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Kuhn-Tucker Conditions and Integral Functionals

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Let X be a decomposable set, h a convex function defined on $\mathbb{R}^n$ with values in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. We show that, under transversality assumptions, the problem $\inf\{f(x) + h(g(x)),\ x \in X\}$ admits generalized or exact Kuhn-Tucker multipliers. We consider the case where f is a scalar $\bar{\mathbb{R}}$-valued integral functional and g is a vector integral functional with values in $(\bar{\mathbb{R}})^n$. These properties are related to growth conditions between integrands.

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1. Introduction

Kuhn-Tucker conditions are classical in convex analysis [2],[14],[30]. They appear in some nonconvex optimization problems too. In [5] and [25], one can find existence results about minimisation of integral functionals under integral contraints. In [11], [12] and [13], we show that in many cases such local optima are global. V.I. Arkin [1] studies the maxima of integral functionals under inequality constraints. In[3] J.P Aubin and I. Ekeland give a duality result. A. Fougères [9], R. Vaudene [10] and A. Bourass [6] show that when there is inclusion between level sets of two scalar integral functionals, then we necessarily have a growth condition between the integrands. In this paper, we widen some statements of [1], [3], [9],[10] and [6]. We obtain not only necessary and sufficient conditions, but we also give new results using non usual techniques. In section 2, the notions of stability and of "epi-convex" function are introduced. The stability of a minimization problem is related to the existence of a subdifferential (in the sense of a convex analysis) of the performance function. In many cases, the epi-convexity of a vector function coincides with the convex like notion for the cone $\{0\} \times \mathbb{R}_+$ of empty interior. A stability result is given; among other things it allows to characterize a global minimum by Kuhn-Tucker conditions. We establish a relation with J.P. Aubin's work [2] and with recent studies by M. Moussaoui, M. Volle [25] and J-P. Penot [20].
In section 3, we carry out a result of interchange of minimization and integration. We use the notion of essential infimum given by Neveu [19]. The result is similar to theorem 3.A

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of R.T. Rockafellar [22], though the infimum is taken on a decomposable set in the sense of [15]. We also use the notion of richness introduced by M. Valadier in [6].
In section 4, we show that when the measure is atomless, the vector integral functionals are epi-convex. Non emptiness criteria of the interior, of the relative interior, of the range of a decomposable set by an integral functional are given.
In section 5, we use the previous results, notably the existence of multipliers. For the integral functionals, we characterize the inclusion of level lines in a level set and even in a level line. In the same way, we characterize the inclusion of level sets in a level set. These results are translated in terms of growth conditions. A connection with some results obtained by V.I. Arkin is also provided.

2. Stability and epi-convex functions

We adopt the following notation : $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$.
Let X be a set, Y a hausdorff locally convex vector space with dual $Y^\star$. If $h : Y \longrightarrow \bar{\mathbb{R}}$ is a mapping, its conjugate $h^\star$ is defined on $Y^\star$ by

$$h^\star(y^\star) = \sup\{< y, y^\star > -h(y),\ y \in Y\}.$$

Given a mapping $f : X \longrightarrow \bar{\mathbb{R}}$ and an operator $g : X \longrightarrow Y$, the effective domain of f is $domf = \{x \in X,\ f(x) < \infty\}$ and the strict epigraph of f is the subset of $X \times \mathbb{R}$ defined by

$$epi^+(f) = \{(x, r) \in X \times \mathbb{R},\ f(x) < r\}.$$

The domain of the operator g, that is $Domg$ is a given subset of X, and its range $r(g)$ is $\{g(x),\ x \in Domg\}$.
Given the optimization problem

$$(\mathcal{P}) \qquad\qquad \inf\{f(x) + h(g(x)),\ x \in X\}$$

where $h(g(x)) = h(g(x))$ if $x \in Domg$, $+\infty$ if not, we classically introduce the performance function defined on Y by

$$p(y) = \inf\{f(x) + h(g(x) + y),\ x \in X,\ g(x) \in domh - y\}.$$

Of course, we have $p(0) = \inf(\mathcal{P})$ and we suppose this quantity is finite in the sequel.
First, let us give some elementary characteristics of the performance function of the problem $(\mathcal{P})$.

