inequality of the same nature of (1): there exists a suitable function $\omega : K \times K \to \mathbb{R}^+$ such that for all $x, y \in K$, for all normal vectors $v \in H$ to K at x, it holds

$$\langle v, y - x \rangle \leq \omega(x, y)\,\|v\|\,\|y - x\|\,. \tag{2}$$

The function ω in (2) controls the lack of monotonicity of the normal cone. We show that 1) $\div$ 4), as well as the property (ω), can be characterized by means of different properties of the function ω, concerning its regularity and its asymptotic behavior around the diagonal of $K \times K$. In particular, asking ω to be upper semicontinuous outside the diagonal, and $\lim_{K \ni y \to x} \omega(x, y) = 0$, is equivalent – in finite dimensional spaces – to the σ-regularity of K; requiring ω to be continuous and $\omega(x, x) = 0$ for all $x \in K$ means (by definition) the property (ω), while $\omega(x, y) = \varphi(x, y)\,\|y - x\|$ together with the continuity of φ characterizes φ-convexity. In other words, we allow the right-hand side of (2) being positive, contrarily to (1), but we require that it tends to zero with a suitable - and suitably uniform - order as $y \to x$. One can see from its variational characterization that σ-regularity is a first order condition, which means – roughly speaking – that if some x in the boundary of K is a corner point, then the corner must be outwards. Instead φ-convexity is of a second order nature; in particular, it implies that all normals are proximal. It turns out that some properties which hold globally for convex sets are still valid - but only in some neighborhood - for φ-convex sets. For example, it is well known (see, e.g., [5]) that a closed subset of a Hilbert space is convex if and only if the metric projection into it is globally nonempty, single valued and continuous. On the other hand, it was proved in [10] that the metric projection into a φ-convex set K is locally nonempty, unique and Lipschitz continuous; as a consequence the distance from K is of class $\mathcal{C}^{1,1}_{\mathrm{loc}}$ in a neighborhood of K. We show here the converse implication, hence characterizing as

φ-convex those sets which are locally proximally smooth. Similar results were obtained in [15, 29]. We present here some different, and possibly simpler, proofs.

The paper is organized as follows: the relevant concepts of nonsmooth analysis can be found in §2; §3 treats some geometrical properties of Fréchet and strong Bouligand normals; §4 ÷ §6 are concerned – respectively – with σ-regularity, the property (ω) and φ-convexity; §7 is devoted to counterexamples.

2. Basic definitions

In a Hilbert space H, the *polar* of $A \subset H$ is the set $A^0 = \{v : \langle v, x\rangle \le 1 \quad \forall x \in A\}$. If A is a cone, the polar set coincides with the negative polar $A^- := \{y \in H : \langle y, x\rangle \le 0 \quad \forall x \in A\}$. Let $y \in H$; the *projection of* y *into* A is $\pi_A(y) := \{x \in A : d_A(y) = \|y - x\|\}$, where $d_A(y) := \inf\{\|y - z\| : z \in A\}$. This set is always nonempty if A is weakly closed, and it is a singleton if A is convex. We say that A is *proximinal* if $\pi_A(x) \ne \emptyset$ for all $x \in H$, and that A is *Chebyshev* if $\pi_A(x)$ is a singleton for all $x \in H$. The open (resp. closed) unit ball in H is denoted by B (resp. $\overline{B}$); co$A$ is the convex hull, cl$A$ is the closure of A, and bd$A$, int$A$ its boundary and its interior, respectively. The *domain* of a multifunction Γ, i.e. the set where it has nonempty values, is denoted by $\operatorname{dom}\Gamma$.

Let $\Gamma : H \to H$ be a multifunction. In what follows by $w - \limsup_{x' \to x} \Gamma(x')$ we mean the *sequential* $(s \times w)$-Hausdorff upper limit in H, i.e. the set of all vectors $v \in H$ for which there exist sequences $x_n \to x$ strongly and $v_n \to v$ weakly, such that $v_n \in \Gamma(x_n)$.

The *Bouligand* and *Clarke tangent cones* to a set $K \subset H$ at $x \in K$ are, respectively,

$$T_K^b(x) = \{v : \liminf_{h \to 0^+} d_K(x + hv)/h = 0\},$$

$$T_K^c(x) = \{v : \lim_{\substack{(h,x') \to (0,x),\\ h>0,\, x' \in K}} d_K(x' + hv)/h = 0\}.$$

Oserve that $T_K^b(x) = \{v : v = \lim_{n\to\infty} v_n, \text{ and } x + h_n v_n \in K \text{ for some } h_n \to 0^+\}$. Along with the Bouligand cone one considers its weak version $T_K^\sigma(x)$. By definition $v \in T_K^\sigma(x)$ iff v is a weak limit of $(y_{h_n} - x)/h_n$ when $h_n \to 0^+$ and $y_{h_n} \in K$. In finite dimension $T_K^b(x)$ and $T_K^\sigma(x)$ of course coincide. It is known that $T_K^b(x)$ is closed, that $T_K^c(x)$ is closed and convex, and that $T_K^c(x) \subseteq T_K^b(x) \subseteq T_K^\sigma(x)$ for all x [3, Proposition 4.1.6]. Normal cones can be defined as polars to tangent ones:

$$N_K^b(x) = (T_K^b(x))^0, \; N_K^\sigma(x) = (T_K^\sigma(x))^0, \; N_K^c(x) = (T_K^c(x))^0, \quad x \in K.$$

Both Bouligand and Clarke normal cones at $x \in K$ contain *proximal normals*, i.e. vectors of the form $y - x$, with $x \in \pi_K(y)$ (see, e.g., Proposition 4.1.2 in [3], p. 3 in [14], or p. 213 in [31]). We denote by $N_K^p(x)$ the cone generated by proximal normals. It enjoyes the following property (see Proposition 1.5 (a) in [14, p.25]):

v belongs to $N_K^p(x)$ if and only if there exists $\sigma = \sigma(x, v)$ such that

$$\langle v, y - x\rangle \le \sigma\|y - x\|^2 \qquad \forall y \in K. \tag{3}$$

It follows also that if $x \in \pi_K(y)$ then

$$\pi_K(z) = \{x\} \qquad \forall z \in \{tx + (1-t)y \,:\, t \in (0,1]\}\,. \tag{4}$$

It is useful to recall the following representation of Clarke normal cone [3, Theorem 4.4.4]

$$N_K^c(x) = \mathrm{cl\,co}\,\tilde{N}_K(x), \quad x \in K\,, \tag{5}$$

where

$$\tilde{N}_K(x) := w\!-\!\limsup_{K \ni x' \to x} N_K^\sigma(x')\,. \tag{6}$$

The cone $\tilde{N}_K(x)$ indeed coincides with the Mordukhovich normal cone, see [27, Theorem 2.9 (i)]. We remark that the inclusions

$$N_K^p(x) \subseteq N_K^\sigma(x) \subseteq N_K^b(x) \subseteq N_K^c(x) \qquad \forall x \in K \tag{7}$$

hold. We observe also that the map $x \mapsto \tilde{N}_K(x)$ has always sequentially $(s \times w)$-closed graph, i.e. its graph is sequentially closed in $K \times H$, where H is endowed with the weak topology, while neither Bouligand nor Clarke normal cones enjoy this property, in general (see [30, Counterexample 2]). By definition we set $N_K^p(x) = N_K^\sigma(x) = N_K^b(x) = N_K^c(x) = \emptyset$ for $x \notin K$.

We recall now the definition of Fréchet (see, e.g., [25, p. 61], or [4, p. 29]) and Clarke subdifferential (see [15]). Let $\Omega \subset H$ be open, and $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ be any function; if $f(x)$ is finite, we set

$$\partial^- f(x) = \left\{ v \in H \,:\, \liminf_{y \to x, y \neq x} \frac{f(y) - f(x) - \langle v, y - x\rangle}{\|y - x\|} \geq 0 \right\},$$

while $\partial^- f(x) = \emptyset$ if $f(x) = +\infty$. The set $\partial^- f(x)$ is closed and convex; some other properties can be found, for example, in [4] (see Lemmas 1.7, 1.8). If $f = I_K$, the indicator function of a set K, i.e. $I_K(x) = 0$ if $x \in K$, $I_K(x) = +\infty$ otherwise, then

$$\partial^- I_K(x) = \left\{ v \in H \,:\, \limsup_{K \ni y \to x, y \neq x} \frac{\langle v, y - x\rangle}{\|y - x\|} \leq 0 \right\}. \tag{8}$$

The elements of $\partial^- I_K(x)$ are often called the *Fréchet normals* to K at x. It follows from Proposition 6.5 in [31] that $\partial^- I_K(x) = N_K^b(x)$ for each closed $K \subset \mathbb{R}^d$. The Clarke subdifferential of a Lipschitz function $f : H \to \mathbb{R}$ at $x \in H$ is defined as

$$\partial^c f(x) = \{\zeta \in H \,:\, \langle \zeta, v\rangle \leq f^o(x; v) \quad \forall v \in H\}\,,$$

where $f^o(x; v) := \limsup_{t \to 0^+, y \to x} (f(y + tv) - f(y))/t$. Among well known properties of the Clarke subdifferential (see, e.g., [15]) we will use the fact that $\partial^c f(x)$ is always nonempty, and it is contained in $L\overline{B}$, where L is the Lipschitz constant of f.

3. Regular normals

The following variational characterization of weak Bouligand normal cones is at the basis of our analysis. It appeared in [24]; see also [8, Proposition 3.1].

