Kuhn-Tucker Conditions and Integral Functionals

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Let $X$ be a decomposable set, $h$ a convex function defined on $\mathbb{R}^n$ with values in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. We show that, under transversality assumptions, the problem $\inf \{ f(x) + h(g(x)), x \in X \}$ admits generalized or exact Kuhn-Tucker multipliers. We consider the case where $f$ is a scalar $\bar{\mathbb{R}}$-valued integral functional and $g$ is a vector integral functional with values in $(\bar{\mathbb{R}})^n$. These properties are related to growth conditions between integrands.

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1. Introduction

Kuhn-Tucker conditions are classical in convex analysis [2],[14],[30]. They appear in some nonconvex optimization problems too. In [5] and [25], one can find existence results about minimisation of integral functionals under integral contraints. In [11], [12] and [13], we show that in many cases such local optima are global. V.I. Arkin [1] studies the maxima of integral functionals under inequality constraints. In[3] J.P Aubin and I. Ekeland give a duality result. A. Fougeres [9], R. Vaudene [10] and A. Bourass [6] show that when there is inclusion between level sets of two scalar integral functionals, then we necessarily have a growth condition between the integrands. In this paper, we widen some statements of [1], [3], [9],[10] and [6]. We obtain not only necessary and sufficient conditions, but we also give new results using non usual techniques. In section 2, the notions of stability and of "epi-convex" function are introduced. The stability of a minimization problem is related to the existence of a subdifferential (in the sense of a convex analysis) of the performance function. In many cases, the epi-convexity of a vector function coincides with the convex like notion for the cone $\{0\} \times \mathbb{R}_+$ of empty interior. A stability result is given; among other things it allows to characterize a global minimum by Kuhn-Tucker conditions. We establish a relation with J.P. Aubin’s work [2] and with recent studies by M. Moussaoui, M. Volle [25] and J-P. Penot [20].

In section 3, we carry out a result of interchange of minimization and integration. We use the notion of essential infimum given by Neveu [19]. The result is similar to theorem 3.A
of R.T. Rockafellar [22], though the infimum is taken on a decomposable set in the sense of [15]. We also use the notion of richness introduced by M. Valadier in [6].

In section 4, we show that when the measure is atomless, the vector integral functionals are epi-convex. Non emptiness criteria of the interior, of the relative interior, of the range of a decomposable set by an integral functional are given.

In section 5, we use the previous results, notably the existence of multipliers. For the integral functionals, we characterize the inclusion of level lines in a level set and even in a level line. In the same way, we characterize the inclusion of level sets in a level set. These results are translated in terms of growth conditions. A connection with some results obtained by V.I. Arkin is also provided.

2. Stability and epi-convex functions

We adopt the following notation: \( \mathbb{R} = \mathbb{R} \cup \{ \infty, -\infty \} \).

Let \( X \) be a set, \( Y \) a hausdorff locally convex vector space with dual \( Y^* \). If \( h : Y \rightarrow \mathbb{R} \) is a mapping, its conjugate \( h^* \) is defined on \( Y^* \) by

\[
h^*(y^*) = \sup \{ \langle y, y^* \rangle - h(y), \ y \in Y \}.
\]

Given a mapping \( f : X \rightarrow \mathbb{R} \) and an operator \( g : X \rightarrow Y \), the effective domain of \( f \) is \( dom f = \{ x \in X, \ f(x) < \infty \} \) and the strict epigraph of \( f \) is the subset of \( X \times \mathbb{R} \) defined by

\[
epi^+(f) = \{ (x, r) \in X \times \mathbb{R}, \ f(x) < r \}.
\]

The domain of the operator \( g \), that is \( Dom g \) is a given subset of \( X \), and its range \( r(g) \) is \( \{ g(x), \ x \in Dom g \} \).

Given the optimization problem

\[
(P) \quad \inf \{ f(x) + h(g(x)), \ x \in X \}
\]

where \( h(g(x)) = h(g(x)) \) if \( x \in Dom g \), \( +\infty \) if not, we classically introduce the performance function defined on \( Y \) by

\[
p(y) = \inf \{ f(x) + h(g(x)) + y, \ x \in X, \ g(x) \in dom h - y \}.
\]

Of course, we have \( p(0) = \inf(P) \) and we suppose this quantity is finite in the sequel.

First, let us give some elementary characteristics of the performance function of the problem \( (P) \).

**Lemma 2.1.** If \( r_f(g) = \{ g(x), \ x \in Dom g \cap dom f \} \), then

\[
domp = dom h - r_f(g).
\]

**Proof.**

\[
y \in domp \iff \exists x \in X : h(g(x) + y) < \infty \text{ and } f(x) < \infty \iff \\
\iff \exists x \in Dom g \cap dom f : y \in dom h - g(x) \iff \\
\iff y \in dom h - r_f(g).
\]
Remark 2.2. Let us recall that for $\varepsilon \geq 0$, the $\varepsilon$-approximate subdifferential of the function $p$ at $y_0$ [16], $\partial \varepsilon p(y_0)$ is the set of all $y^* \in Y^*$ verifying one of the following assertions

(i) $\forall y \in Y, \ p(y) \geq p(y_0) - \varepsilon + < y^*, y - y_0 >$;

(ii) $p^*(y^*) + p(y_0) - < y^*, y_0 > \leq \varepsilon$.

For $\varepsilon = 0$, we obtain the subdifferential $\partial p(y_0)$ of $p$ at $y_0$ in the sense of convex analysis.

Definition 2.3. For any $\varepsilon \geq 0$ the set of $\varepsilon$-approximate multipliers of $(P)$ denoted by $M_\varepsilon(P)$, is defined by

$M_\varepsilon(P) = \{y^* \in Y^*, p(0) - \varepsilon \leq \inf \{f(x) + < y^*, g(x) > - h^*(y^*), x \in Domg\}\}$.

The following result is to be compared with [20] Lemma 5.1. and [2] ch. 14.1 theorem 6.

Proposition 2.4. For any $\varepsilon \geq 0$, we have the equality

$\partial_\varepsilon p(0) = M_\varepsilon(P)$.

Proof. Let us first show the inclusion $\partial_\varepsilon p(0) \subset M_\varepsilon(P)$. Given $y^* \in \partial_\varepsilon p(0)$, for any $y \in Y$, we have

$p(y) \geq p(0) - \varepsilon + < y^*, y >$.

Let $x \in Domg$ and $a \in domh$. If $y = a - g(x)$, then:

$h(a) + f(x) \geq p(y) \geq p(0) - \varepsilon + < y^*, y >$.

Whence $f(x) + < y^*, g(x) > \geq p(0) - \varepsilon + < y^*, a > - h(a)$.

Passing through this inequality to the infimum relatively to $x$ in $Domg$ and to the supremum relatively to $a$ in $domh$, we get $y^* \in M_\varepsilon(P)$.

Conversely let $y^* \in M_\varepsilon(P)$. For any $x \in Domg$ and any $a \in domh$, we get

$f(x) + < y^*, g(x) > \geq p(0) - \varepsilon + < y^*, a > - h(a)$.

Given $y \in Y$ such that there exists $x \in Domg : g(x) + y = a \in domh$, then

$f(x) + h(g(x) + y) \geq p(0) - \varepsilon + < y^*, y >$.

Taking the infimum on $x$ such that $g(x) + y \in domh$, we obtain

$p(y) \geq p(0) - \varepsilon + < y^*, y >$.

which proves that $y^*$ is an element of $\partial_\varepsilon p(0)$. Proposition 2.4 justifies the introduction of the following notions:

Definition 2.5. $(P)$ is said to be stable if $\partial p(0)$ is non-empty. $(P)$ is said to be weakly stable if for any positive $\varepsilon$, $\partial_\varepsilon p(0)$ is non-empty.

The notion of stability is standard, see for example [28].
Theorem 2.6. The duality result,
\[ \inf(\mathcal{P}) = \sup_{y^* \in \mathcal{Y}^*} \inf \{ f(x) + \langle y^*, g(x) \rangle - h^*(y^*), \ x \in \text{Dom}g \} \]
holds if and only if \((\mathcal{P})\) is weakly stable.
Moreover, the supremum above is a maximum if and only if \((\mathcal{P})\) is stable.