Lemma 2.1. *If $r_f(g) = \{g(x),\ x \in Domg \cap domf\}$, then*

$$domp = domh - r_f(g).$$

Proof.

$$\begin{aligned} y \in domp &\iff \exists x \in X : h(g(x) + y) < \infty \text{ and } f(x) < infty \\ &\iff \exists x \in Domg \cap domf : y \in domh - g(x) \\ &\iff y \in domh - r_f(g). \end{aligned}$$

Remark 2.2. Let us recall that for $\varepsilon \geq 0$, the ε-approximate subdifferential of the function p at y_0 [16], $\partial_\varepsilon p(y_0)$ is the set of all $y^\star \in Y^\star$ verifying one of the following assertions

$$\begin{array}{ll} (i) & \forall y \in Y,\ p(y) \geq p(y_0) - \varepsilon + < y^\star, y - y_0 >; \\ (ii) & p^\star(y^\star) + p(y_0) - < y^\star, y_0 > \leq \varepsilon. \end{array}$$

For $\varepsilon = 0$, we obtain the subdifferential $\partial p(y_0)$ of p at y_0 in the sense of convex analysis.

Definition 2.3. For any $\varepsilon \geq 0$ the set of ε-approximate multipliers of $(\mathcal{P})$ denoted by $M_\varepsilon(\mathcal{P})$, is defined by

$$M_\varepsilon(\mathcal{P}) = \{y^\star \in Y^\star,\ p(0) - \varepsilon \leq \inf\{f(x) + < y^\star, g(x) > -h^\star(y^\star),\ x \in Domg\}\}.$$

The following result is to be compared with [20] Lemma 5.1. and [2] ch. 14.1 theorem 6.

Proposition 2.4. *For any $\varepsilon \geq 0$, we have the equality*

$$\partial_\varepsilon p(0) = M_\varepsilon(\mathcal{P}).$$

Proof. Let us first show the inclusion $\partial_\varepsilon p(0) \subset M_\varepsilon(\mathcal{P})$. Given $y^\star \in \partial_\varepsilon p(0)$, for any $y \in Y$, we have

$$p(y) \geq p(0) - \varepsilon + < y^\star, y > .$$

Let $x \in Domg$ and $a \in domh$. If $y = a - g(x)$, then:

$$h(a) + f(x) \geq p(y) \geq p(0) - \varepsilon + < y^\star, y > .$$

Whence $f(x) + < y^\star, g(x) > \geq p(0) - \varepsilon + < y^\star, a > -h(a)$.
Passing through this inequality to the infimum relatively to x in $Domg$ and to the supremum relatively to a in $domh$, we get $y^\star \in M_\varepsilon(\mathcal{P})$.
Conversely let $y^\star \in M_\varepsilon(\mathcal{P})$. For any $x \in Domg >$ and any $a \in domh$, we get

$$f(x) + < y^\star, g(x) > \geq p(0) - \varepsilon + < y^\star, a > -h(a).$$

Given $y \in Y$ such that there exists $x \in Domg : g(x) + y = a \in domh$, then

$$f(x) + h(g(x) + y) \geq p(0) - \varepsilon + < y^\star, y > .$$

Taking the infimum on x such that $g(x) + y \in domh$, we obtain

$$p(y) \geq p(0) - \varepsilon + < y^\star, y >$$

wich proves that $y^\star$ is an element of $\partial_\varepsilon p(0)$. Proposition 2.4 justifies the introduction of the following notions:

Definition 2.5. $(\mathcal{P})$ is said to be stable if $\partial p(0)$ is non-empty. $(\mathcal{P})$ is said to be weakly stable if for any positive ε, $\partial_\varepsilon p(0)$ is non-empty.

The notion of stability is standard, see for example [28]

Theorem 2.6. *The duality result,*

$$\inf(\mathcal{P}) = \sup_{y^\star \in Y^\star} \inf\{f(x) + < y^\star, g(x) > -h^\star(y^\star),\ x \in Domg\}$$

holds if and only if $(\mathcal{P})$ *is weakly stable.*
Moreover, the supremum above is a maximum if and only if $(\mathcal{P})$ *is stable.*

Proof. Using the inequality: $< y^\star, g(x) > -h^\star(y^\star) \leq h(g(x))$, we deduce for any $y^\star$:

$$\inf\{f(x) + < y^\star, g(x) > -h^\star(y^\star),\ x \in Domg\} \leq \inf(\mathcal{P})$$

and the following inequality is always valid:

$$\sup_{y^\star \in Y^\star} \inf\{f(x) + < y^\star, g(x) > -h^\star(y^\star),\ x \in Domg\} \leq \inf(\mathcal{P})$$

Theorem 2.6 is now a consequence of the Proposition 2.4.