Proposition 3.1. *If H is an arbitrary Hilbert space, and $K \subset H$ is closed, then*

$$\partial^- I_K(x) = N_K^\sigma(x) \qquad \forall x \in K\,. \tag{9}$$

We now give a geometrical characterization of the cone N_K^σ, as the set of those vectors being normal to K at x in a "reasonable way". In finite dimensional spaces, the elements of $N_K^b = N_K^\sigma$ are called in [31] *regular normals*. Proposition 10 below – together with Example 7.1 – suggests that in infinite dimensional spaces the adjective "regular" is better fit for weak Bouligand normals than for strong ones.

Proposition 3.2. *Let $K \subset H$ be closed, and let $x \in K$, $v \in H$. The following statements are equivalent:*

(i) $v \in N_K^\sigma(x)$;

(ii)

$$\lim_{h\to 0^+} \frac{d_K(x+hv)}{h} = \|v\|\,; \tag{10}$$

(iii) *for each choice of $y_h \in K$ such that*

$$\|x+hv-y_h\|^2 \le d_K^2(x+hv) + o(h^2), \qquad h \to 0^+ \tag{11}$$

(in particular, for each $y_h \in \pi_K(x+hv)$, if any), we have

$$\lim_{h\to 0^+} \frac{\|y_h - x\|}{h} = 0\,. \tag{12}$$

Proof. (i) $\Rightarrow$ (ii). Let $v \in N_K^\sigma(x)$. Take $y_h \in K$ such that $\|x+hv-y_h\|^2 \le d_K^2(x+hv) + o(h^2)$, $h \to 0^+$. Clearly $\|y_h - x\| \le 2h\|v\| + o(h)$, and by (9) $\limsup_{h\to 0^+} \dfrac{\langle v, y_h - x\rangle}{\|y_h - x\|} \le 0$. We have the inequalities

$$\begin{aligned} h^2\|v\|^2 &\ge d_K^2(x+hv) \ge \langle x+hv-y_h, x+hv-y_h\rangle + o(h^2) \\ &= h^2\|v\|^2 - 2h\langle v, y_h - x\rangle + \|x-y_h\|^2 + o(h^2) \\ &\ge h^2\|v\|^2 - 2h\langle v, y_h - x\rangle + o(h^2)\,. \end{aligned}$$

Therefore, if $\langle v, y_h - x\rangle \ge 0$ we obtain

$$\|v\|^2 \ge \frac{d_K^2(x+hv)}{h^2} \ge \|v\|^2 - 4\|v\|\frac{\langle v, y_h - x\rangle}{\|y_h - x\|} + o(1)\,.$$

Otherwise,

$$\|v\|^2 \ge \frac{d_K^2(x+hv)}{h^2} \ge \|v\|^2 + o(1)\,.$$

In both cases, by passing to limits in the above inequalities we prove (10).

(ii) $\Rightarrow$ (i). Let $v \in H$, $\|v\| = 1$, be satisfying (10), and assume by contradiction that $v \notin N_K^\sigma(x)$. Then, by (8), (9), there exist $0 < \eta < 1$ and a sequence $\{y_n\} \subset K$, $y_n \to x$, such that

$$\langle v, y_n - x\rangle \geq \eta \,\|y_n - x\| \qquad \forall n = 1, 2, \ldots .$$

Thus

$$\begin{aligned}\left\| x + \frac{\|y_n - x\|}{\eta} v - y_n \right\|^2 &= \|x - y_n\|^2 + \frac{\|x - y_n\|^2}{\eta^2} + \frac{2}{\eta}\langle v, x - y_n\rangle \,\|x - y_n\| \\ &\leq \|x - y_n\|^2 (\frac{1}{\eta^2} - 1)\,.\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \frac{d_K(x + v\,\|x - y_n\|/\eta)}{\|x - y_n\|/\eta} \leq 1 - \eta^2 < 1\,,$$

a contradiction.

(i) $\Rightarrow$ (iii). Let $v \in N_K^\sigma(x)$ and take $y_h \in K$ satisfying (11). Observe that, since the set $\{(y_h - x)/h\}$ is bounded, one can choose a sequence h_n such that $(y_{h_n} - x)/h_n$ converges weakly to some $\xi \in T_K^\sigma(x)$. Then clearly $\|y_{h_n} - x\|^2/h_n^2 \leq d_K^2(x + h_n v)/h_n^2 - \|v\|^2 + 2\langle v, (y_{h_n} - x)/h_n\rangle + o(1)$; by (10) this sequence converges to $2\langle v, \xi\rangle \leq 0$, and the conclusion follows.

(iii) $\Rightarrow$ (ii) is straightforward. $\square$

Remarks. 1) A characterization of the weak Bouligand normal cone in terms of the Fréchet subdifferential of the distance function was established by Kruger [23] and Ioffe [21]. A simple proof can be found in [9], Theorem 3.1. In particular, the implication (i) $\Rightarrow$ (ii) follows easily from that result. Observe that Proposition 3.2 says also that $v \in N_K^\sigma(x)$ if and only if the directional derivative of $\partial d_K(x)/\partial v$ equals $\|v\|$, i.e. it has the largest possible value.

2) Example 7.1 (c) shows that elements of $N_K^b(x)$ may fail to enjoy property (10). Consequently, in general, $N_K^b(x)$ strictly contains $N_K^\sigma(x)$ (see also the Remark 6.1 in [9], which is based on an example contained in [7]).

3) Comparing (12) with (4) one can see the difference between the elements of $N_K^\sigma(x)$ and of $N_K^p(x)$: in the second case $y_h = x$ for all h small enough.

We now give an equality of the same nature as (9) characterizing the (strong) Bouligand normal cone. To this aim we introduce the following definition.

Definition 3.3. We say that a sequence $\{x_n\} \subset H$ *directionally converges* to a point x (we write $x_n \xrightarrow{d} x$) if there exist sequences $\{v_n\} \subset H$, $v_n \to v \neq 0$, and $h_n \to 0^+$ such that $x_n = x + h_n v_n$ for infinitely many n.

Clearly in finite dimensional spaces directional convergence coincides with the usual convergence. If f is a scalar function defined on H and $K \subset H$, we set

$$\limsup_{K \ni y \xrightarrow{d} x} f(y) := \sup\left\{ \limsup_{n\to\infty} f(x_n) \;:\; \{x_n\} \subset K,\, x_n \xrightarrow{d} x \right\}.$$

Proposition 3.4. *Let $K \subset H$ be closed and $x \in K$. Then*

$$N_K^b(x) = \left\{ v \in H : \limsup_{K \ni y \xrightarrow{d} x, y \neq x} \langle v, \frac{y-x}{\|y-x\|} \rangle \leq 0 \right\} . \tag{13}$$

Proof. Take $v \in N_K^b(x)$ and a sequence $\{x_n\} \subset K$ directionally converging to x. Without loss of generality assume that there exists a sequence $h_n \to 0^+$ for which $(x_n - x)/h_n \to w \neq 0$. Then we have $(x_n - x)/\|x_n - x\| \to w/\|w\|$ and $w \in T_K^b(x)$. Hence, $\lim_{n\to\infty}\langle v, (x_n - x)/\|x_n - x\|\rangle = \langle v, w\rangle/\|w\| \leq 0$.

Conversely, let v belong to the right-hand side of (13), and let $w \in T_K^b(x)$, $w \neq 0$. Then ther exist sequences $\{x_n\} \subset K$ and $h_n \to 0^+$ such that $(x_n - x)/h_n \to w$, i.e. $x_n \xrightarrow{d} x$. Therefore, $\langle v, w\rangle \leq 0$ and $v \in \big(T_K^b(x)\big)^-$. □

By using the concept of directional convergence one can give a characterization of the boundary of N_K^b.

Proposition 3.5. *Let $K \subset H$ be closed and $x \in K$ be such that $T_K^b(x) \neq \{0\}$. Then $v \in \mathrm{bd}N_K^b(x)$ if and only if*

$$\limsup_{K \ni y \xrightarrow{d} x} \langle v, \frac{y-x}{\|y-x\|} \rangle = 0 . \tag{14}$$

More precisely, for each $v \in N_K^b(x)$ the equality

$$\limsup_{K \ni y \xrightarrow{d} x} \langle v, \frac{y-x}{\|y-x\|} \rangle = -d_{\mathrm{bd}\, N_K^b(x)}(v) \tag{15}$$

holds.

Proof. Let v satisfy the equality (14). Observe that $v \in N_K^b(x)$, by (13). Assuming by contradiction that v is in the interior of $N_K^b(x)$ we find $\varepsilon > 0$ with $v + \varepsilon\overline{B} \subset N_K^b(x)$ and a sequence $\{x_n\} \subset K$ converging to x, such that $(x_n - x)/\|x_n - x\| \to w \in T_K^b(x)$, and $\lim_{n\to\infty}\langle v, (x_n - x)/\|x_n - x\|\rangle > -\varepsilon$. Hence $\langle v + \varepsilon w, w\rangle = \langle v, w\rangle + \varepsilon > 0$, a contradiction. The converse implication follows directly from (13).