**Proof.** Using the inequality: \(\langle y^*, g(x) \rangle - h^*(y^*) \leq h(g(x))\), we deduce for any \(y^*\):
\[ \inf \{ f(x) + \langle y^*, g(x) \rangle - h^*(y^*), \ x \in \text{Dom}g \} \leq \inf(\mathcal{P}) \]
and the following inequality is always valid:
\[ \sup_{y^* \in \mathcal{Y}^*} \inf \{ f(x) + \langle y^*, g(x) \rangle - h^*(y^*), \ x \in \text{Dom}g \} \leq \inf(\mathcal{P}) \]

Theorem 2.6 is now a consequence of the Proposition 2.4.

**Proposition 2.7.** Let \(x_0\) be an element of \(X\). If the problem \((\mathcal{P})\) is stable, then the following assertions are equivalent

(i) \(x_0\) is a minimizer of \((\mathcal{P})\);
(ii) there exists \(y^* \in \partial g(x_0)\) such that

\[ f(x_0) + \langle y^*, g(x_0) \rangle = \text{Min} \{ f(x) + \langle y^*, g(x) \rangle, \ x \in \text{Dom}g \}. \]

Moreover, the set of all \(y^*\) satisfying (ii) is exactly \(\partial p(0)\).

**Proof.** First suppose that (ii) holds for some \(y^*\). Using 2.2, we get \(\langle y^*, g(x_0) \rangle = h(g(x_0)) + h^*(y^*)\) and consequently
\[ f(x_0) + h(g(x_0)) = f(x_0) + \langle y^*, g(x_0) \rangle - h^*(y^*) \leq \inf \{ f(x) + \langle y^*, g(x) \rangle - h^*(y^*), \ x \in \text{Dom}g \} \leq \inf \{ f(x) + h(g(x)), \ x \in \text{Dom}g, g(x) \in \text{dom}h \} \]
Hence \(x_0\) is a minimizer of \((\mathcal{P})\).

For the converse, we need the following lemma.

**Lemma 2.8.** Let \(x_0\) be a minimizer of \((\mathcal{P})\). If \(y^*\) is an element of \(\mathcal{Y}^*\), then the following assertions are equivalent

(i) \(y^*\) is an element of \(M_0(\mathcal{P})\)
(ii) \(y^*\) is an element of \(\partial g(x_0)\) which satisfies

\[ f(x_0) + \langle y^*, g(x_0) \rangle = \text{Min} \{ f(x) + \langle y^*, g(x) \rangle, \ x \in \text{Dom}g \}. \]

**Proof of 2.8.** Applying the Definition 2.3 we have
\[ (1)(y^* \in M_0(\mathcal{P})) \iff (h^*(y^*) + h(g(x_0)) + f(x_0)) \leq \inf \{ f(x) + \langle y^*, g(x) \rangle, \ x \in \text{Dom}g \}. \]

Given \(y^* \in M_0(\mathcal{P})\). If in (1) we take \(x = x_0\), we obtain \(h^*(y^*) + h(g(x_0)) \leq \langle y^*, g(x_0) \rangle\). Consequently, \(h^*(y^*) + h(g(x_0)) = \langle y^*, g(x_0) \rangle\) and according to 2.2(ii), the second
Corollary 2.9. Let $A$ be a subset of $Y$. If the problem
\[(Q) \quad \inf\{f(x), \ x \in X, \ g(x) \in A\} \text{ is stable, then we have:} \]
\[\inf(Q) = \max_{y^* \in Y^*} \inf \{f(x) + < y^*, g(x) > - \sup_{a \in A} < y^*, a >, \ x \in \text{Dom}g\} \]

**Proof.** We use 2.6 with $h = \psi_A$, where $\psi_A(y) = 0$ if $y \in A$ and $\psi_A(y) = +\infty$ if $y \notin A$.

If $f : X \to \mathbb{R}$ is an $\mathbb{R}$-valued mapping and $r$ a real number, we adopt the notation $f^{\leq r}$ for the level set of order $r$ of $f$ defined by
\[f^{\leq r} = \{x \in X, \ f(x) \leq r\}.\]

We say that $f$ is proper if it does not take the values $-\infty$. For a function $g : X \to Y$, we set $g^{-1}(a) = \{x \in X, \ g(x) = a\}$; this is the level line of order $a$, where $a$ is an element of $Y$.

Corollary 2.10. (Inclusion of a level line in a level set). Let $a$ be an element of $Y$. If the problem $\inf\{-f(x), \ x \in X, \ g(x) = a\}$ is stable, then the following assertions are equivalent
\[(i) \quad g^{-1}(a) \subset f^{\leq r}; \]
\[(ii) \quad \text{there exists } y^* \in Y^* \text{ such that} \]
\[-r \leq \inf\{-f(x) + < y^*, g(x) > - < y^*, a >, \ x \in \text{Dom}g\}.\]

**Proof.** The inclusion of (i) is equivalent to:
\[-r \leq \inf\{-f(x), \ x \in X, \ g(x) = a\}.\]

Since this problem is stable, this inequality is equivalent, by 2.9, to the existence of a $y^*$ in $Y^*$ which verifies
\[-r \leq \inf\{-f(x) + < y^*, g(x) > - < y^*, a >, \ x \in \text{Dom}g\}.\]

This ends the proof.

Corollary 2.11. (Inclusion of a level line in a level line). If the problems $\inf\{-f(x), \ x \in X, \ g(x) = a\}$ and $\inf\{f(x), \ x \in X, g(x) = a\}$ are stable, then the following assertions are equivalent
\[(i) \quad g^{-1}(a) \subset f^{-1}(r); \]
\[(ii) \quad \text{there exists } y^*, z^* \text{ in } Y^* \text{ such that} \]
\[-r \leq \inf\{-f(x) + < y^*, g(x) > - < y^*, a >, \ x \in \text{Dom}g\}; \]
\[r \leq \inf\{f(x) + < z^*, g(x) > - < z^*, a >, \ x \in \text{Dom}g\}.\]
Proof. The inclusion of (i) is equivalent to \( g^{-1}(a) \subset f \leq \) and, by 2.10, this assertion is equivalent to (ii).

As usual, we denote by \( \mathbb{R}_+ \) the set of non negative real numbers.

**Corollary 2.12.** Let \((f_i)_{0 \leq i \leq n}\) be a finite family of mappings defined on \( X \) with values in \( \mathbb{R} \) and let \((r_i)_{0 \leq i \leq n}\) be a sequence of real numbers. If the problem

\[
\inf_{x \in X} f_i(x), \quad x \in X, \quad -\infty < f_i(x) \leq r_i, \quad i = 1, ..., n
\]

is stable, then

\[
\inf(\mathcal{R}) = \max_{y^* \in (\mathbb{R}_+)^n} \inf f_i(x) + \sum_{i=1}^n y_i^* f_i(x) - \sum_{i=1}^n y_i^* r_i, \quad x \in \cap_{i \geq 1} \text{Dom} f_i
\]

Proof. Problem \( (\mathcal{R}) \) can be written:

\[
\inf_{x \in X} f_i(x), \quad x \in X, \quad g(x) \in A; \quad g = (f_i)_{1 \leq i \leq n}, \quad r = (r_i)_{1 \leq i \leq n} \text{ and } A = r - (\mathbb{R}_+)^n. \]

The \((\mathbb{R})^n\)-valued map \( g \) is considered as an \((\mathbb{R})^n\)-valued operator.

For \( y^* \in \mathbb{R}^n \) we have, \( \sup \langle y^*, a \rangle, \quad a \in A \rangle = \langle y^*, r \rangle \) if \( y^* \in \mathbb{R}_+^n \), and \(+\infty \) if not.

Since \( \text{Dom} g = \cap_{i \geq 1} \text{Dom} f_i \), with \( \text{Dom} f_i = \{x \in X : f_i(x) \in \mathbb{R}\} \), using Corollary 2.9 we obtain the desired result.