Proposition 2.7. *Let* x_0 *be an element of* X*. If the problem* $(\mathcal{P})$ *is stable, then the following assertions are equivalent*

(i) x_0 *is a minimizer of* $(\mathcal{P})$;
(ii) *there exists* $y^\star \in \partial h(g(x_0))$ *such that*

$f(x_0) + < y^\star, g(x_0) >= Min\{f(x) + < y^\star, g(x) >,\ x \in Domg\}$.
Moreover, the set of all $y^\star$ *satisfying* (ii) *is exactly* $\partial p(0)$.

Proof. First suppose that (ii) holds for some $y^\star$. Using 2.2, we get $< y^\star, g(x_0) >= h(g(x_0)) + h^\star(y^\star)$ and consequently

$$\begin{aligned} f(x_0) + h(g(x_0)) &= f(x_0) + < y^\star, g(x_0) > -h^\star(y^\star) \\ &\leq \inf\{f(x) + < y^\star, g(x) > -h^\star(y^\star),\ x \in Domg\} \\ &\leq \inf\{f(x) + h(g(x)),\ x \in Domg,\ g(x) \in domh\} \end{aligned}$$

Hence x_0 is a minimizer of $(\mathcal{P})$.
For the converse, we need the following lemma.

Lemma 2.8. *Let* x_0 *be a minimizer of* $(\mathcal{P})$*. If* $y^\star$ *is an element of* $Y^\star$*, then the following assertions are equivalent*

(i) $y^\star$ *is an element of* $M_0(\mathcal{P})$
(ii) $y^\star$ *is an element of* $\partial h(g(x_0))$ *which satisfies*

$f(x_0) + < y^\star, g(x_0) >= Min\{f(x) + < y^\star, g(x) >,\ x \in Domg\}$.

Proof of 2.8. Applying the Definition 2.3 we have

$(1)(y^\star \in M_0(\mathcal{P})) \Longleftrightarrow (h^\star(y^\star) + h(g(x_0)) + f(x_0)) \leq \inf\{f(x) + < y^\star, g(x) >,\ x \in Domg\}$.

Given $y^\star \in M_0(\mathcal{P})$. If in (1) we take $x = x_0$, we obtain $h^\star(y^\star) + h(g(x_0)) \leq < y^\star, g(x_0) >$. Consequently, $h^\star(y^\star) + h(g(x_0)) =< y^\star, g(x_0) >$ and according to $2.2(ii)$, the second

assertion is satisfied.
Conversely if 2.8(ii) is verified then $h^\star(y^\star) + h(g(x_0)) =< y^\star, g(x_0) >$, and
$h^\star(y^\star) + h(g(x_0)) + f(x_0) = Min\{f(x) + < y^\star, g(x) >,\ x \in Domg\}$.
Since $y^\star$ verifies (1), we have $y^\star$ in $M_0(\mathcal{P})$.
End of the proof of 2.7. Let x_0 be a minimum of $(\mathcal{P})$. Since $(\mathcal{P})$ is stable, any element $y^\star$ of $\partial p(0) = M_0(\mathcal{P})$ verifies 2.7(ii), by 2.8. The last assertion follows from 2.4 and 2.8.

Corollary 2.9. *Let A be a subset of Y. If the problem*
(Q) $\quad \inf\{f(x),\ x \in X,\ g(x) \in A\}$ *is stable, then we have:*

$$\inf(Q) = Max_{y^\star \in Y^\star} \inf\{f(x) + < y^\star, g(x) > - \sup_{a \in A} < y^\star, a >,\ x \in Domg\}$$

Proof. We use 2.6 with $h = \psi_A$, where $\psi_A(y) = 0$ if $y \in A$ and $\psi_A(y) = +\infty$ if $y \notin A$.