To prove (15), take $v \in N_K^b(x)$. Let us show that there exists $\xi^* \in T_K^b(x)$ with $\|\xi^*\| = 1$ satisfying

$$\limsup_{K \ni y \xrightarrow{d} x} \langle v, \frac{y-x}{\|y-x\|} \rangle = \sup \left\{ \langle v, \xi\rangle : \xi \in T_K^b(x),\ \|\xi\| = 1 \right\} = \langle v, \xi^*\rangle . \tag{16}$$

Indeed, write $S := \sup\{\langle v, \xi\rangle : \xi \in T_K^b(x), \|\xi\| = 1\}$, fix $\varepsilon > 0$ and choose $\xi_\varepsilon \in T_K^b(x)$, $\|\xi_\varepsilon\| = 1$, with $\langle v, \xi_\varepsilon\rangle \geq S - \varepsilon$. By the definition of Bouligand tangent cone there exists a sequence $\{x_n\} \subset K$ converging to x and such that $(x_n - x)/\|x_n - x\| \to \xi_\varepsilon$. Consequently, $x_n \xrightarrow{d} x$ and we have $\limsup_{K \ni y \xrightarrow{d} x}\langle v, (y - x)/\|y - x\|\rangle \geq \langle v, \xi_\varepsilon\rangle \geq S - \varepsilon$. On the other hand, take a sequence $\{y_n\} \subset K$, $y_n \xrightarrow{d} x$ such that $\limsup_{K \ni y \xrightarrow{d} x}\langle v, (y - x)/\|y - x\|\rangle = \lim_{n\to\infty}\langle v, (y_n - x)/\|y_n - x\|\rangle$. Without loss of generality we can assume that

$\{(y_n - x)/\|y_n - x\|\}$ converges to some $\overline{\xi} \in T_K^b(x)$, $\|\overline{\xi}\| = 1$. Hence $\limsup_{K \ni y \xrightarrow{d} x} \langle v, (y - x)/\|y - x\|\rangle = \langle v, \overline{\xi}\rangle \le S$, and the equality (16) follows. Assume now $v \in \mathrm{int} N_K^b(x)$, and set $\lambda = 1/(1 - \langle v, \xi^* \rangle)$, where ξ^* is given by (16). It is easy to see from (14) and (16) that $0 < \lambda < 1$, and the point $u = \lambda v + (1 - \lambda)\xi^*$ belongs to the boundary of $N_K^b(x)$. Let p be the projection of v into the line through 0 and u. From elementary geometric considerations, taking into account that $\langle u, \xi^* \rangle = 0$, we deduce that

$$\|v - p\| = \frac{\|u - v\|}{\|u - \xi^*\|} = \frac{1 - \lambda}{\lambda} = -\langle v, \xi^* \rangle\,,$$

so that $d_{\mathrm{bd}\, N_K^b(x)}(v) \le -\limsup_{K \ni y \xrightarrow{d} x} \langle v, (y-x)/\|y-x\|\rangle$. To see the converse inequality, let $w \in \mathrm{bd}\, N_K^b(x)$; by the same argument as above, choose $\eta^* \in T_K^b(x)$, $\|\eta^*\| = 1$, such that

$$\langle w, \eta^* \rangle = \sup\big\{\langle w, \eta\rangle \,:\, \eta \in T_K^b(x),\ \|\eta\| = 1\big\} = \limsup_{K \ni y \xrightarrow{d} x} \langle w, (y - x)/\|y - x\|\rangle = 0\,.$$

Then by Cauchy-Schwartz inequality and the definition of ξ^* we have

$$\|w - v\| + \langle v, \xi^* \rangle \ge \langle w - v, \eta^* \rangle + \langle v, \eta^* \rangle = \langle w, \eta^* \rangle = 0,$$

and the proof is concluded. □

We observe that in a finite dimensional space the Bouligand normal cone at x is large (i.e. has nonempty interior) if and only if x is an "outwards corner point". More precisely, it is straightforward to show the following fact.

Corollary 3.6. *Let $K \subset \mathbb{R}^d$ be closed, and $x \in K$. Then $v \in \mathrm{int}\, N_K^b(x)$ if and only if there exist positive δ and η such that*

$$\langle v, y - x\rangle \le -\eta\|y - x\| \qquad \forall y \in K \cap (x + \delta B)\,. \tag{17}$$

Proof. We can assume that x is a non isolated point, i.e. $T_K^b(x) \ne \{0\}$, since otherwise the property 17 holds trivially. By Proposition 3.5, $v \in \mathrm{int}\, N_K^b(x)$ if and only if

$$\limsup_{K \ni y \to x} \langle v, \frac{y - x}{\|y - x\|}\rangle = -2\eta$$

for some $\eta > 0$, and the remainder follows from the definition of "lim sup". □

Remark. 1) A characterization of vectors internal to the Bouligand normal cone similar to Corollary 3.6 is not valid in infinite dimensional Hilbert spaces (see Example 7.1 (b)).

2) It is a natural question finding assumptions under which the strong and weak Bouligand cones coincide. In the next sections sufficient conditions will be provided. Here we observe only that $N_K^b(x) = N_K^\sigma(x)$ for some x does not imply $T_K^b(x) = T_K^\sigma(x)$ (see Example 7.1 (d)).

4. Sleek, regular and σ-regular sets

This section has a partial overlapping with [9]. Since our arguments are short – because we work in a Hilbert space, while [9] deals with Banach spaces – we prefer to prove all the stated results.

Definition 4.1. We say that a closed set $K \subset H$ is sleek if the set-valued map $x \mapsto T_K^b(x)$ is lower semicontinuous on K.

Observe that, in particular, both convex sets and sets with $\mathcal{C}^1$ boundary are sleek. The above concept is studied, under the same name, in [3, §4.1.4], or – in finite dimensional spaces – in [31, Corollary 6.29], as an item of a list of characterizations of Clarke regularity. Some of the implications in [31, Corollary 6.29] hold also in infinite dimensional spaces.

Proposition 4.2. *For a closed set $K \subset H$ consider the following statements:*

(a) *K is sleek;*
(b) *the map $x \mapsto N_K^b(x)$ has sequentially $(s \times w)$-closed graph;*
(c) *there exists a function $\omega : K \times K \to \mathbb{R}^+$ such that*
  1) *for all $x, y \in K$, $v \in N_K^b(x)$ it holds*

$$\langle v, y - x\rangle \le \omega(x, y)\,\|v\|\,\|y - x\|\,; \tag{18}$$

  2) *ω is upper semicontinuous in $(K \times K) \cap \{(x, y) : x \ne y\}$;*
  3) $\lim_{K \ni y \xrightarrow{d} x} \omega(x, y) = 0$.

(d) *$N_K^b(x) = N_K^c(x)$ for all $x \in K$;*
(e) *$T_K^b(x) = T_K^c(x)$ for all $x \in K$.*

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e).

Proof. (a) $\Rightarrow$ (b). Let sequences $\{x_n\} \subset K$ and $v_n \in N_K^b(x_n)$ be such that $x_n \to x$ strongly and $v_n \to v$ weakly. Let $w \in T_K^b(x)$ and let, by lower semicontinuity, $w_n \in T_K^b(x_n)$ be such that $w_n \to w$. Since $\langle v_n, w_n\rangle \le 0$, also $\langle v, w\rangle \le 0$, i.e. $v \in N_K^b(x)$.

(b) $\Rightarrow$ (c). Set, for $x, y \in K$, $x \ne y$

$$\omega(x, y) = \sup_{\substack{\|v\| \le 1,\\ v \in N_K^b(x)}} \frac{\langle v, y - x\rangle}{\|y - x\|} \vee 0\,.$$

By weak compactness and graph closedness of $x \mapsto N_K^b(x)$, ω is upper semicontinuous in $(K \times K) \cap \{(x, y) : y \ne x\}$ (see Theorem 5 in [2, p. 53]). It remains to show that $\lim_{K \ni y \xrightarrow{d} x} \omega(x, y) = 0$. Assume by contradiction that there exist sequences $\{y_n\} \subset K$, $y_n \xrightarrow{d} x$, and $v_n \in N_K^b(x)$, $\|v_n\| \le 1$, such that

$$\langle v_n, \frac{y_n - x}{\|y_n - x\|}\rangle \to \eta > 0\,. \tag{19}$$

By compactness we can assume that v_n weakly converges to $v \in N_K^b(x)$, and by directional convergence that $(y_n - x)/\|y_n - x\|$ converges to $w \in T_K^b(x)$ (strongly). Then (19) implies that $\langle v, w\rangle > 0$, which is impossible.

(c) $\Rightarrow$ (b). Let $K \ni x_n \to x$, $v_n \in N_K^b(x_n)$, $v_n \to v$ weakly. We want to show that $v \in N_K^b(x)$. Fix $y \in K$, $y \neq x$. Then, by 1),

$$\langle v, y - x_n\rangle \le \omega(x_n, y)\|v_n\|\|y - x_n\| + \|v - v_n\|\|x - x_n\| + \langle v - v_n, y - x\rangle .$$

Since the sequence $\{v_n\}$ is bounded, say $\|v_n\| \le M$, we can pass to the limsup for $n \to \infty$, and obtain by 2)

$$\langle v, \frac{y - x}{\|y - x\|}\rangle \le \omega(x, y)\, M .$$

By passing now to the limsup for $y \xrightarrow{d} x$ in the above inequality and using the property 3) we obtain the result, recalling Proposition 3.4.

(b) $\Rightarrow$ (a). It follows from the duality theorem [3, Theorem 1.1.8], since $T_K^b(x)$ is a convex cone (see (e)).