**Corollary 2.13.** (Inclusion of a level set in a level set). Let \((f_i)_{0 \leq i \leq n}\) be a finite family of proper \( \mathbb{R} \)-valued mappings defined on \( X \) and \((r_i)_{0 \leq i \leq n}\) real numbers. If the problem

\[
\inf \{ -f_0(x), \quad x \in X, \quad f_i(x) \leq r_i, \quad i = 1, ..., n \}
\]

is stable, then the following assertions are equivalent

(i) \( \cap_{i \geq 1} f_i^{\leq r_i} \subset f_0^{\leq r_0} \); 
(ii) there exists \( y^* \) in \((\mathbb{R}_+)^n\) such that for all \( x \) in \( \cap_{i \geq 1} \text{Dom} f_i \)

\[-r_0 \leq -f_0(x) + \sum_{i=1}^n y_i^* f_i(x) - \sum_{i=1}^n y_i^* r_i.\]

Proof. Since the \( f_i \) are proper, the inclusion of the assertion (i) is equivalent to \( -r_0 \leq \inf \{-f_0(x), \quad x \in X, \quad -\infty < f_i(x) \leq r_i, \quad i = 1, ..., n\} \). By Assumption this last problem being stable, the inclusion of (i) is equivalent, by virtue of 2.12, to the existence of some \( y^* \) in \((\mathbb{R}_+)^n\) such that for all \( x \in \cap_{i \geq 1} \text{Dom} f_i, -r_0 \leq -f_0(x) + \sum_{i=1}^n y_i^* f_i(x) - \sum_{i=1}^n y_i^* r_i.\)

But, since the \( f_i \) are proper, we have \( \text{Dom} f_i = \text{Dom} f_i \). That proves the equivalence between the given assertions.

The results 2.6,...,2.13 show the importance of the stability of an optimization problem. Whence the question : under what conditions the performance function is convex ?. In order to answer this question, it seems necessary to introduce the following definition.

**Definition 2.14.** The strict epigraph of the mapping \((g, f)\) denoted by \( epi^{+}(g, f) \), is defined by

\[ epi^{+}(g, f) = \{(y, r) \in Y \times \mathbb{R}, \quad \exists x \in X, \quad g(x) = y \quad \text{and} \quad f(x) < r\}. \]

We say that \((g, f)\) is epi-convex if \( epi^{+}(g, f) \) is convex.
Corollary 2.16. If \( \Rightarrow \Leftrightarrow \exists \Rightarrow \exists \Rightarrow \exists \Leftrightarrow \exists \), \( \Rightarrow \Leftrightarrow \exists \Rightarrow \exists \Rightarrow \exists \Leftrightarrow \exists \), \( \Rightarrow \Leftrightarrow \exists \Rightarrow \exists \Rightarrow \exists \Leftrightarrow \exists \), \( \Rightarrow \Leftrightarrow \exists \Rightarrow \exists \Rightarrow \exists \Leftrightarrow \exists \). Consequently, \( \lambda > 0 \).

Remark. 2.17. Lemma 2.18. If \((g, f)\) is epi-convex then the set \( r_f(g) = \{g(x), x \in \text{dom}g \cap \text{dom}f\} \) is convex.
Proof. If \((g, f)\) is epi-convex, then \(e pi^+(g, f)\) is convex. If \(q\) stands for the projection of \(Y \times \mathbb{R}\) on \(Y\) defined by \(q(y, r) = y\), then we get the following equality:

\[ r_f(g) = q(e pi^+(g, f)), \]

from which the convexity of \(r_f(g)\) follows.

**Theorem 2.19.** We suppose \(Y\) is finite dimensional. Let \(h : Y \to \mathbb{R}\) be a convex function. Under the following assumptions:

(i) the function \((g, f)\) is epi-convex;
(ii) the origin is in the relative interior of \(\text{dom} h - r_f(g)\),

the problem \((P)\) is stable.

Proof. By virtue of 2.16 the performance function of the problem \((P)\) is convex and \(p(0) = \inf(P)\) is finite by assumption. Using the second assumption, 2.1 and 2.17, we can deduce that the performance function is subdifferentiable at the origin. Therefore the problem \((P)\) is stable by definition.

3. On the interchange of minimization and integration

Let \((\Omega, \tau, \mu)\) be a measured space by a \(\sigma\)-finite positive measure \(\mu\), \(E\) a separable Banach space with Borel tribe \(\mathcal{B}(E)\). Consider a subset \(X\) of the space \(L_0(\Omega, E)\) of classes of measurable functions \((\text{for } \mu\text{-a.e equality})\) defined on \(\Omega\) and with values in \(E\). The set \(X\) is supposed to be decomposable in the following sense [29]: for all \(x, y \in X\) and all \(A \in \tau\) the function \(y 1_A + x 1_{\Omega \setminus A}\) is in \(X\), where \(1_S\) stands for the characteristic function of \(S \in \tau\).

Given \(v \in L_0(\Omega, \mathbb{R})\), we denote by \(I_v\) or \(\int_\Omega^* v d\mu\) the upper integral of \(v\) defined by:

\[ I_v = \int_\Omega^* v d\mu = \inf \{ \int_\Omega u d\mu, u \in L_1(\Omega, \mathbb{R}), v \leq u \mu - \text{a.e} \}. \]

If \(f : \Omega \times E \to \mathbb{R}\) is an \(\tau \otimes \mathcal{B}(E)\) measurable scalar integrand, the functional integral \(I_f\) is defined on \(X\) by:

\[ I_f(x) = I_{f(x)} = \int_\Omega^* f(x) d\mu, \]

where \(f(x)\) stands for the function \(\omega \in \Omega \mapsto f(\omega, x(\omega))\). If \(g : \Omega \times E \to (\mathbb{R})^n\) is an \(\tau \otimes \mathcal{B}(E)\) measurable vector integrand, then \(g = (g_i)_{1 \leq i \leq n}\) where \(g_i\) is a scalar measurable integrand. Likewise we consider the integral functional \(I_g\) defined on \(X\) by

\[ I_g(x) = (I_{g_i}(x))_{1 \leq i \leq n}. \]

We also adopt the notation \(g(x)\) for the mapping \(\omega \in \Omega \mapsto g(\omega, x(\omega))\). Let us recall that Neveu [19], II.4 shows that any family \(\{v_i, i \in I\}\) of elements of \(L_0(\Omega, \mathbb{R})\) has an essential infimum \(\text{ess } \inf_I v_i\) defined by:

\[ (\forall i \in I, v_i \geq v \mu - \text{a.e}) \iff (\text{ess } \inf_I v_i \geq v \mu - \text{a.e}). \]
Theorem 3.1. Let $X$ be a decomposable set and $f : \Omega \times E \to \mathbb{R}$ a measurable integrand. We have
\[
\inf_X I_f(x) = I \text{ess inf}_X f(x)
\]
provided the left hand side is distinct from $\infty$.

Proof. Let $y$ be an element of $X$ such that $I_f(y) < \infty$. There exists $v \in L_1(\Omega, \mathbb{R})$ such that $f(y) \leq v \mu - a.e$. We consider the sets
\[
f(X) = \{ f(x), x \in X \}; U = \{ u \in L_1(\Omega, \mathbb{R}), \exists x \in X, f(x) \leq u \leq v \mu - a.e \}.
\]

Lemma 3.2. The sets $f(X)$ and $U$ are non empty complete lattices.

Proof of 3.2. Let us first verify that $f(X)$ is a complete lattice. Let $x, y$ be two elements of $X$. If we consider the set,
\[
A = \{ \omega \in \Omega, f(\omega, x(\omega)) \leq f(\omega, y(\omega)) \}
\]
then,
\[
\inf \{ f(x), f(y) \} = f(x)1_A + f(y)1_{\Omega \setminus A} = f(x1_A + y1_{\Omega \setminus A}).
\]
Since $X$ is decomposable we deduce that $\inf \{ f(x), f(y) \} \in X$. Similarly $\sup \{ f(x), f(y) \} \in X$.