If $f : X \longrightarrow \bar{\mathbb{R}}$ is an $\bar{\mathbb{R}}-$valued mapping and r a real number, we adopt the notation $f^{\leq r}$ for the level set of order r of f defined by

$$f^{\leq r} = \{x \in X,\ f(x) \leq r\}.$$

We say that f is proper if it does not take the values $-\infty$. For a function $g : X \longrightarrow Y$, we set $g^{-1}(a) = \{x \in X,\ g(x) = a\}$; this is the level line of order a, where a is an element of Y.

Corollary 2.10. *(Inclusion of a level line in a level set). Let a be an element of Y. If the problem $inf\{-f(x),\ x \in X,\ g(x) = a\}$ is stable, then the following assertions are equivalent*

$$(i) \quad g^{-1}(a) \subset f^{\leq r};$$
$$(ii) \quad \textit{there exists } y^\star \in Y^\star \textit{ such that}$$
$$-r \leq \inf\{-f(x) + < y^\star, g(x) > - < y^\star, a >,\ x \in Domg\}.$$

Proof. The inclusion of (i) is equivalent to :

$$-r \leq \inf\{-f(x),\ x \in X,\ g(x) = a\}.$$

Since this problem is stable, this inequality is equivalent, by 2.9, to the existence of a $y^\star$ in $Y^\star$ which verifies

$$-r \leq \inf\{-f(x) + < y^\star, g(x) > - < y^\star, a >,\ x \in Domg\}.$$

This ends the proof.

Corollary 2.11. *(Inclusion of a level line in a level line). If the problems* $\inf\{-f(x),\ x \in X,\ g(x) = a\}$ *and* $\inf\{f(x),\ x \in X, g(x) = a\}$ *are stable, then the following assertions are equivalent*

$$(i) \quad g^{-1}(a) \subset f^{-1}(r)$$
$$(ii) \quad \textit{there exists } y^\star, z^\star \textit{ in } Y^\star \textit{ such that}$$
$$-r \leq \inf\{-f(x) + < y^\star, g(x) > - < y^\star, a >,\ x \in Domg\}.$$
$$r \leq \inf\{f(x) + < z^\star, g(x) > - < z^\star, a >,\ x \in Domg\}.$$

Proof. The inclusion of (i) is equivalent to $g^{-1}(a) \subset f^{\leq r} \cap (-f)^{\leq -r}$ and, by 2.10, this assertion is equivalent to (ii).

As usual, we denote by $\mathbb{R}_+$ the set of non negative real numbers.

Corollary 2.12. *Let $(f_i)_{0\leq i\leq n}$ be a finite family of mappings defined on X with values in $\bar{\mathbb{R}}$ and let $(r_i)_{0\leq i\leq n}$ be a sequence of real numbers. If the problem*

$$(\mathcal{R}) \qquad \inf\{f_0(x),\ x \in X,\ -\infty < f_i(x) \leq r_i,\ i = 1, ..., n\}$$

is stable, then

$$\inf(\mathcal{R}) = Max_{y^\star \in (\mathbb{R}_+)^n} \inf\{f_0(x) + \Sigma_{i=1}^n y_i^\star f_i(x) - \Sigma_{i=1}^n y_i^\star r_i,\ x \in \bigcap_{i\geq 1} Dom f_i\}$$

Proof. Problem $(\mathcal{R})$ can be written:
$\inf\{f_0(x),\ x \in X,\ g(x) \in A\}$, where $g = (f_i)_{1\leq i\leq n}, r = (r_i)_{1\leq i\leq n}$ and $A = r - (\mathbb{R}_+)^n$. The $(\bar{\mathbb{R}})^n$-valued map g is considered as an $(\mathbb{R})^n$-valued operator.
For $y^\star \in \mathbb{R}^n$ we have, $\sup\{< y^\star, a >,\ a \in A\} =< y^\star, r >$ if $y^\star \in {\mathbb{R}_+}^n$, and $+\infty$ if not.
Since $Dom g = \bigcap_{i\geq 1} Dom f_i$, with $Dom f_i = \{x \in X : f_i(x) \in \mathbb{R}\}$, using Corollary 2.9 we obtain the desired result.