(b) $\Rightarrow$ (d). By (5), (6), (7) and the convexity of $N_K^b(x)$ we have

$$N_K^c(x) \subset \operatorname{cl}\operatorname{co}(w-\limsup_{K \ni x' \to x} N_K^b(x')) \subset \operatorname{cl}\operatorname{co} N_K^b(x) = N_K^b(x) \subset N_K^c(x) \qquad \forall x \in K .$$

(d) $\Rightarrow$ (e). By the bipolar theorem

$$T_K^c(x) = (N_K^c(x))^0 = (N_K^b(x))^0 = \operatorname{cl}\operatorname{co} T_K^b(x) \supset T_K^b(x) .$$

(e) $\Rightarrow$ (d) by definition. □

Remark. The equivalence (e) $\Rightarrow$ (d) appears also in [9, Theorem 6.1], while (a) $\Rightarrow$ (d) is [3, Theorem 4.1.8].

In [13, Definition 2.4.6], a set K is said to be *regular* if the condition (e) in the above Proposition holds. This notion is introduced in order to find a class of sets where Clarke normal cone has good geometrical properties, or Bouligand normal cone has a good calculus. If K is sleek then it is regular (this is (a) $\Rightarrow$ (e) of Proposition 4.2), while the converse implication holds only if H is finite dimensional (see Example 7.1 e). Some properties of regular sets can be found also in [28]. We introduce now a stronger regularity condition, for which it holds the same set of equivalences characterizing finite dimensional sleek sets.

Definition 4.3. We say that a closed set $K \subset H$ is σ-regular if for each $x \in K$ the equality $N_K^\sigma(x) = N_K^c(x)$ holds.

Remark. The above condition is called *Fréchet normal regularity* in [9], and proved to be equivalent to another concept of regularity defined earlier by Mordukhovich (see [26] and [9, Theorem 3.4]). Thanks to (7), we obtain that a σ-regular set is regular. Example 7.1 c) shows that the converse is not true in infinite dimensional spaces.

The next result is the announced characterization of σ-regular sets: the chain of implications in Proposition 4.2 can be closed, if the strong Bouligand normal cone is substituted by the weak one. We add to that list of equivalent conditions a further characterization based on a variational inequality.

Proposition 4.4. *Let $K \subset H$ be closed. Consider the statements:*

(a) *K is σ-regular;*
(b) *the map $x \mapsto N_K^\sigma(x)$ has sequentially $(s \times w)$-closed graph;*
(c) *the map $x \mapsto T_K^\sigma(x)$ is lower semicontinuous;*
(d) *$T_K^\sigma(x) = T_K^c(x)$ for all $x \in K$;*
(e) *there exists a function $\omega : K \times K \to \mathbb{R}^+$ such that*
 1) *for all $x, y \in K$, $v \in N_K^\sigma(x)$ it holds*

$$\langle v, y - x\rangle \le \omega(x, y)\, \|v\|\, \|y - x\|\,; \tag{20}$$

 2) *ω is upper semicontinuous in $(K \times K) \cap \{(x, y) : x \neq y\}$;*
 3) $\lim_{K \ni y \to x} \omega(x, y) = 0$.

Then (a) $\div$ (d) *are equivalent, and follow from* (e). *Furthermore, in finite dimensional spaces* (e) *is equivalent to the other properties.*

Proof. (a) $\Leftrightarrow$ (b) follows readily from (6) and (5), while (d) follows from both (a) and (b) by the same argument of Proposition 4.2. Now (b) $\Leftrightarrow$ (c) from the Duality Theorem [3, Theorem 1.1.8] and (d) $\Rightarrow$ (a) by definition.

(e) $\Rightarrow$ (b). Let $K \ni x_n \to x$, $v_n \in N_K^\sigma(x_n)$, $v_n \to v$ weakly. We want to show that $v \in N_K^\sigma(x)$. Fix $y \in K$, $y \neq x$, and let $\|v_n\| \le M$ for $n = 1, 2, \ldots$ Then, by 1), by the same argument of Proposition 4.2 (b) $\Rightarrow$ (c), we obtain

$$\langle v, \frac{y - x}{\|y - x\|}\rangle \le \omega(x, y)\, M\,.$$

By passing now to the limsup for $y \to x$ in the above inequality and using the property 3) we obtain the result, recalling Proposition 3.1.

(a) $\Rightarrow$ (e). It is trivial by Proposition 4.2, since in finite dimensional spaces sleekness and σ-regularity do coincide. $\square$

From Propositions 4.4 and 4.2 we immediately deduce

Corollary 4.5. *Each closed σ-regular set $K \subset H$ is sleek, and the property* (e) *of Proposition* 4.4 *holds, provided* 3) *is substituted by*

3') $\lim_{K \ni y \xrightarrow{d} x} \omega(x, y) = 0,$

Remark. 1) The equivalences (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (d) are proved also in [9], Theorems 3.4 and 6.3, respectively.

2) Examples 7.1 show that in infinite dimensional spaces

- a sleek set may not be σ-regular;
- the implication (a) $\Rightarrow$ (e) does not hold.

Moreover, one can see from Example 7.2 that no further regularity of ω w.r.t. x, even in the finite dimensional case, can be expected.

5. The property (ω)

We introduce now a class of closed sets satisfying a variational inequality of the same nature as (20).

Definition 5.1. We say that a closed set $K \subset H$ satisfies the property (ω) if there exists a continuous function $\omega : K \times K \to \mathbb{R}^+$ with $\omega(x,x) = 0$ for all $x \in K$ such that

$$\langle v, y - x \rangle \leq \omega(x,y) \, \|v\| \, \|y - x\| \tag{21}$$

for all $x, y \in K$, $v \in N_K^\sigma(x)$.

Remark. A set K is convex if and only if it has the property (ω), with $\omega \equiv 0$ (see [34, Theorem 4.10]). Furthermore, a set whose boundary is a $\mathcal{C}^1$-manifold has the property (ω); more in general a set K such that for all $x \in K$ there exists a neighborhood $\mathcal{U}$ with either $\mathcal{U} \cap K$ convex, or $\mathcal{U} \cap \operatorname{bd} K$ a $\mathcal{C}^1$-manifold has the property (ω) (see Theorem 5.8 (iii) $\Rightarrow$ (ii)).

Proposition 4.4 (e) immediately yields

Proposition 5.2. *Let $K \subset H$ be a closed set satisfying the property (ω). Then K is σ-regular.*

The converse is not true, as the Example 7.2 shows. In fact the property (ω) implies stronger regularity conditions.

Proposition 5.3. *Let $K \subset H$ be closed and satisfying the property (ω). Then for each $\{x_n\} \subset K \setminus \{x\}$, $x_n \xrightarrow{d} x$, and for each bounded sequence $v_n \in N_K^\sigma(x_n)$, one has*

$$\lim_{n\to\infty} \langle v_n, \frac{x_n - x}{\|x_n - x\|} \rangle = 0\,. \tag{22}$$

Proof. Let $\|v_n\| \leq M$ for all n. Then, by the property (ω),

$$\langle v_n, x - x_n \rangle \leq M\omega(x_n, x) \, \|x_n - x\|\,.$$

On the other hand, without loss of generality we can assume that $\{v_n\}$ converges weakly to some $v \in H$, which belongs to $N_K^\sigma(x)$ by Propositions 5.2 and 4.4 (b). Consequently, $\langle v, (x_n - x)/\|x_n - x\| \rangle \leq \omega(x, x_n) \, \|v\|$. Assuming, moreover, that $(x_n - x)/\|x_n - x\| \to \xi$ we have:

$$\left| \langle v_n, \frac{x_n - x}{\|x_n - x\|} \rangle \right| \leq \omega(x, x_n) \, \|v\| + |\langle v_n - v, \xi \rangle| + \|v_n - v\| \left\| \xi - \frac{x_n - x}{\|x_n - x\|} \right\| .$$

Thus $|\langle v_n, (x_n - x)/\|x_n - x\| \rangle| \to 0$ as $n \to \infty$ and the Proposition is proved. $\square$

As a corollary we obtain the following regularity property of the boundary of the normal cone.

Corollary 5.4. *Assume that a closed set $K \subset H$ satisfies the property (ω) and a point $x \in K$ is such that $T_K^b(x) \neq \{0\}$. Then the inclusion*

$$w-\limsup_{K\setminus\{x\}\ni y \xrightarrow{d} x} N_K^b(y) \subseteq \operatorname{bd} N_K^b(x)$$

holds. In particular, the map $x \mapsto \mathrm{bd}N_K^b(x)$ *has (sequentially)* $(d \times w)$*-closed graph.*

Proof. Let us take $v \in H$ for which there exist sequences $\{x_n\} \subset K \setminus \{x\}$, $x_n \xrightarrow{d} x$, and $\{v_n\}$ converging to v weakly such that $v_n \in N_K^b(x_n)$. Then $v \in N_K^b(x)$ and by Proposition 5.2 $\langle v_n, (x_n - x)/\|x_n - x\|\rangle$ converges to zero. Hence, as it is easy to see, also $\langle v, (x_n - x)/\|x_n - x\|\rangle \to 0$, and $\limsup_{K \ni y \xrightarrow{d} x} \langle v, (y - x)/\|y - x\|\rangle = 0$. This means that $v \in \mathrm{bd}N_K^b(x)$ (see Proposition 3.5). □

Observe that in a finite dimensional space the Corollary above gives a property stronger than sleekness, and implies graph closedness of $\mathrm{bd}N_K^b$.

Definition 5.5. Let $\Gamma : H \to H$ be a multivalued map. We say that Γ is locally monotone at $x \in \mathrm{dom}\,(\Gamma)$ if for all sequences $\{x_n\}, \{y_n\} \subset \mathrm{dom}\,(\Gamma)$, with $x_n \neq y_n$, converging to x, and for all bounded sequences $u_n \in \Gamma(x_n)$, $v_n \in \Gamma(y_n)$ we have

$$\liminf_{n\to\infty} \langle u_n - v_n, \frac{x_n - y_n}{\|x_n - y_n\|}\rangle \geq 0\,. \tag{23}$$

We say that Γ is locally monotone if it is so at all $x \in \mathrm{dom}\,\Gamma$.