By means of a similar proof, let us prove that $U$ is a complete lattice. Let $u_1$ and $u_2$ be two elements of $U$. By definition there exists two elements $x_1$ and $x_2$ of $X$ such that $f(x_i) \leq u_i \leq v \mu - a.e$. If $B = \{ \omega \in \Omega, u_1(\omega) \leq u_2(\omega) \mu - a.e \}$, we have
\[
v \geq \inf \{ u_1, u_2 \} = u_11_B + u_21_{\Omega \setminus B} \geq f(x_1)1_B + f(x_2)1_{\Omega \setminus B} = f(x_11_B + x_21_{\Omega \setminus B}).
\]
Since $X$ is decomposable we obtain
\[
\inf \{ u_1, u_2 \} \in U.
\]
In the same way $\sup \{ u_1, u_2 \} \in U$ can be shown.

Lemma 3.3. If $\text{ess inf} u$ is integrable, then for any $x \in X$, $f(x)$ does not take $\mu$-a.e the value $-\infty$.

Proof. Suppose the contrary. Let $x$ be in $X$ such that the set $A = \{ \omega \in \Omega, f(\omega, x(\omega)) = -\infty \}$ is of positive measure. If $y$ and $v$ are defined as in the beginning of the proof of 3.1, by lemma 3.2, we consider $z$ in $X$ such that:
\[
f(z) = \inf \{ f(x), f(y) \}.
\]
Thus we get $f(z) \leq v \mu - a.e$ and for $\omega \in A, f(\omega, z(\omega)) = -\infty$. If $\alpha$ is a positive integrable function, then we set, for $n \in \mathbb{N}
\[
u_n = v1_{\Omega \setminus A} - n \sup \{ \alpha, \|v\| \} 1_A.
\]
It is obvious that $f(z) \leq u_n \leq v$. Consequently $u_n$ is in $U$ and we have
\[
I \text{ess inf}_U u \leq \inf \int_\Omega u_n d\mu = -\infty
\]
which leads to a contradiction.
Lemma 3.4. If $\text{ess inf}_U u$ is integrable, then

$$\text{ess inf}_U u = \text{ess inf}_X f(x) \quad \mu - \text{a.e.}$$

**Proof.** By virtue of the definition of $U$ we have

$$\text{ess inf}_X f(x) \leq \text{ess inf}_U u \quad \mu - \text{a.e.}$$

Conversely, let us show that for any $x \in X$ we have

$$\text{ess inf}_U u \leq f(x) \quad \mu - \text{a.e.}$$

For this, let $y$ and $v$ be as above. $f(X)$ being a complete lattice, by lemma 3.2, we consider $z \in X$ which satisfies $f(z) = \inf\{f(x), f(y)\}$. Let us prove that

$$\text{ess inf}_U u \leq f(z) \quad \mu - \text{a.e.}$$

If this is not the case, then the set $A = \{\omega \in \Omega, f(\omega, z(\omega)) < \text{ess inf}_U u(\omega)\}$ is of positive measure. Let $\alpha$ be a positive integrable function on $\Omega$. We set for $n \in \mathbb{N}$,

$$A_n = \{\omega \in A, -n\alpha(\omega) \leq f(\omega, z(\omega))\}.$$ 

By lemma 3.3, the union of the $A_n$ is $A$. If $A_k$ has a positive measure, then we set $u = v1_{\Omega\setminus A_k} + f(z)1_{A_k}$.

By construction, $u \in L_1(\Omega, \mathbb{R})$ and $f(z) \leq u \leq v \quad \mu - \text{a.e.}$ Hence $u$ is an element of $U$. But for any $\omega \in A_k$, we get the contradiction

$$\text{ess inf}_U u(\omega) \leq u(\omega) = f(\omega, z(\omega)) < \text{ess inf}_U u(\omega).$$

This shows that $A$ is of null measure. Moreover, for any $x \in X$ we have

$$\text{ess inf}_U u \leq f(z) \leq f(x) \quad \mu - \text{a.e.}$$

and the proof of 3.4 is complete.

Lemma 3.5. If $\inf_U \int_{\Omega} u \, d\mu$ is a real number then $\text{ess inf}_U u$ is integrable and we get

$$\inf_U \int_{\Omega} u \, d\mu = \int_{\Omega} \text{ess inf}_U u \, d\mu.$$ 

**Proof.** Let $r = \inf_U \int_{\Omega} u \, d\mu$ and $u_n \in U$ be such that $\lim_n \int_{\Omega} u_n \, d\mu = r$. Since $U$ is a complete lattice (Lemma 3.2), the sequence $\bar{u}_n = \inf\{u_i, 1 \leq i \leq n\}$ is in $U$ and satisfies $\lim_n \int_{\Omega} \bar{u}_n \, d\mu = r$. Let $\bar{u} = \lim_n \bar{u}_n$. By the monotone convergence theorem $\bar{u}$ is integrable and $\int_{\Omega} \bar{u} \, d\mu = r$. In order to prove 3.5 all we have to do is to show that $\bar{u}$ is the essential infimum of the family $U$. 


Let $w \in L_0(\Omega, \mathbb{R})$ be such that for any $u \in U, w \leq u \mu - a.e.$ It is obvious that $w \leq \bar{u} \mu - a.e.$ One needs only to show that $\bar{u}$ is an essential lower bound of the family $U$. That is to say for any $u \in U$ we have $\bar{u} \leq u \mu - a.e.$

For this, let us consider an element $u \in U$ and the sequence $v_n = \inf\{u, \bar{u}_n\}$. Since $U$ is a complete lattice, $v_n$ is in $U$. The sequence $(v_n)$ is decreasing and we have

$$r \leq \lim_n \int_{\Omega} v_n d\mu \leq \lim_n \int_{\Omega} \bar{u}_n d\mu = r.$$ 

Thus by the monotone convergence theorem, the sequence $(v_n)$ converges to $\inf\{u, \bar{u}\}$ in $L_1(\Omega, \mathbb{R})$ with

$$\int_{\Omega} \inf\{u, \bar{u}\} d\mu = \lim_n \int_{\Omega} v_n d\mu = r.$$ 

As a result

$$\int_{\Omega} \bar{u} - \inf\{u, \bar{u}\} d\mu = 0;$$ 

and consequently $\inf\{u, \bar{u}\} = \bar{u} \mu - a.e$ or $\bar{u} \leq u \mu - a.e$. Therefore we have shown that $\bar{u} = \text{ess inf}_U u$, and 3.5 is proved.

End of the proof of Theorem 3.1. We obviously have the inequality

$$I_{\text{ess inf}_X f(x)} \leq \inf_X I_f(x).$$

If $\inf_X I_f(x) = -\infty$, then the two terms coincide. Consequently, let us suppose $-\infty < \inf_X I_f(x) \leq \inf_U \int_{\Omega} u d\mu < \infty$. In this case, we obtain by Lemma 3.4 and Lemma 3.5

$$\inf_X I_f(x) \leq \inf_U \int_{\Omega} u d\mu = \int_{\Omega} \text{ess inf}_U u d\mu = \int_{\Omega} \text{ess inf}_X f(x) d\mu.$$ 

This proves the previous equality and ends the proof of 3.1.

Theorem 3.1 gives rise to the question of the calculation of the essential infimum of the range by an integrand of a decomposable set.

**Definition 3.6.** An increasing sequence $(\Omega_n)_n$ of measurable sets of finite measure is said to be a $\sigma$-finite covering of $\Omega$ if $\mu(\Omega \setminus \bigcup_n \Omega_n) = 0$.

Given a multifunction $M$ defined on $\Omega$ with values in $E$ and with measurable graph. We denote by $S_M$ the decomposable set of measurable selections of $M$, i.e,

$$S_M = \{x \in L_0(\Omega, E), \forall \omega \in \Omega, x(\omega) \in M(\omega)\}.$$ 

**Definition 3.7.** Let $X$ and $Y$ be two decomposable sets of $L_0(\Omega, E)$. $X$ is said to be rich in $Y$ if $X$ is a subset of $Y$ and if for any $y$ in $Y$, there exists a $\sigma$-finite covering $(\Omega_n)_n$ of $\Omega$ and a sequence $(x_n)$ of elements of $X$ verifying, $y1_{\Omega_n} = x_n1_{\Omega_n}$ for all $n \in \mathbb{N}$.
Let us give an example. Let $M$ be a multifunction defined on $\Omega$ with values in $E$ and with a measurable graph. If $L_p(\Omega, E) \cap S_M$ is non empty, then it is a decomposable set which is rich in $S_M$.