Corollary 2.13. *(Inclusion of a level set in a level set). Let $(f_i)_{0\leq i\leq n}$ be a finite family of proper $\bar{\mathbb{R}}$-valued mappings defined on X and $(r_i)_{0\leq i\leq n}$ real numbers. If the problem* $\inf\{-f_0(x),\ x \in X,\ f_i(x) \leq r_i,\ i = 1, ..., n\}$ *is stable, then the following assertions are equivalent*

(i) $\cap_{i\geq 1} f_i^{\leq r_i} \subset f_0^{\leq r_0}$;

(ii) *there exists $y^\star$ in $(\mathbb{R}_+)^n$ such that for all x in $\bigcap_{i\geq 1} dom f_i$*

$$-r_0 \leq -f_0(x) + \Sigma_{i=1}^n y_i^\star f_i(x) - \Sigma_{i=1}^n y_i^\star r_i.$$

Proof. Since the f_i are proper, the inclusion of the assertion (i) is equivalent to $-r_0 \leq \inf\{-f_0(x),\ x \in X,\ -\infty < f_i(x) \leq r_i,\ i = 1, ..., n\}$. By Assumption this last problem being stable, the inclusion of (i) is equivalent, by virtue of 2.12, to the existence of some $y^\star$ in $(\mathbb{R}_+)^n$ such that for all $x \in \bigcap_{i\geq 1} Dom f_i, -r_0 \leq -f_0(x) + \Sigma_{i=1}^n y_i^\star f_i(x) - \Sigma_{i=1}^n y_i^\star r_i$.
But, since the f_i are proper, we have $dom f_i = Dom f_i$. That proves the equivalence between the given assertions.

The results 2.6,...,2.13 show the importance of the stability of an optimization problem. Whence the question : under what conditions the performance function is convex ?. In order to answer this question, it seems necessary to introduce the following definition.

Definition 2.14. The strict epigraph of the mapping (g, f) denoted by $epi^+(g, f)$, is defined by

$$epi^+(g, f) = \{(y, r) \in Y \times \mathbb{R},\ \exists x \in X,\ g(x) = y \text{ and } f(x) < r\}$$

We say that (g, f) is epi-convex if $epi^+(g, f)$ is convex.

Proposition 2.15. *Let $\pi : Y \times \mathbb{R} \longrightarrow Y \times \mathbb{R}$ be the mapping defined by $\pi(y, r) = (-y, r)$. If p is the performance function of the problem $(\mathcal{P})$, then*

$$epi^{+}(p) = epi^{+}(h) + \pi(epi^{+}(g, f)).$$

Proof.$(y, r) \in epi^{+}(p) \iff \exists x \in X : h(g(x) + y) + f(x) < r$
$\iff \exists x \in X, s, t \in \mathbb{R} : s + t = r, h(g(x) + y) < s$ and $f(x) < t$
$\iff \exists x \in X, s, t \in \mathbb{R} : s + t = r, ((g(x), t) \in epi^{+}(g, f)$ and $(g(x) + y, s) \in epi^{+}(h)$
$\iff \exists x \in X, s, t \in \mathbb{R} : s + t = r, ((g(x), t) \in epi^{+}(g, f)$ and $(y, s) \in epi^{+}(h) + (-g(x), 0)$
$\iff \exists x \in X, s, t \in \mathbb{R} : s + t = r, ((g(x), t) \in epi^{+}(g, f)$ and $(y, r) \in epi^{+}(h) + (-g(x), t)$
$\iff \exists x \in X, s, t \in \mathbb{R} : s + t = r, ((g(x), t) \in epi^{+}(g, f)$ and $(y, r) \in epi^{+}(h) + \pi(g(x), t)$
$\iff (y, r) \in epi^{+}(h) + \pi(epi^{+}(g, f))$.

Corollary 2.16. *If h is convex and (g, f) is epi-convex then the performance function of the problem $(\mathcal{P})$ is convex.*

With the help of Corollary 2.16, the known theorems [17], [18], [24], [25], can be used to ensure the stability of the problem $(\mathcal{P})$. In this paper, we consider the case where Y is finite dimensional. The following result can be deduced from Corollary 2 of [25], but for greater convenience, we will give a geometrical proof of it. We denote by riC the relative interior of the convex C [21].