A straightforward equivalent condition for local monotonicity of cones is the following

Proposition 5.6. *Let $K \subset H$ be closed, and let $\Gamma(x)$ be a closed set containing the origin for each $x \in K$. Then Γ is locally monotone at $x \in K$ if and only if for all sequences $\{x_n\}, \{y_n\} \subset K$, with $x_n \neq y_n$ converging to x, and for all bounded sequences $u_n \in \Gamma(x_n)$ one has*

$$\limsup_{n\to\infty} \langle u_n, \frac{y_n - x_n}{\|y_n - x_n\|}\rangle \leq 0\,. \tag{24}$$

Using the concept of local monotonicity we now characterize property (ω). Before stating the result we recall another definition.

Definition 5.7 ([32]). We say that a closed $K \subset H$ is $o(1)$-convex if for all $x \in K$ there exist a neighborhood U of x and a function $\psi_x : (K \cap U) \times (K \cap U) \to \mathbb{R}^+$ such that $\lim_{K \ni y, z \to x} \psi_x(y, z) = 0$ and for all $y, z \in K \cap U$, $y \neq z$, one has

$$d_{T_K^b(y)}\left(\frac{z - y}{\|z - y\|}\right) \leq \psi_x(y, z)\,.$$

Theorem 5.8. *Let H be separable and $K \subset H$ be closed. The following statements are equivalent:*

(i) *K has the property (ω);*
(ii) *N_K^c is locally monotone;*
(iii) *N_K^b is locally monotone;*
(iv) *N_K^σ is locally monotone and $N_K^\sigma(x) = N_K^b(x)$ for all $x \in K$;*
(v) *K is $o(1)$-convex.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) follow immediately from (7) and Propositions 5.6 and 5.2.

(i) $\Rightarrow$ (iv) follows from (7) and Proposition 5.2, while (iv) $\Rightarrow$ (iii) is trivial.

(ii) $\Rightarrow$ (i). Consider the function

$$\varphi(x,v,y)=\begin{cases}\langle v,(y-x)/\|y-x\|\rangle & \text{if } y\neq x\\ 0 & \text{if } y=x,\end{cases}$$

defined on $\operatorname{graph} N_K^c \times K$. This function is sequentially upper semicontinuous on its domain if $\operatorname{graph} N_K^c$ is endowed with the $(s\times w)$-topology (the upper semicontinuity at points (x,v,x) follows from the local monotonicity, using Proposition 5.6). We recall (see (5) and (6)) that the cone $\tilde N_K \subset N_K^c$ has sequentially $(s\times w)$-closed graph. Consider the function

$$\psi(x,y)=\sup_{v\in \tilde N_K(x)\cap\overline{B}} \varphi(x,v,y)\,,$$

which is upper semicontinuous by the sequential $(s\times w)$-upper semicontinuity of the map $x\mapsto \tilde N_K(x)\cap\overline{B}$ (Theorem 5 in [2, p. 53]) and the metrizability of $\overline{B}$ endowed with the weak topology. Set

$$\Omega(x,y)=\begin{cases}[\psi(x,y),+\infty) & \text{if } y\neq x\\ \{0\} & \text{if } x=y\ .\end{cases}$$

Clearly, Ω is lower semicontinuous (as a multivalued map), with closed convex values, therefore it admits a continuous selection $\omega(x,y)$ by Michael's theorem. Since $N_K^\sigma\subset \tilde N_K$, this fact establishes property (ω).

(iii) $\Rightarrow$ (ii). In view of Proposition 4.2, it suffices to show that (iii) implies that K is sleek. To this aim, let $\{x_n\}\subset K\setminus\{x\}$, $x_n\to x$ and $v_n\in N_K^b(x_n)$, $v_n\to v$ weakly. We want to prove that $v\in N_K^b(x)$, i.e., recalling Proposition 3.4,

$$\limsup_{K\ni y\xrightarrow{d}x}\langle v,\frac{y-x}{\|y-x\|}\rangle\le 0\,. \tag{25}$$

Let $\{y_k\}\subset K\setminus\{x\}$, $y_k\xrightarrow{d}x$ and for all $k=1,2,\ldots$ choose n_k such that $\|(y_k-x_{n_k})/\|y_k-x_{n_k}\|-(y_k-x)/\|y_k-x\|\|\le 1/k$. Then,

$$\begin{aligned}\langle v,\frac{y_k-x}{\|y_k-x\|}\rangle &= \langle v-v_{n_k},\frac{y_k-x}{\|y_k-x\|}\rangle+\\ &\quad+\langle v_{n_k},\frac{y_k-x_{n_k}}{\|y_k-x_{n_k}\|}\rangle+\langle v_{n_k},\frac{y_k-x}{\|y_k-x\|}-\frac{y_k-x_{n_k}}{\|y_k-x_{n_k}\|}\rangle\,.\end{aligned}$$

There is no loss of generality in assuming that $(y_k-x)/\|y_k-x\|$ is strongly converging. Therefore, recalling (24), we obtain (25) from the above inequality.

(i) $\Rightarrow$ (v). Observe that, by Fenchel duality, one has $d_{T_K^c(y)}((z-y)/\|z-y\|)=\sup_{v\in N_K^c(y),\|v\|=1}\langle v,(z-y)/\|z-y\|\rangle$. Now, property (ω) implies $T_K^c=T_K^b$ and $N_K^c=N_K^b$, so that K is $o(1)$-convex with $\psi_x(y,z)=\omega(y,z)$ for all $x\in K$.

(v) $\Rightarrow$ (i) follows from the fact (see Lemma 2.1 in [32]) that for a $o(1)$-convex set the map $x\mapsto N_K^b(x)$ is locally monotone. $\square$

The regularity of the boundary of a class of plane sets with the property (ω) can now be described. We recall that a cone C is said to be *pointed* if $0 \neq z \in C$ implies $-z \notin C$. In [30, Theorem 3] closed sets $K \subset \mathbb{R}^d$ such that $N_K^c(x)$ is pointed are characterized as those sets for which, after a linear change of coordinates, a neighborhood of x intersected with K coincides with the intersection of this neighborhhod with the epigraph of a Lipschitz continuous function from $\mathbb{R}^{d-1}$ into $\mathbb{R}$. If this property holds for all $x \in K$, the set is called *epi-Lipschitzian*.

Proposition 5.9. *Let $K \subset \mathbb{R}^2$ be closed, epi-Lipschitzian and satisfying property (ω). Then there exists a dense set $R \subset \operatorname{bd} K$ and a continuous function $v : R \to \mathbb{R}^2$, $\|v(x)\| = 1$, such that $N_K^b(x) = \mathbb{R}^+ v(x)$ for all $x \in R$, while for all $x \in \operatorname{bd} K \setminus R$ the cone $N_K^b(x)$ has nonempty interior.*

Proof. By Propositions 5.2 and 4.2 the map $\Gamma : K \ni x \mapsto N_K^b(x) \cap \{\|v\| = 1\}$ is upper semicontinuous (see [2, p. 41]). By [22] there exists a dense set $R \subset \operatorname{bd} K$ where Γ is lower semicontinuous. We show that in such points $\Gamma(x)$ must be a singleton, namely $\Gamma(x) = \{v(x)\}$ for all $x \in R$, with $v : R \to \mathbb{R}^2$ continuous. Indeed, since K has the property (ω), by lower semicontinuity and Proposition 5.3 we have for all $x \in R$

$$\Gamma(x) \subset \liminf_{R\setminus\{x\}\ni y \to x} \Gamma(y) \subset \limsup_{K\setminus\{x\}\ni y \to x} \Gamma(y) \subset (\operatorname{bd} N_K^b(x)) \cap \{\|v\| = 1\}. \tag{26}$$

Assume that there are two different points $v_1, v_2 \in \Gamma(x)$, and observe that by epi-Lipschitzianity they are linearly independent. Then any nontrivial convex combination of v_1, v_2 belongs to the interior of $N_K^b(x)$: this contradicts (26). □

6. φ-convexity

The last regularity property we consider concerns sets satisfying an external sphere condition, with locally uniform radius.

Definition 6.1. We say that a closed set $K \subset H$ is φ-convex if there exists a continuous $\varphi : K \times K \to \mathbb{R}^+$ such that for all $x, y \in K$, $v \in N_K^\sigma(x)$

$$\langle v, y - x\rangle \leq \varphi(x, y)\,\|v\|\,\|y - x\|^2. \tag{27}$$

The φ-convexity of a set K means the φ-convexity of the indicator function $I_K(\cdot)$ as defined in [25] (see (9)). Such sets (under the name of "sets with positive reach") were thoroughly studied in finite dimensional spaces by Federer [20], in connection with local uniqueness of the metric projection and smoothness of the distance function. They were also investigated in [35]. In infinite dimensional spaces, φ-convex sets were introduced and studied by A. Canino [10, 11], in connection with global analysis. They were characterized in several ways in [29, Theorem 1.3], which in particular extends to infinite dimensional spaces the results in [20]. The particular case of $\varphi \equiv$ const. was treated in [15].