**Theorem 3.8.** Suppose the tribe $\tau \mu$-complete. Let $M$ be a multifunction with non empty values and with $\tau \otimes \mathcal{B}(E)$ measurable graph, and $f : \Omega \times E \rightarrow \mathbb{R}$ a measurable integrand. If $X$ is a decomposable set which is rich in $S_M$, then

$$\text{ess inf}_X f(x)(\omega) = \inf_{e \in M(\omega)} f(\omega, e) \quad \mu - a.e.$$  

**Proof.** Let $u(\omega) = \inf_{e \in M(\omega)} f(\omega, e)$. By virtue of [7] III.39, $u$ is $\tau$-measurable and therefore satisfies $u \leq \text{ess inf}_X f(x) \quad \mu - a.e$

Let us show the inverse inequality. For all $n \in \mathbb{N}$, we consider the multifunction $N_n$ defined by:

$$N_n(\omega) = \begin{cases} 
\{ e \in M(\omega), f(\omega, e) \leq u(\omega) + \frac{1}{n} \} & \text{if } \omega \in u^{-1}(\mathbb{R}) \\
M(\omega) & \text{if } \omega \in u^{-1}(\{ \infty \}) \\
\{ e \in M(\omega), f(\omega, e) \leq -n \} & \text{if } \omega \in u^{-1}(\{ -\infty \})
\end{cases}$$

We check that $N_n$ has an $\tau \otimes \mathcal{B}(E)$ measurable graph and it is with non empty values. Using [7] III.22, we get that $N_n$ admits a measurable selection $x_n \in S_M$. Since $X$ is rich in $S_M$, let $(\Omega^n_p)_p$ be a $\sigma$-finite covering of $\Omega$ and $(x^n_p)_p$ a sequence of elements of $X$ such that for any $p \in \mathbb{N}$

$$x^n_1\Omega^n_p = x^n_p 1_{\Omega^n_p} \quad \mu - a.e.$$  

For all $p \in \mathbb{N}$, we have

$$\text{ess inf}_X f(x) 1_{\Omega^n_p} \leq f(x^n_p) 1_{\Omega^n_p} \mu - a.e.$$  

Taking the limit in $p \in \mathbb{N}$, we obtain for all $n \in \mathbb{N}$

$$\text{ess inf}_X f(x) \leq f(x_n) \quad \mu - a.e.$$  

Hence

$$\text{ess inf}_X f(x) \leq \inf_{n \in \mathbb{N}} f(x_n) = u \quad \mu - a.e.$$  

Which proves 3.8.

**Corollary 3.9.** Suppose that the tribe $\tau$ is $\mu$-complete. Let $M$ be a $\tau \otimes \mathcal{B}(E)$ measurable multifunction with non empty values and $f : \Omega \times E \rightarrow \mathbb{R}$ a measurable integrand. If $X$ is a decomposable subset which is rich in $S_M$, we have

$$\inf_{x \in X} I_f(x) = \inf_{e \in M(\cdot)} f(\cdot, e),$$

provided the left hand side is distinct from $\infty$.

**Proof.** This follows from the theorems 3.1 and 3.8.

A result in relation with the Corollary 3.9 is the Theorem 2.2 of [29] by F. Hiai and H.
Umegaki. In this last theorem $X$ has been replaced by $L_p(\Omega, E) \cap S_M$ and the integrand is supposed to be normal. When the tribe is $\mu$-complete, Corollary 3.9 is a natural extension of the result of Hiai and Umegaki.

Examples of rich decomposable sets which will be useful in the sequel. Let $f : \Omega \times E \to \mathbb{R}$ and $g : \Omega \times E \to \mathbb{R}^n$ be two measurable integrands. The multifunctions defined by $\forall \omega \in \Omega$, $domf(\omega) = domf(\omega, \cdot)$ and $Domg(\omega) = Domg(\omega, \cdot)$ have a measurable graph. We set

$$L_f = \{ x \in L_0(\Omega, E), I_f(x) < \infty \},$$

$$L_g^1 = \{ x \in L_0(\Omega, E), I_g(x) \in \mathbb{R}^n \}.$$

**Lemma 3.10.**

(i) If $L_f$ is non empty, then it is a decomposable subset which is rich in $S_{domf}$.

(ii) If $L_g^1$ is non empty, then it is a decomposable subset which is rich in $S_{Domg}$.

**Proof.** We prove (i). Let us first show that $L_f$ is decomposable. Let $x, y$ be two elements of $L_f$, then there exists $u, v$ in $L_1(\Omega, \mathbb{R})$ such that

$$f(x) \leq u \quad \mu - a.e \text{ and } f(y) \leq v \quad \mu - a.e.$$

If $A$ is an element of $\tau$, then

$$f(1_A + y1_{\Omega \setminus A}) = f(x)1_A + f(y)1_{\Omega \setminus A} \leq u1_A + v1_{\Omega \setminus A} \quad \mu - a.e,$$

which shows that $x1_A + y1_{\Omega \setminus A}$ is in $L_f$. Therefore $L_f$ is decomposable. We now prove the richness of $L_f$ in $S_{domf}$. Let $y$ be in $L_f$ and $x$ in $S_{domf}$. If $(\Omega_n)_{n}$ is a $\sigma$-finite covering of $\Omega$, then the sequence $S_n = \{ \omega \in \Omega_n, f(\omega, x(\omega)) \leq n \}$ is also a $\sigma$-finite covering of $\Omega$. For all $n \in \mathbb{N}, x_n = x1_{S_n} + y1_{\Omega \setminus S_n}$ is in $L_f$ and we have

$$x1_{S_n} = x_n1_{S_n}.$$

Which proves the first assertion. The decomposability of $L_g^1$ is verified thanks to the classical Lemma:

**Lemma 3.11.** Let $x$ be in $L_0(\Omega, E)$. $I_g(x)$ is in $\mathbb{R}^n$ if and only if the map $g(x)$ is integrable.

By 3.11, if $x, y$ are in $L_g^1$ and $A$ is in $\tau$, we have

$$g(x1_A + y1_{\Omega \setminus A}) = g(x)1_A + g(y)1_{\Omega \setminus A} \in L_1(\Omega, \mathbb{R}^n).$$

Consequently $x1_A + y1_{\Omega \setminus A}$ is in $L_g^1$. Let us show the richness of $L_g^1$ in $S_{Domg}$. Let $y$ be in $L_g^1, x \in S_{Domg}$ and let $(\Omega_n)_{n}$ be a $\sigma$-finite covering of $\Omega$. Set $S_n = \{ \omega \in \Omega_n, |g(\omega, x(\omega))| \leq n \}$, where $|.|$ is a norm of $\mathbb{R}^n$. The sequence $(S_n)_{n}$ is a $\sigma$-finite covering of $\Omega$. The sequence $x_n = x1_{S_n} + y1_{\Omega \setminus S_n}$ is in $L_g^1$ and satisfies $x1_{S_n} = x_n1_{S_n} \mu - a.e$. Therefore $L_g^1$ is rich in $S_{Domg}$.