Proposition 2.17. *Let Y be finite dimensional and $p : Y \longrightarrow \bar{\mathbb{R}}$ be a convex function. If y_0 is in the relative interior of $domp$ and $p(y_0)$ is finite, then the subdifferential $\partial p(y_0)$ is non empty.*

Proof. Without any loss of generality, we can suppose $(y_0, p(y_0)) = (0, 0)$. Since $(0, 0) \notin epi^{+}(p)$, we can properly separate these two sets [21] Theorem 1.3. There exists $(y^{\star}, \lambda) \in Y \times \mathbb{R}$ such that
(1) for any $(y, r) \in epi^{+}(p), 0 \leq < y^{\star}, y > + \lambda r$;
(2) there exists $(y_0, r_0) \in epi^{+}(p), 0 << y^{\star}, y_0 > + \lambda r_0$.
From (1), we deduce that λ is non negative. Let us show that $\lambda \neq 0$. If $\lambda = 0$ then (1) and (2) can be written
(3) for any $y \in dom(p), 0 \leq < y^{\star}, y >$;
(4) there exists $y_0 \in dom(p), 0 << y^{\star}, y_0 >$.
From (3) we deduce that $0 \leq < y^{\star}, \mathbb{R}_{+} dom(p) >$. Since $0 \in ridom(p)$, one has $\mathbb{R}_{+} dom(p)$ is a vector space and thus $< y^{\star}, \mathbb{R}_{+} dom(p) >= 0$; which contradicts (4).
Consequently $\lambda > 0$ and by setting $y_0^{\star} = -\lambda^{-1} y^{\star}$, we obtain for any $(y, r) \in epi^{+}(p)$, $< y_0^{\star}, y > \leq r$. Thus

$$< y_0^{\star}, y > \leq p(y), \text{ for any } y \in dom(p).$$

Since this last inequality is valid for all $y \in Y$, we obtain that $y_0^{\star} \in \partial p(0)$. Which proves 2.17.

Remark. When p is supposed to be a proper convex function, Proposition 2.17 is classical; see theorem 23.4 of [21].

Before stating the result of stability, let us prove the following lemma:

Lemma 2.18. *If (g, f) is epi-convex then the set $r_f(g) = \{g(x),\ x \in Domg \cap domf\}$ is convex.*

Proof. If (g, f) is epi-convex, then $epi^+(g, f)$ is convex. If q stands for the projection of $Y \times \mathbb{R}$ on Y defined by $q(y, r) = y$, then we get the following equality:

$$r_f(g) = q(epi^+(g, f),$$

from which the convexity of $r_f(g)$ follows.

Theorem 2.19. *We suppose Y is finite dimensional. Let $h : Y \longrightarrow \bar{\mathbb{R}}$ be a convex function. Under the following assumptions:*

(i) the function (g, f) is epi-convex;
(ii) the origin is in the relative interior of $domh - r_f(g)$,
the problem $(\mathcal{P})$ is stable.

Proof. By virtue of 2.16 the performance function of the problem $(\mathcal{P})$ is convex and $p(0) = \inf(\mathcal{P})$ is finite by assumption. Using the second assumption, 2.1 and 2.17, we can deduce that the performance function is subdifferentiable at the origin. Therefore the problem $(\mathcal{P})$ is stable by definition.

3. On the interchange of minimization and integration

Let (Ω, τ, μ) be a measured space by a σ-finite positive measure μ, E a separable Banach space with Borel tribe $\mathcal{B}(E)$. Consider a subset X of the space $L_0(\Omega, E)$ of classes of measurable functions (for μ-a.e equality) defined on Ω and with values in E. The set X is supposed to be decomposable in the following sense [29]: for all $x, y \in X$ and all $A \in \tau$ the function $y1_A + x1_{\Omega\setminus A}$ is in X, where 1_S stands for the characteristic function of $S \in \tau$. Given $v \in L_0(\Omega, \bar{\mathbb{R}})$, we denote by I_v or $\displaystyle\int_\Omega^\star v\, d\mu$ the upper integral of v defined by:

$$I_v = \int_\Omega^\star v\, d\mu = \inf\{\int_\Omega u\, d\mu,\ u \in L_1(\Omega, \mathbb{R}),\ v \leq u\ \mu - a.e\}.$$