Each φ-convex set has obviously the property (ω), with $\omega(x, y) = \varphi(x, y)\|y - x\|$, so, in particular, it is σ-regular. Examples of φ-convex sets are convex sets, or sets with a $\mathcal{C}^{1,1}$ boundary; more in general a set K such that for all $x \in K$ there exists a neighborhood $\mathcal{U}$ with either $\mathcal{U} \cap K$ convex, or $\mathcal{U} \cap \operatorname{bd} K$ a $\mathcal{C}^{1,1}$-manifold is φ-convex. The most interesting example is contained in [10, Proposition 1.9]. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, and c, ρ be positive numbers and set $K_c = \{u \in H_0^1(\Omega) : \int_\Omega |\nabla u|^2 \le c^2\}$, $M_\rho = \{u \in L^2(\Omega) : \int_\Omega |u|^2 = \rho^2\}$. Then $M_\rho \cap K_c$ is φ-convex in L^2, provided c^2/ρ^2 is not an eigenvalue of $-\Delta$ in H_0^1.

Let us give a few preliminary results related to φ-convex sets.

Proposition 6.2. *Let $K \subset H$ be a closed set. Consider the statements:*

(a) *there exists a continuous $\psi : K \to \mathbb{R}^+$ such that for each $x \in K$ and for each $v \in N_K^p(x)$ it holds*

$$\langle v, y - x\rangle \le \psi(x)\,\|v\|\,\|y - x\|^2 \qquad \forall y \in K\,; \tag{28}$$

(b) *K is φ-convex;*

(c) *there exists a function $\omega : K \times K \to \mathbb{R}^+$ with the properties*

1) *for all $x, y \in K$, $v \in N_K^p(x)$ it holds*

$$\langle v, y - x\rangle \le \omega(x, y)\,\|v\|\,\|y - x\|^2\,; \tag{29}$$

2) *ω is upper semicontinuous in $(K \times K) \cap \{(x, y) : y \neq x\}$;*

3) $\limsup_{K \ni y \to x} \omega(x, y) < +\infty$.

(d) *all the normal cones coincide, i.e.*

$$N_K^p(x) = N_K^\sigma(x) = N_K^b(x) = N_K^c(x) \qquad \forall x \in K\,. \tag{30}$$

Then the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) *hold. Furthermore,* (d) $\Rightarrow$ (c), *provided* 3) *is substituted by*

3') $\limsup_{K \ni y \xrightarrow{d} x} \omega(x, y) < +\infty$,

and in finite dimensional spaces (c) $\Leftrightarrow$ (d).

Proof. (a) $\Rightarrow$ (b). Clearly (28) implies sequential $(s \times w)$-closedness of the graph of $x \mapsto N_K^p(x)$ (see the proof of (e) $\Rightarrow$ (b) of Proposition 4.2 and the characterization (3)), i.e. $w-\limsup_{K \ni x' \to x} N_K^p(x') \subset N_K^p(x)$. Then by Theorem 2.6.1 (b) [14] we have

$$N_K^c(x) = \operatorname{cl}\operatorname{co}\,(w-\limsup_{K \ni x' \to x} N_K^p(x')) \subset \operatorname{cl}\operatorname{co} N_K^p(x) = N_K^p(x)\,, \tag{31}$$

where the last equality holds since $N_K^p(x)$ is sequentially weakly closed and convex. Thus K is φ-convex.

(c) $\Rightarrow$ (d). One can see as above that 1) $\div$ 3) imply sequential $(s \times w)$-closedness of the graph of $x \mapsto N_K^p(x)$, and (d) follows from the relation (31).

(b) $\Rightarrow$ (a). Let K be φ-convex. For each $x_0 \in K$ there exist $p(x_0) > 0$, $r(x_0) > 0$ such that

$$\langle v, y - x\rangle \le p(x_0)\,\|v\|\,\|y - x\|^2 \qquad \forall x, y \in K \cap B(x_0, 2r(x_0)), \quad \forall v \in N_K^p(x)\,.$$

Then $\langle v, y - x\rangle \le (p(x_0) + 1/r)\,\|v\|\,\|y - x\|^2$ for all $x \in K \cap B(x_0, r(x_0))$, $y \in K$ and $v \in N_K^p(x)$. Let $\{U_\alpha\}$ be a locally finite refinement of the covering $\{B(x_0, r(x_0)) \cap K : x_0 \in K\}$ with a continuous partition of unity $\{\xi_\alpha(\cdot)\}$ subordinate to it. Let $x_\alpha \in K$ be such that $U_\alpha \subset B(x_\alpha, r(x_\alpha)) \cap K$, and set $\psi(x) = \sum_\alpha \xi_\alpha(x) p(x_\alpha)$, $x \in K$. Then the property (28) holds.

(d) $\Rightarrow$ (c). Define, for $x, y \in K$, $x \neq y$,

$$\omega(x,y) = \sup_{\|v\| \le 1, v \in N_K^p(x)} \frac{\langle v, y-x \rangle}{\|y-x\|^2} \vee 0\,.$$

By the same argument of Proposition 4.2 (b) $\Rightarrow$ (c), properties 1) and 2) hold. To show 3), observe that the cone $N_K^p(x)$ has closed graph. Take sequences $\{y_n\} \subset K$, $y_n \overset{d}{\longrightarrow} x$, and $v_n \in N_K^p(x)$, $\|v_n\| = 1$ such that $\langle v_n, (y_n - x)/\|y_n - x\|^2 \rangle \to \limsup_{K \ni y \overset{d}{\longrightarrow} x} \omega(x,y) := L$. Without loss of generality we can assume that

$$\limsup_{n \to \infty} \langle v_n, (y_n - x)/\|y_n - x\| \rangle \le 0.$$

Thus $L < +\infty$, which concludes the proof. □

Remark. i) The above Proposition immediately shows that there are σ-regular sets which are not φ-convex. A simple example is the epigraph of the function $x \mapsto -|x|^{3/2}$.

ii) The property that all Clarke normals to a set are proximal is only necessary for φ-convexity, as Example 7.2 shows.

We conclude the section with a list of characterizations of φ-convex sets. They are essentially known (see [10, 11, 29]), but some of them appear together for the first time, and we provide a few alternative and simpler proofs. In particular, we observe that the local existence and uniqueness of the metric projection into K are straighforward consequences of the geometrical property (ii) in the below statement: in fact the classical argument of Convex Analysis can still be applied. Moreover, the proof of (iv) $\Rightarrow$ (i) is an infinite dimensional version of an argument developed by Federer in [20]: in his paper Peano's existence theorem for ordinary differential equations was applied; instead we use the monotonicity of the metric projection in order to obtain the existence of solutions to an infinite dimensional Cauchy problem. A global argument of the same nature appeared in [5], where the author was concerned with the necessity of convexity for a Chebyshev set (i.e. a set the metric projection into which is everywhere nonempty and unique) with a continuous metric projection. We point out that φ-convex sets appear to be exactly those sets which satisfy locally the above property.

Theorem 6.3. *Let $K \subset H$ be closed. The following statements are equivalent:*

(i) *K is φ-convex;*

(ii) *for each $x \in K$ there exist $r > 0$ and $p \ge 0$ such that for all $x_1, x_2 \in K \cap B(x,r)$ one has*

$$d_K\left(\frac{x_1 + x_2}{2}\right) \le p\,\|x_1 - x_2\|^2\,; \tag{32}$$

(iii) *there exists an open set $\mathcal{U} \supset K$ such that each $x \in \mathcal{U}$ has a unique metric projection $\pi_K(x)$ into K, and the map $x \mapsto \pi_K(x)$ is continuous in $\mathcal{U}$.*

(iv) *there exists an open set $\mathcal{U} \supset K$ such that $d_K \in \mathcal{C}^1(\mathcal{U} \setminus K)$.*

Moreover, the gradient of the distance function can be expressed through the (unique) projection as

$$\nabla d_K(x) = \frac{x - \pi_K(x)}{d_K(x)}, \qquad x \in \mathcal{U} \setminus K\,. \tag{33}$$

Finally, the function φ can be chosen to be $1/(2d_{\mathrm{bd}\mathcal{U}})$, and the projection π_K as well as ∇d_K are locally Lipschitzian in $\{y \in \mathcal{U} \setminus K : d_K(y)\varphi(\pi_K(y)) < 1/2\}$.

Proof. (i) $\Leftrightarrow$ (ii) is [11, Proposition 1.12]. We give here an alternative proof of (i) $\Rightarrow$ (ii), based on a direct geometrical argument. Fix $M > \varphi(x)$ and let $r > 0$ be such that $\varphi(x') < M$ for all $x \in K \cap B(x, 2r)$. Take $x_1, x_2 \in K \cap B(x, r)$ and set $z = (x_1 + x_2)/2$. Suppose that $z \notin K$ and take by Edelstein's theorem [19] a sequence $\{z_n\} \subset H \setminus K$ such that $z_n \to z$ and $\pi_K(z_n)$ is a singleton, say y_n. For all $n = 1, 2, \dots$ set $w_n = y_n + (z_n - y_n)/(2M\,\|z_n - y_n\|)$ and let B_n be the ball centered at w_n with radius $1/(2M)$. We claim that

$$\overline{B}_n \cap K = \{y_n\} \qquad \forall n \text{ large enough.} \tag{34}$$

In fact, observe first that $\|y_n - x\| < 2r$ for n large enough, so that $\varphi(y_n) < M$. Let $y_n' \in K$ be such that $\|y_n' - w_n\| \le 1/(2M)$, and set $v_n = w_n - y_n$. Then

$$\begin{aligned} \|v_n\|^2 &\ge \|w_n - y_n'\|^2 = \|v_n + y_n - y_n'\|^2 \\ &= \|v_n\|^2 + \|y_n - y_n'\|^2 - 2\langle v_n, y_n' - y_n\rangle\,, \end{aligned}$$

i. e.

$$\|y_n - y_n'\|^2 \le 2\langle v_n, y_n' - y_n\rangle\,. \tag{35}$$

On the other hand, by φ-convexity we have

$$\langle v_n, y_n' - y_n\rangle < M\|v_n\|\,\|y_n - y_n'\|^2\,.$$

The above inequality together with (35) and the choice of v_n force y_n' to be y_n.