**Lemma 3.12.** Let $M$ and $N$ be two multifunctions with non empty values and with an $\tau \otimes \mathcal{B}(E)$ measurable graph. Let $X$ be a decomposable set which is rich in $S_M$ and $Y$ a decomposable set which is rich in $S_N$. If $X \cap Y$ is non empty, then it is a decomposable set which is rich in $S_{M \cap N}$.
Proof. Let \( y \) be in \( X \cap Y \) and \( x \) be in \( S_{M \cap N} \). Since \( X \) is rich in \( S_M \), there exists a \( \sigma \)-finite covering \((\Omega_{1,n})_n\) and a sequence \((x_n)_n\) of elements of \( X \) verifying
\[
x_1\Omega_{1,n} = x_n1_{\Omega_{1,n}} \mu - a.e., \text{ for any } n \in \mathbb{N}
\]
In the same way, there exists a \( \sigma \)-finite covering \((\Omega_{2,n})_n\) and a sequence \((y_n)_n\) of elements of \( Y \) verifying
\[
x_1\Omega_{2,n} = y_n1_{\Omega_{2,n}} \mu - a.e. \text{ for any } n \in \mathbb{N}
\]
The sequence \( \Omega_n = \Omega_{1,n} \cap \Omega_{2,n} \mu - a.e. \) is a \( \sigma \)-finite covering of \( \Omega \). Let \( z_n = x_1\Omega_n + y_1\Omega/\Omega_n \). Since \( X \) and \( Y \) are decomposable, by using (1) and (2), we deduce that \( z_n \) is in \( X \cap Y \). Besides, from (1) and (2), \( z_n1_{\Omega_n} = x_1\Omega_n \). Which proves that \( X \cap Y \) is rich in \( S_{M \cap N} \).

Proposition 3.13. Let \( M \) be a multifunction with non empty values and with an \( \tau \otimes B \) measurable graph and \( X \) a rich decomposable subset in \( S_M \). We consider the integral functionals \( I_f \) and \( I_g \) defined on \( X \).

(i) If \( Dom I_g \) is non empty, then it is decomposable set which is rich in \( S_{M \cap Dom g} \).
(ii) If \( Dom I_g \cap dom I_f \) is non empty, then it is decomposable set which is rich in \( S_N \) with
\[
N = M \cap Dom g \cap dom f.
\]

Proof. For the first assertion, we notice the equality
\[
Dom I_g = X \cap L^1_g,
\]
and we use 3.10 and 3.12. The second assertion can be proved in the same way, noting that \( dom I_f = X \cap L_f \).

4. Epi-convexity and other properties of the range of a vector integral functional

Throughout this section, the measure \( \mu \) is assumed to be atomless and the tribe \( \tau \mu \)-complete.

Let \( f : \Omega \times E \rightarrow \mathbb{R} \), \( g : \Omega \times E \rightarrow \mathbb{R}^n \) be two measurable integrands, \( M \) be a multifunction with an \( \tau \otimes B(E) \) measurable graph and \( X \) be a decomposable subset which is rich in \( S_M \). The functional \((I_g, I_f)\) is defined on \( X \).

Theorem 4.1. The functional \((I_g, I_f)\) is epi-convex.

Proof. Let us prove that the strict epigraph
\[
epi^+(I_g, I_f) = \{(y, r) \in \mathbb{R}^{n+1}, \exists x \in X, y = I_g(x) \text{ and } I_f(x) < r\},
\]
is convex. Given \((y_i, r_i)\) in \(epi^+(I_g, I_f)\), \(i = 0, 1\), and \( t \) in the unit interval \([0,1]\), let us verify that
\[
t(y_1, r_1) + (1-t)(y_0, r_0) \in epi^+(I_g, I_f).
\]
There exists two elements \( x_i, i = 0, 1 \) of \( X \) such that
\[
y_i = I_g(x_i) \text{ and } I_f(x_i) < r_i.
\]
By the definition of $I_f$ there exists two elements $u_i, i = 0, 1$ of $L_1(\Omega, \mathbb{R})$ satisfying

$$f(x_i) \leq u_i \mu - a.e \quad \text{and} \quad \int_{\Omega} u_i \, d\mu \leq r_i, \ i = 0, 1. \quad (1)$$

By Lemma 3.11, the vector measure $\nu$ defined on $\tau$ by

$$\nu(A) = \left( \int_A (g(x_1) - g(x_0)) \, d\mu, \int_A (u_1 - u_0) \, d\mu \right)$$

is well defined and atomless. Using Lyapunov’s theorem [8], we obtain the convexity of the range of $\nu$. Therefore, there exists a measurable set $A \in \tau$ such that

$$\nu(A) = t\nu(\Omega) + (1-t)\nu(\emptyset) = t\nu(\Omega). \quad (2)$$

Let $x = x_11_A + x_01_{\Omega \setminus A}$, then $x \in X$ and we have

$$I_g(x) = \int_A g(x_1) \, d\mu + \int_{\Omega \setminus A} g(x_0) \, d\mu = \int_A (g(x_1) - g(x_0)) \, d\mu + \int_{\Omega} g(x_0) \, d\mu$$

$$= t \int_{\Omega} g(x_1) - g(x_0) \, d\mu + \int_{\Omega} g(x_0) \, d\mu \quad \text{(see (2))}$$

$$= tI_g(x_1) + (1-t)I_g(x_0) = ty_1 + (1-t)y_0.$$

Besides

$$I_f(x) = \int_A^* f(x_1) \, d\mu + \int_{\Omega \setminus A}^* f(x_0) \, d\mu$$

$$\leq \int_A u_1 \, d\mu + \int_{\Omega \setminus A} u_0 \, d\mu \quad \text{(see (1))}$$

$$= \int_A (u_1 - u_0) \, d\mu + \int_{\Omega} u_0 \, d\mu = t \int_{\Omega} (u_1 - u_0) \, d\mu + \int_{\Omega} u_0 \, d\mu \quad \text{(see (2))}$$

$$= t \int_{\Omega} u_1 \, d\mu + (1-t) \int_{\Omega} u_0 \, d\mu < tr_1 + (1-t)r_0. \quad \text{(see (1))}$$

Hence $(ty_1 + (1-t)y_0, tr_1 + (1-t)r_0) \in epi^+(I_g, I_f)$. This completes the proof of 4.1.

By Lemma 2.18, the set $r_{I_f}(I_g) = \{I_g(x), \ x \in Dom I_g \cap dom I_f\}$ is convex. We will give some criteria that will allow to attest that the interior or the relative interior of $r_{I_f}(I_g)$ are non empty. $C^{int}$ denotes the interior of a subset $C$ of $\mathbb{R}^n$ and $\overline{C}$ its closure.

**Proposition 4.2.** Let $x_0$ be in $Dom I_g \cap dom I_f$ and $N(\omega) = (M \cap Dom g \cap dom f)(\omega)$. If $A_{x_0} = \{\omega \in \Omega, \ g(\omega, x_0(\omega)) \in \overline{g(\omega, N(\omega))}^{int}\}$ is of positive measure, then $I_g(x_0)$ is an element of $r_{I_f}(I_g)^{int}$.

**Proof.** Let us prove firstly the following lemma.

**Lemma 4.3.** The multifunction $L(\omega) = \overline{g(\omega, N(\omega))}^{int}$ has an $\tau \otimes \mathcal{B}(E)$ measurable graph. Its domain is measurable and, as a consequence, $A_{x_0}$ is measurable.
Proof of 4.3. The multifunction $N$ has a measurable graph. If $F(\omega) = g(\omega, N(\omega))$, then $F$ has non empty values. Moreover for any $y \in \mathbb{R}^n$, the function $d(y, F(\omega)) = \inf\{|y-g(\omega,e)|, e \in N(\omega)\}$ is measurable by virtue of [7] III.39. Therefore $F$ is measurable in the sense of [7] III.30. If $G(\omega) = \mathbb{R}^n \setminus F(\omega)$ then $G$ has a measurable graph and by virtue of [7] III.23, the domain $\Omega_0$ of $G$ is $\tau$ measurable. The multifunction $G$ has non empty values on $\Omega_0$ and a measurable graph, hence as a consequence of [7] III.40, $G$ has a measurable graph. Finally $L(\omega) = \mathbb{R}^n \setminus G(\omega)$ has a measurable graph. By virtue of [7] III.23, the domain of $L$ is measurable.