If $f : \Omega \times E \longrightarrow \bar{\mathbb{R}}$ is an $\tau \otimes \mathcal{B}(E)$ measurable scalar integrand, the functional integral I_f is defined on X by:

$$I_f(x) = I_{f(x)} = \int_\Omega^\star f(x)\, d\mu,$$

where $f(x)$ stands for the function $\omega \in \Omega \mapsto f(\omega, x(\omega))$. If $g : \Omega \times E \longrightarrow (\bar{\mathbb{R}})^n$ is an $\tau \otimes \mathcal{B}(E)$ measurable vector integrand, then $g = (g_i)_{1 \leq i \leq n}$ where g_i is a scalar measurable integrand. Likewise we consider the integral functional I_g defined on X by

$$I_g(x) = (I_{g_i}(x))_{1 \leq i \leq n}.$$

We also adopt the notation $g(x)$ for the mapping $\omega \in \Omega \mapsto g(\omega, x(\omega))$. Let us recall that Neveu [19], II.4 shows that any family $\{v_i,\ i \in I\}$ of elements of $L_0(\Omega, \bar{\mathbb{R}})$ has an essential infimum $ess \inf_I v_i$ defined by:

$$(\forall\, i \in I,\ v_i \geq v\ \mu - a.e) \Longleftrightarrow (ess \inf_I v_i \geq v\ \mu - a.e).$$

Theorem 3.1. *Let X be a decomposable set and $f : \Omega \times E \longrightarrow \bar{\mathbb{R}}$ a measurable integrand. We have*

$$\inf_X I_f(x) = I\, ess \inf_X f(x)$$

provided the left hand side is distinct from ∞.

Proof. Let y be an element of X such that $I_f(y) < \infty$. There exists $v \in L_1(\Omega, \mathbb{R})$ such that $f(y) \leq v \quad \mu - a.\, e.$ We consider the sets

$$f(X) = \{f(x),\ x \in X\}; U = \{u \in L_1(\Omega, \mathbb{R}),\ \exists x \in X, f(x) \leq u \leq v \quad \mu - a.e\}.$$

Lemma 3.2. *The sets $f(X)$ and U are non empty complete lattices.*

Proof of 3.2. Let us first verify that $f(X)$ is a complete lattice. Let x, y be two elements of X. If we consider the set,

$$A = \{\omega \in \Omega,\ f(\omega, x(\omega)) \leq f(\omega, y(\omega))\}$$

then,

$$\inf\{f(x), f(y)\} = f(x)1_A + f(y)1_{\Omega \setminus A} = f(x1_A + y1_{\Omega \setminus A}).$$

Since X is decomposable we deduce that $\inf\{f(x), f(y)\} \in X$. Similarly $\sup\{f(x),\ f(y)\} \in X$.

By means of a similar proof, let us prove that U is a complete lattice. Let u_1 and u_2 be two elements of U. By definition there exists two elements x_1 and x_2 of X such that $f(x_i) \leq u_i \leq v\, \mu - a.\, e.$

If $B = \{\omega \in \Omega,\ u_1(\omega) \leq u_2(\omega)\, \mu - a.\, e\}$, we have

$$v \geq \inf\{u_1,\ u_2\} = u_1 1_B + u_2 1_{\Omega \setminus B} \geq f(x_1)1_B + f(x_2)1_{\Omega \setminus B} = f(x_1 1_B + x_2 1_{\Omega \setminus B}).$$

Since X is decomposable we obtain

$$\inf\{u_1,\ u_2\} \in U.$$

In the same way $\sup\{u_1,\ u_2\} \in U$ can be shown.

Lemma 3.3. *If $ess \inf_U u$ is integrable, then for any $x \in X, f(x)$ does not take μ-a.e the value $-\infty$.*

Proof. Suppose the contrary. Let x be in X such that the set $A = \{\omega \in \Omega,\ f(\omega, x(\omega)) = -\infty\}$ is of positive measure. If y and v are defined as in the beginning of the proof of 3.1, by lemma 3.2, we consider z in X such that:

$$f(z) = \inf\{f(x),\ f(y)\}.$$

Thus we get $f(z) \leq v \qquad \mu - a.e$ and for $\omega \in A, f(\omega, z(\omega)) = -\infty$. If α is a positive integrable function, then we set, for $n \in \mathbb{N}$

$$u_n = v1_{\Omega \setminus A} - n \sup\{\alpha, |v|\}1_A.$$

It is obvious that $f(z) \leq u_n \leq v$. Consequently u_n is in U and we have

$$I\, ess \inf_U u \leq \inf_{\mathbb{N}} \int_\Omega u_n\, d\mu = -\infty$$

which leads to a contradiction.