Call now ξ_n the intersection of the segment joining x_1 and z_n with the boundary of B_n. By elementary considerations we have that

$$d_K(z_n) \le \mathrm{const}\|z_n - \xi_n\|^2\,.$$

By passing to the limit in the above inequality and using (34) we obtain (32).

(ii) $\Rightarrow$ (iii). Given $x \in K$, we find a neighborhood $U(x)$ such that whenever $z \in U(x)$ each sequence minimizing the function $y \mapsto \|z - y\|$ in K is a Cauchy sequence, which therefore converges to an element of the set $\pi_K(z)$; the uniqueness as well as the continuity of the projection are also immediate consequence of the Cauchy property of minimizing sequences. In order to obtain the neighborhood $U(x)$, let p, r be as in (ii). Without loss of generality we can choose $r > 0$ so small that

$$4rp < 1\,. \tag{36}$$

Fix $z \in U(x) := B(x, r/2)$, and let $\{x_n\} \subset K$ be such that

$$\|z - x_n\|^2 \le d_K^2(z) + \varepsilon_n,\ n = 1, 2, \dots$$

for some $\varepsilon_n \to 0^+$. There is no loss of generality in assuming that $\|x_n - x\| < r$. Recall that for all $m, n = 1, 2, \ldots$ one has

$$\left\| z - \frac{x_n + x_m}{2} \right\|^2 + \left\| \frac{x_n - x_m}{2} \right\|^2 = \frac{1}{2} \left(\|z - x_n\|^2 + \|z - x_m\|^2 \right) .$$

Then, by (32),

$$\begin{aligned} \left\| \frac{x_n - x_m}{2} \right\|^2 &\leq d_K^2(z) + \frac{\varepsilon_n + \varepsilon_m}{2} - \left\| z - \frac{x_n + x_m}{2} \right\|^2 \\ &\leq \frac{\varepsilon_n + \varepsilon_m}{2} + 2 d_K(z) d_K((x_n + x_n)/2) \\ &\leq \frac{\varepsilon_n + \varepsilon_m}{2} + rp \, \|x_n - x_m\|^2 . \end{aligned}$$

The Cauchy property now follows from (36).

(iii) $\Rightarrow$ (iv) and (33). Fix $x \in \mathcal{U}$, and let $\zeta \in \partial^c d_K(x)$. Then, in particular,

$$\begin{aligned} \langle \zeta, \pi_K(x) - x \rangle &\leq \limsup_{h \to 0^+,\, y \to x} \frac{d_K(y + h(\pi_K(x) - x)) - d_K(y)}{h} \\ &\leq \limsup_{h \to 0^+,\, y \to x} \frac{d_K(y + h(\pi_K(y) - y)) - d_K(y)}{h} \\ &\qquad + \limsup_{h \to 0^+,\, y \to x} \frac{d_K(y + h(\pi_K(x) - x)) - d_K(y + h(\pi_K(y) - y))}{h} \\ &\leq \limsup_{h \to 0^+,\, y \to x} (-d_K(y)) + \limsup_{h \to 0^+,\, y \to x} \|\pi_K(y) - \pi_K(x) + y - x\| \\ &= -\|x - \pi_K(x)\| . \end{aligned}$$

Since $\|\zeta\| \leq 1$, it follows that ζ is parallel to $\pi_K(x) - x$. Thus $\partial^c d_K(x)$ reduces to the singleton $(x - \pi_K(x))/\|x - \pi_K(x)\|$, which is the Gâteaux derivative of $d_K(\cdot)$ at x (see [15, p. 122]). Now the continuity of the Gâteaux derivative w.r.t. x implies the Fréchet differentiability of d_K, and (33) follows.

(iv) $\Rightarrow$ (i). For all $x \in H$, consider the set

$$\Phi(x) = \bigcap_{r > d_K(x)} \overline{\mathrm{co}}\, (B(x, r) \cap K) ,$$

which is the subdifferential of a convex function $g(x)$ (see [1, 5, 6]). Obviously, $\overline{\mathrm{co}}\, \pi_K(x) \subset \Phi(x)$ for all $x \in H$. In [15, Proposition 3.6 (2)] it was proved that under our assumptions the projection $\pi_K(x)$ is a singleton, continuously depending on $x \in \mathcal{U}$, and (33) holds. We claim that the multivalued map

$$\tilde{\Phi}(x) := \begin{cases} \{\pi_K(x)\} & \text{if } x \in \mathcal{U} \\ \Phi(x) & \text{if } x \notin \mathcal{U} \end{cases}$$

is maximal monotone, i.e.

$$\tilde{\Phi}(x) = \Phi(x) \qquad \forall x \in H . \tag{37}$$

In fact, fix $x_0 \in \mathcal{U}$, take $q_0 \neq \pi_K(x_0)$, and set $y_\tau = x_0 + \tau(q_0 - \pi_K(x_0))$, $\tau > 0$. Then, for τ sufficiently small,

$$\begin{aligned} \langle \pi_K(y_\tau) - q_0, y_\tau - x_0 \rangle &= \langle \pi_K(x_0) + o(1) - q_0, y_\tau - x_0 \rangle \\ &\leq \tau \left(-\|\pi_K(x_0) - q_0\|^2 + o(1) \|\pi_K(x_0) - q_0\| \right) < 0 \,, \end{aligned}$$

so that no proper extension of $\tilde{\Phi}$ can be monotone.

Fix now $x \in K$ with $N_K^p(x) \neq \emptyset$. Let $v \in N_K^p(x)$, $\|v\| = 1$, be such that

$$0 < \tau_0 := \sup\{t > 0 \,:\, \pi_K(x + tv) = x\} < +\infty \,.$$

We claim that $y := x + \tau_0 v \notin \mathcal{U}$.

Indeed, assuming that $y \in \mathcal{U}$, by continuity we would have $\pi_K(y) = x$ and $d_K(y) = \tau_0$. It is easy to see that for each $\xi \in H$ the set $\Phi(\xi) - \xi$ is the (Fréchet) subdifferential of the function $\xi \mapsto g(\xi) - \|\xi\|^2/2$, and for all $\xi_i \in H$, $u_i \in \Phi(\xi_i) - \xi_i$ $(i = 1, 2)$ it holds

$$\langle u_1 - u_2, \xi_1 - \xi_2 \rangle \geq -\|\xi_1 - \xi_2\|^2 \,.$$

This implies that the Cauchy problem

$$\begin{aligned} \dot{\xi} &\in \xi - \Phi(\xi), \\ \xi(0) &= y \end{aligned} \tag{38}$$

admits a (unique) solution ξ in some interval $I = [0, T]$, $T > 0$, in the sense of Definition 2.1 in [18] (see Definition 1.4 and Theorem 3.2 in [18]). In particular, ξ is continuous on I and absolutely continuous in all compact subintervals of $(0, T)$. Let $r > 0$ be so small that $\xi(t) \in \mathcal{U}$ for all $t \in [0, r]$. By (37) and (33), ξ is the solution of the Cauchy problem

$$\begin{aligned} \dot{\xi} &= \xi - \pi_K(\xi) \, (= \frac{1}{2} \nabla d_K^2(\xi)), \\ \xi(0) &= y. \end{aligned} \tag{39}$$

It follows from (39) that

$$\frac{d}{dt} \frac{1}{2} d_K^2(\xi(t)) = \|\dot{\xi}(t)\|^2 = d_K^2(\xi(t)) \,, \tag{40}$$

so that

$$d_K(\xi(t)) = \tau_0 e^t \,. \tag{41}$$

We observe that, by (40) and (41), $\|\xi(t) - y\| \leq \tau_0 \left(e^t - 1 \right) = d_K(\xi(t)) - \tau_0$. On the other hand, obviously,

$$d_K(\xi(t)) \leq \tau_0 + \|\xi(t) - y\| \,.$$

Therefore,

$$d_K(\xi(t)) = \tau_0 + \|\xi(t) - y\| \,, \tag{42}$$

which means that $t \mapsto \xi(t)$ parametrizes a straight line segment in the direction $v = \nabla d_K(y)$, namely for $0 < t < r$ there exists $s > 0$ small enough such that $\xi(t) = y + sv$. Thus $\pi_K(+(\tau_0 + s)v) = x$, contradicting the definition of τ_0.

Fix now $z \in \mathcal{U} \setminus K$ and $y \in K$. Let $x = \pi_K(z)$, $w = (z-x)/\|z-x\|$ and set $S = \{t > 0 : \pi_K(x+tw) = x\}$. It follows from the previous claim that

$$\sup S \geq d_{\mathrm{bd}\mathcal{U}}(x) > 0 . \tag{43}$$

Fix any $t \in S$. From

$$\|x + tw - y\| \geq d_K(x+tw) = t$$

it follows

$$2t\langle w, y - x\rangle \leq \|y - x\|^2 .$$

The above inequality and Proposition 6.2 imply the φ-convexity of K, with $\varphi(x) = 1/(2d_{\mathrm{bd}\mathcal{U}}(x))$.