Let us prove 4.2. If $I_g(x_0)$ is not in $r_{I_f}(I_g)^{\text{int}}$ then $I_g(x_0)$ is on the boundary of this convex set. By [4] 2.3.7, there exists $y^*$ in $\mathbb{R}^n \setminus \{0\}$ such that for all $y \in r_{I_f}(I_g)$,

$$< y^*, I_g(x_0) > \leq < y^*, y >$$

or equivalently,

$$0 = \text{Min}\{< y^*, I_g(x) - I_g(x_0) >, x \in \text{Dom}I_g \cap \text{dom}I_f\}.$$ 

But $\text{Dom}I_g \cap \text{dom}I_f$ is a decomposable set which is rich in $S_N$ (3.13(ii)). Using 3.9, this last equality is equivalent to

$$0 = \text{Min}\{< y^*, g(\omega, e) - g(\omega, x_0(\omega)), e \in N(\omega) \text{ for } \mu - a.e \ \omega \in \Omega \}$$

which proves that for any $\omega \in A_{x_0}$, $y^*$ attains its minimum on $\overline{g(\omega, N(\omega))}$ at $g(\omega, x_0(\omega))$.

Since $g(\omega, x_0(\omega))$ is in $\overline{g(\omega, N(\omega))}^{\text{int}}$, we obtain the contradiction $y^* = 0$. Therefore $I_g(x_0)$ is in $r_{I_f}(I_g)^{\text{int}}$.

Definition 4.4. Let $C$ be a subset of $\mathbb{R}^n$. An element $c_0$ of $C$ is in the relative interior of $C$ if $\mathbb{R}_+(C - c_0)$ is a vector space.

This generalization of the notion of relative interior is due to J.M.Borwein and A.S.Lewis in the convex case [27], here is justified by the following two results.

Lemma 4.5. If a linear functional $l$ assumes its minimum on $C$ at a point $c_0$ of the relative interior of $C$, then it is constant on $C$.

Proof. As a matter of fact, we have $l(\mathbb{R}_+(C - c_0)) \geq 0$. Therefore $l(\mathbb{R}_+(C - c_0)) = 0$. As a consequence $l$ is identically equal to $l(c_0)$ on $C$.

Proposition 4.6. Let $x_0$ be in $\text{Dom}I_g \cap \text{dom}I_f$. We set $N(\omega) = (M \cap \text{Dom}g \cap \text{dom}f)(\omega)$. If $g(x_0)$ is a selection of $\omega \mapsto ri(g(\omega, N(\omega)))$, then $I_g(x_0)$ is an element of $ri(r_{I_f}(I_g))$.

Proof. Suppose that $I_g(x_0)$ is not in $ri(r_{I_f}(I_g))$. Then $I_g(x_0)$ is on the boundary of this convex set. By [21], Theorem 1.3, There exists $y^* \in \mathbb{R}^n \setminus \{0\}$ such that

1) for any $y$ in $r_{I_f}(I_g): < y^*, I_g(x_0) > \leq < y^*, y >$,

2) there exists $y_0$ in $r_{I_f}(I_g)$ such that: $< y^*, I_g(x_0) > < < y^*, y_0 >$.

Assertion (1) is equivalent to

$$0 = \text{Min}\{< y^*, I_g(x) - I_g(x_0) >, x \in \text{Dom}I_g \cap \text{dom}I_f\}.$$ 

By 3.9 and 3.13(ii), this last assertion is equivalent to

$$0 = \text{Min}\{< y^*, g(\omega,e) - g(\omega, x_0(\omega)) >, e \in N(\omega) \} \text{ for } \mu - a.e \ \omega \in \Omega.$$
Since \( g(\omega, x_0(\omega)) \) is in \( \text{ri}(g(\omega, N(\omega))) \), we deduce that \( y^* \) is constant on \( g(\omega, N(\omega)) \), by virtue of 4.5. Assertion (2) is equivalent to the existence of an element \( x \) of \( \text{Dom}I_g \cap \text{dom}I_f \) such that the set \( A = \{ \omega \in \Omega, < y^*, g(\omega, x_0(\omega)) > < y^*, g(\omega, x(\omega)) > \} \) is of positive measure. As \( x(\omega) \) is in \( N(\omega) \), this shows that \( y^* \) is not constant on \( g(\omega, N(\omega)) \) and gives a contradiction. This completes the proof of 4.6.

5. **Kuhn-Tucker conditions and growth conditions**

The same assumptions as in section 4 are made.

Let \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) be a convex function. Let us consider the problem

\[
(P) \quad \inf\{I_f(x) + h(I_g(x)), \ x \in X, \ I_g(x) \in \text{dom}h\}.
\]

As in section 2, \( p \) stands the performance function of the problem \( (P) \) and \( \inf(P) \) is supposed to be finite.

The following Theorem extends a result of J.P.Aubin and I.Ekeland[3],[2] chapter 14.2.7.

**Theorem 5.1.** Suppose that the origin is in \( \text{ri}(\text{dom}h - rI_f(I_g)) \). Then the problem \( (P) \) is stable and we have:

\[
\inf(P) = \max_{y^* \in \mathbb{R}^n} \inf\{I_f(x) + < y^*, I_g(x) > - h^*(y^*)\}
\]

where \( u_{y^*}(\omega) = \inf\{f(\omega, e) + < y^*, g(\omega, e) >, \ e \in M \cap \text{Dom}g(\omega)\} \).

**Proof.** By 4.1, \( (I_g, I_f) \) is epi-convex and the assumptions of 2.19 are verified. Therefore \( (P) \) is stable. Using 2.6, we deduce

\[
\inf(P) = \max_{y^* \in \mathbb{R}^n} \inf\{I_f(x) + < y^*, I_g(x) > - h^*(y^*)\}, \ x \in \text{Dom}I_g\}.
\]

By using 3.9, 3.11 and 3.13(i), we have

\[
\inf\{I_f(x) + < y^*, I_g(x) >, \ x \in \text{Dom}I_g\} = I_{u_{y^*}}.
\]

**Corollary 5.2.** Let \( C \) be a convex subset of \( \mathbb{R}^n \). Suppose that the origin is in \( \text{ri}(C - rI_f(I_g)) \). Then

\[
\inf\{I_f(x), \ x \in X, \ I_g(x) \in C\} = \max_{y^* \in \mathbb{R}^n} I_{u_{y^*}} - \sup\{< y^*, c >, c \in C\}
\]

where \( u_{y^*}(\omega) = \inf\{f(\omega, e) + < y^*, g(\omega, e) >, \ e \in M \cap \text{Dom}g(\omega)\} \).

**Proof.** We use 5.1 with \( h = \psi_C \).

**Corollary 5.3.** (Inclusion of a level line in a level set). Suppose that \( a \) is in \( \text{ri}(rI_{I_f}(I_g)) \). Given a real number \( r \), the following assertions are equivalent

\[
(i) \quad I_g^{-1}(a) \subset I_f^r;
(ii) \quad \text{there exists } y^* \in \mathbb{R}^n \text{ such that the function }
\omega \mapsto u(\omega) = \inf\{-f(\omega, e) + < y^*, g(\omega, e) >, e \in M \cap \text{Dom}g(\omega)\} \text{ is integrable and satisfies }-r \leq \int_\Omega u \, d\mu - < y^*, a >.
\]
Proof. 

Lemma 5.4. If $I_f(x) < \infty$ then $I_f(x) = -I_{-f}(x)$.

Proof of 5.4. If $I_f(x) < \infty$ then $f^+(x) = \sup \{0, f(x)\}$ is integrable. Either $f(x)$ is integrable and in this case $I_f(x) = -I_{-f}(x)$; or $f(x)$ is not integrable. In the latter case $f^-(x) = \inf \{0, f(x)\}$ is not integrable and $I_f(x) = -\infty = -I_{-f}(x)$.

Proof of 5.3. By 5.4, if the assertion $(i)$ is satisfied then we have

(i) \quad $I_g^{-1}(a) \subset (-I_{-f})^{\leq r}$; \quad which is equivalent to the next one

(i)" \quad $-r \leq \inf \{I_{-f}(x), x \in X, I_g(x) = a\}$.

By 5.2, $(i)"$ is equivalent to $(ii)$, the reciprocal is immediate.

For $e \in M \cap \text{Dom}(\omega)$, we get

\[ f(\omega, e) \leq < y^*, g(\omega, e) > - u(\omega). \]

If $x$ is an element of $X$ satisfying $I_g(x) = a$, we obtain

\[ I_f(x) \leq < y^*, I_g(x) > - \int_{\Omega} u d\mu = < y^*, a > - \int_{\Omega} v d\mu \leq r. \]

Corollary 5.5. (Inclusion of a level line in a level line). Suppose that $a$ is an element of \( \text{ri}(r_{I_g}(I_g)) \cap \text{ri}(r_{I_f}(I_g)) \). Given a real number $r$, the following assertions are equivalent

(i) \quad $I_g^{-1}(a) \subset I_f^{-1}(r)$

(ii) \quad there exists $y^*, z^*$ such that the functions

\[ u(\omega) = \inf \{-f(\omega, e) + < y^*, g(\omega, e) \}, e \in M \cap \text{Dom}(\omega)\} \]

and

\[ v(\omega) = \inf \{f(\omega, e) + < y^*, g(\omega, e) \}, e \in M \cap \text{Dom}(\omega)\} \]

are integrable and satisfies

\[-r \leq \int_{\Omega} u d\mu - < y^*, a > \quad \text{and} \quad r \leq \int_{\Omega} v d\mu - < z^*, a >.\]

Proof. The assertion $(i)$ is equivalent to the two following conditions $I_g^{-1}(a) \subset I_f^{-1}(r)$ and $r \leq \inf \{I_f(x), x \in X, I_g(x) = a\}$. Then we use 5.3 and 5.2.

Corollary 5.6. We consider a family $(f_i)_{0 \leq i \leq n}$ of measurable scalar integrands. We suppose that there exists real numbers $c_i, 0 \leq i \leq n$, and $x$ of $X$ such that

\[ (H) \quad \text{for every } i = 1, ..., n, - \infty < I_{f_i}(x) < c_i \text{ and } I_{f_0}(x) < \infty. \]

Then if $r = \inf \{I_{f_0}(y), y \in X, - \infty < I_{f_i}(y) \leq c_i, i = 1, ..., n\}$ is finite, we have

\[ r = Max_{y^* \in (\mathbb{R}^+)^n} u_{y^*} - < y^*, e > \]

where $e = (c_i)_{1 \leq i \leq n}$, and

\[ u_{y^*}(\omega) = \inf \{f_0(\omega, e) + \sum_{i=1}^{n} y^*_i f_i(\omega, e), e \in M(\omega) \cap \{\cup_{i=1}^{n} \text{Dom}(f_i(\omega, .))\}\} \]
Proof. Take $C = c - (\mathbb{R}_+)^n$, and $g = (f_i)_{1 \leq i \leq n}$.

The assumption $(H)$ ensures that the origin is in the interior of $C - r_{	ext{fin}}(I_g)$. For $y^* \in \mathbb{R}^n$ we have $\sup \{< y^*, \cdot >, \cdot \in C \} = < y^*, c >$ if $y^* \in \mathbb{R}_+^n$ and $+\infty$ if not.

Moreover, we have $\text{Dom} g(\omega, \cdot) = \cap_{i \geq 1} \text{Dom} f_i(\omega, \cdot)$, and therefore 5.6 is a consequence of 5.2.

The following result generalizes and completes some key theorems of [9], [10] and [6].

**Corollary 5.7.** (Inclusion of a level set in a level set). Consider a family $(f_i)_{0 \leq i \leq n}$ of measurable scalar integrands. We suppose that for all $1 \leq i \leq n$, $f_i$ is proper. Besides, we suppose that there exists real numbers $(c_i)_{0 \leq i \leq n}$ and an element $x$ of $X$ such that

$$(H') \quad \text{for every } i = 1, \ldots, n, -\infty < I_{f_i}(x) < c_i \text{ and } I_{-f_0}(x) < \infty.$$ 

The following assertions are equivalent

(i) $\bigcap_{i \geq 1} I_{f_i}^{\leq c_i} \subset I_{f_0}^{\leq c_0}$

(ii) there exists $y^* \in (\mathbb{R}_+)^n$ such that the function $u(\omega) = \inf \{-f_0(\omega, e) + \sum_{i=1}^n y_i^* f_i(\omega, e), e \in M(\omega) \cap \bigcap_{i \geq 1} \text{dom} f_i(\omega, \cdot)\}$ is integrable and verifies $-c_0 \leq \int_\Omega u \, d\mu - < y^*, c >$,

where $c = (c_i)_{1 \leq i \leq n}$.

Proof. If the assertion $(ii)$ is satisfied, then for any $e \in M(\omega)$, we have

$$f_0(\omega, e) \leq \sum_{i=1}^n y_i^* f_i(\omega, e) - u(\omega).$$

Consequently, for $z \in X$ verifying $I_{f_i}(z) \leq c_i, i = 1, \ldots, n$, by integration, we obtain

$$I_{f_0}(z) \leq \sum_{i=1}^n y_i^* I_{f_i}(z) - \int_\Omega u \, d\mu \leq < y^*, c > + c_0 - < y^*, c > = c_0.$$

With the convention : $0 \times (-\infty) = 0$.

Conversely, by 5.4, if assertion $(i)$ is true, then the following holds

$$\bigcap_{i \geq 1} I_{f_i}^{\leq c_i} \subset -I_{-f_0}^{\leq c_0}.$$

This assertion is equivalent to the following one

$$-c_0 \leq \inf \{I_{-f_0}(y), y \in X, I_{f_i}(y) \leq c_i, i = 1, \ldots, n\}.$$

Therefore we get

$$-c_0 \leq \inf \{I_{-f_0}(y), y \in X, -\infty < I_{f_i}(y) \leq c_i, i = 1, \ldots, n\} = r.$$

By $(H')$ and corollary 5.6, there exists $y^* \in (\mathbb{R}_+)^n$ such that $-c_0 \leq r = I_{u^*}, -< y^*, c >$, where we have $u^*_y(\omega) = \inf \{-f_0(\omega, e) + \sum_{i=1}^n y_i^* f_i(\omega, e), e \in M(\omega) \cap \bigcap_{i \geq 1} \text{Dom} f_i(\omega, \cdot)\}$.

Since the $f_i, i = 1, \ldots, n$, are proper, we deduce $\text{Dom} f_i = \text{dom} f_i$, moreover, since $I_{u^*}$ is finite, $u^*_y$ is integrable and satisfies the second assertion of Corollary 5.7. This completes the proof.

**Proposition 5.8.** We consider the problem $(\mathcal{P})$. Suppose that the origin is an element of $ri(\text{dom} h - r_{f_i}(I_g))$. The following assertions are equivalent
(i) $x_0$ is a minimizer of $(P)$;
(ii) there exists $y^*$ in $\partial h(I_g(x_0))$ such that the function
\[ v(\omega) = \min \{ f(\omega, e) + < y^*, g(\omega, e) > - (f(\omega), x_0(\omega)) + < y^*, g(\omega, x_0(\omega)) >, \]
\[ e \in M \cap \text{Dom}g(\omega) \}
\]is the null mapping almost everywhere.

Moreover, the set of all $y^*$ verifying (ii) is exactly $\partial p(0)$.

**Proof.** By 4.1 and 2.19, the problem $(P)$ is stable. By virtue of 2.7, $x_0$ is a minimizer of $(P)$ if and only if there exists $y^*$ in $\partial h(I_g(x_0))$ such that
\[ I_f(x_0) + < y^*, I_g(x_0) > = \min \{ I_f(x) + < y^*, I_g(x) >, x \in \text{Dom} I_g \}. \]

Using 3.9, 3.13(i) and 3.11, this is equivalent to $\int_\Omega v \, d\mu = 0$, with
\[ v(\omega) = \min \{ f(\omega, e) + < y^*, g(\omega, e) > - (f(\omega), x_0(\omega)) + < y^*, g(\omega, x_0(\omega)) >, \]
\[ e \in M \cap \text{Dom}g(\omega) \}
\]Since $v$ is non positive, we deduce that $v$ is null almost everywhere. Therefore the assertions are equivalent. Besides, the set of all $y^*$ satisfying (ii) is exactly $\partial p(0)$, by 2.7.

When $h = \psi_P$, where $P$ is a cone of $\mathbb{R}^n$ of the following type
\[ P = \{ 0 \}^k \times (\mathbb{R}^+)\ell, k + \ell = n, \]
we complete some results of [1].

**References**