Now we prove the local Lipschitz continuity of π_K on $\mathcal{U}^* := \{y \in \mathcal{U} \setminus K : d_K(y)\varphi(\pi_K(y)) < 1/2\}$. To this aim, fix $x, y \in \mathcal{U}^*$; by (i)

$$\begin{aligned}
\langle x - \pi_K(x), \pi_K(y) - \pi_K(x)\rangle &\leq \varphi(\pi_K(x))\|\pi_K(y) - \pi_K(x)\|^2 d_K(x) , \\
\langle y - \pi_K(y), \pi_K(x) - \pi_K(y)\rangle &\leq \varphi(\pi_K(y))\|\pi_K(y) - \pi_K(x)\|^2 d_K(y) ,
\end{aligned}$$

so that

$$\begin{aligned}
&\|\pi_K(x) - \pi_K(y)\|^2 - \langle x - y, \pi_K(x) - \pi_K(y)\rangle \leq \\
&\quad \leq (d_K(x)\varphi(\pi_K(x)) + d_K(y)\varphi(\pi_K(y)))\,\|\pi_K(y) - \pi_K(x)\|^2 .
\end{aligned}$$

By the Cauchy-Schwartz inequality we have

$$(1 - (d_K(x)\varphi(\pi_K(x)) + d_K(y)\varphi(\pi_K(y))))\,\|\pi_K(y) - \pi_K(x)\| \leq \|x - y\| ,$$

whence the local Lipschitzianity follows. □

Corollary 6.4. *Let $K \subset H$ be closed. Then K is φ-convex if and only if there exists an open set $\mathcal{U} \supset K$ such that the unique (bounded from below) viscosity solution of the boundary value problem*

$$\begin{cases} \|\nabla u(x)\| = 1 & \textit{on } H \setminus K , \\ u(x) = 0 & \textit{for } x \in \mathrm{bd}\, K \end{cases} \tag{44}$$

is of class $\mathcal{C}^{1,1}_{\mathrm{loc}}$ in $\mathcal{U} \setminus K$. Moreover, in such case, for all $x \in \mathrm{bd}\, K$ and all $v \in N^c_K(x)$, $v \neq 0$, the directional derivative $\partial u(x)/\partial v$ exists and

$$\frac{\partial u(x)}{\partial v} = \|v\| . \tag{45}$$

Proof. The unique viscosity solution of (44) bounded from below is $u(x) = d_K(x)$ (see [17]), and its $\mathcal{C}^{1,1}_{\mathrm{loc}}$-regularity follows from Theorem 6.3. Now (45) follows from Proposition 6.2. □

Remark. Theorem 6.3 implies that a closed set K is convex if and only if it is Chebyshev, and the metric projection into it is continuous (Asplund's theorem [1]). We feel it is a natural question whether the projection into a φ-convex set is *globally* nonempty, though not necessarily unique. The answer is negative, as the Example 7.3 shows.

7. Counterexamples

Example 7.1. Sets in infinite dimensional spaces exhibiting differences with the finite dimensional case.

Let H be separable, with an orthonormal basis written as $\{e^*, e_1, e_2, \dots\}$.

a). Let $K = \{0\} \cup \{e_n/n \, : \, n = 1, 2, \dots\}$. Obviously K is compact. Observe that $N^\sigma_K(0) = H$, because, if $v \in H$,

$$\lim_{n\to\infty} \langle v, e_n \rangle = 0 \,. \tag{46}$$

Thus also $N^b_K(0) = N^c_K(0) = H$, and so K is σ-regular. Observe that 0 is not an isolated point, but $T^\sigma_K(0) = T^b_K(0) = T^c_K(0) = \{0\}$. Moreover, by (46), $\lim_{K\ni y\to 0}\langle v, y/\|y\|\rangle = 0$ for all v, but $\mathrm{bd}\, N^\sigma_K(0) = \emptyset$. Finally, take $v_n = y_n = e_n/n$, and observe that, since $v_n \in N^\sigma_K(0)$, the condition (e) 3) of Proposition 4.4 is violated. On the other hand, 0 is an isolated point of K w.r.t. the directional convergence, so that 3') of Corollary 5.2 holds trivially.

b). Consider the compact set $K = [0,1]e^* \cup \{e_n/n \, : \, n = 1, 2, \dots\}$. We have $T^b_K(0) = T^\sigma_K(0) = \mathbb{R}^+ e^*$. Then $-e^* \in \mathrm{int}\, N^b_K(0)$ but $\langle -e^*, e_n/n\rangle = 0$ for all n, so that the property (17) does not hold.

c). Let $K = \{0\}\cup\{(e_n+e^*)/n\}$. Then K is compact and, as it is easy to see, $T^\sigma_K(0) = \mathbb{R}^+e^*$, while $T^b_K(0) = T^c_K(0) = \{0\}$. This shows that K is sleek, but not σ-regular. Moreover, $d(e^*/n, K) = \|e^*/n - (e_{2n} + e^*)/(2n)\| = 1/(\sqrt{2}n)$, so that $e^* \in N^b_K(0)$ does not satisfy the condition (10).

d). Set $K = \bigcup_{n\geq 1}\mathbb{R}(e_1+e_n)$. Observe that K is a closed cone and e_1+e_n weakly converges to e_1. Thus $T^b_K(0) = K$, while $T^\sigma_K(0) = K \cup \mathbb{R}\, e_1$. However, $N^b_K(0) = N^\sigma_K(0) = \mathbb{R}\, e^*$.

e). This example was inspired by Counter-example 3.1 in [33].

For $n = 1, 2, \dots$ set

$$A_n = \left\{\frac{e_n}{n}\right\} \cup \left\{ \frac{e_n}{n} + \bigcup_{m=1}^\infty \left[\frac{1}{m}, +\infty\right)\left\{\frac{e_m}{n} + e^*\right\}\right\}$$

and

$$K = \mathbb{R}^+e^* \cup \left[\bigcup_{n=1}^\infty A_n\right].$$

Observe that K is closed, and, for all n, $T^b_K(e_n/n) = \{0\} = T^c_K(e_n/n)$. Thus,

$$\liminf_{K\ni y\to 0} T^b_K(y) = \{0\}\,.$$

On the other hand, $T^c_K(0) = T^b_K(0) = \mathbb{R}^+e^*$, so that K is regular in the sense of Clarke [13, Definition 2.4.6], but it is not sleek. Notice that here $N^\sigma_K(0) = N^c_K(0)$ but σ-regularity is violated since $T^\sigma_K(e_n/n) = \mathbb{R}^+e^* \neq T^c_K(e_n/n)$. We have also $-e^* \in \mathrm{int}\, N^\sigma_K(0)$, but $\limsup_{K\ni y\to 0}\langle -e^*, y/\|y\|\rangle = 0$, although in this case $\mathrm{bd}\, N^\sigma_K(0) \neq \emptyset$.

Example 7.2. A plane sleek set with some pathologies.

Consider the two curves $\gamma_1 = \{(x, \sqrt{x}), x \geq 0\}$ and $\gamma_2 = \{(x, 2\sqrt{x}), x \geq 0\}$, and the decreasing sequences

$$x'_n = \frac{1}{n^2} - \frac{2}{n^4}, \quad x''_n = \frac{1}{n^2} - \frac{1}{n^4}, \quad n \geq 2\,.$$

Observe that $x''_{n+1} < x'_n$ for all n and the points

$$P_n = (x'_n, \frac{1}{n}), \quad Q_n = (x''_n, \frac{1}{n})$$

belong to the open region between γ_1 and γ_2. Taking into account that the segments $[x'_n, x''_n]$ are disjoint we define a $\mathcal{C}^2$ function $\psi : (0, +\infty) \to \mathbb{R}^+$ such that $\psi(x) = 1/n$ for $x \in [x'_n, x''_n]$ and $\sqrt{x} < \psi(x) < 2\sqrt{x}$ for all $x > 0$. Clearly, ψ can be continuously extended to the whole of $\mathbb{R}^+$ by setting $\psi(0) = 0$. Set $K = \{(x, y) : \psi(x) \leq y \leq 2\sqrt{x},\, x \geq 0\}$. Then K is sleek, and every normal vector to K is proximal (at $(0,0)$ by construction, at other points by $\mathcal{C}^2$-smoothness of ψ and γ_2). However K does not enjoy property (ω). Indeed, let x_n be in the interval (x'_n, x''_n) and $X_n = (x_n, 1/n)$; then $(0, -1) \in N^b_K(X_n)$, and $\langle (0, -1), -X_n/\|X_n\| \rangle \to 1$. Thus property (ω) is violated, even if the function ω is required only to be separately upper semicontinuous. We see also that $(0, -1) \in \operatorname{int} N^b_K(0, 0)$, i.e. the statements of Proposition 5.3 and of Corollary 5.4 are not valid in this case. Observe that any neighborhood of $(0, 0)$ must contain points whose projection into K is not a singleton.

Example 7.3. A φ-convex set which is not proximinal.

This example was inspired by [7, Theorem 4.1].

Set $x_n = (1 + 2^{-n})e_n$, $n = 1, 2, \ldots$, and let $K = \{x_n\}$. Observe that K is φ-convex, with $\varphi \equiv 1/\sqrt{2}$. Indeed, for all $v \in H$ it holds

$$\langle v, x_n - x_m \rangle \leq \|v\|\,\|x_n - x_m\| \leq \frac{\|v\|}{\sqrt{2}} \|x_n - x_m\|^2\,.$$

On the other hand, $d_K(0) = 1 < \|x_n\|$ for all $n = 1, 2, \ldots$

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