Lemma 3.4. *If* $ess\inf_U u$ *is integrable, then*

$$ess\inf_U u = ess\inf_X f(x) \qquad \mu - a.\,e.$$

Proof. By virtue of the definition of U we have

$$ess\inf_X f(x) \leq ess\inf_U u \qquad \mu - a.\,e.$$

Conversely, let us show that for any $x \in X$ we have

$$ess\inf_U u \leq f(x) \qquad \mu - a.\,e.$$

For this, let y and v be as above. $f(X)$ being a complete lattice, by lemma 3.2, we consider $z \in X$ which satisfies $f(z) = \inf\{f(x),\ f(y)\}$. Let us prove that

$$ess\inf_U u \leq f(z) \qquad \mu - a.\,e.$$

If this is not the case, then the set $A = \{\omega \in \Omega,\ f(\omega, z(\omega)) < ess\inf_U u(\omega)\}$ is of positive measure. Let α be a positive integrable function on Ω. We set for $n \in \mathbb{N}$,
$A_n = \{\omega \in A,\ -n\alpha(\omega) \leq f(\omega, z(\omega))\}$.
By lemma 3.3, the union of the A_n is A. If A_k has a positive measure, then we set $u = v1_{\Omega\setminus A_k} + f(z)1_{A_k}$.
By construction, $u \in L_1(\Omega, \mathbb{R})$ and $f(z) \leq u \leq v \qquad \mu - a.\,e.$ Hence u is an element of U. But for any $\omega \in A_k$, we get the contradiction

$$ess\inf_U u(\omega) \leq u(\omega) = f(\omega, z(\omega)) < ess\inf_U u(\omega).$$

This shows that A is of null measure. Moreover, for any $x \in X$ we have

$$ess\inf_U u \leq f(z) \leq f(x) \qquad \mu - a.\,e$$

and the proof of 3.4 is complete.

Lemma 3.5. *If* $\inf_U \int_\Omega u\,d\mu$ *is a real number then* $ess\inf_U u$ *is integrable and we get*

$$\inf_U \int_\Omega u\,d\mu = \int_\Omega ess\inf_U u\,d\mu.$$

Proof. Let $r = \inf_U \int_\Omega u\,d\mu$ and $u_n \in U$ be such that $\lim_n \int_\Omega u_n\,d\mu = r$. Since U is a complete lattice (Lemma 3.2), the sequence $\bar{u}_n = \inf\{u_i,\ 1 \leq i \leq n\}$ is in U and satisfies $\lim_n \int_\Omega \bar{u}_n\,d\mu = r$. Let $\bar{u} = \lim_n \bar{u}_n$. By the monotone convergence theorem $\bar{u}$ is integrable and $\int_\Omega \bar{u}\,d\mu = r$. In order to prove 3.5 all we have to do is to show that $\bar{u}$ is the essential infimum of the family U.

Let $w \in L_0(\Omega, \bar{\mathbb{R}})$ be such that for any $u \in U, w \leq u \quad \mu - a.\ e.$ It is obvious that $w \leq \bar{u}\ \mu - a.\ e.$ One needs only to show that $\bar{u}$ is an essential lower bound of the family U. That is to say for any $u \in U$ we have $\bar{u} \leq u\ \mu - a.e.$
For this, let us consider an element $u \in U$ and the sequence $v_n = \inf\{u,\ \bar{u}_n\}$. Since U is a complete lattice, v_n is in U. The sequence (v_n) is decreasing and we have

$$r \leq \lim_n \int_\Omega v_n\, d\mu \leq \lim_n \int_\Omega \bar{u}_n\, d\mu = r.$$

Thus by the monotone convergence theorem, the sequence (v_n) converges to $\inf\{u,\ \bar{u}\}$ in $L_1(\Omega, \mathbb{R})$ with

$$\int_\Omega \inf\{u,\ \bar{u}\}\, d\mu = \lim_n \int_\Omega v_n\, d\mu = r.$$

As a result

$$\int_\Omega \bar{u} - \inf\{u,\ \bar{u}\}\, d\mu = 0;$$

and consequently $\inf\{u, \bar{u}\} = \bar{u} \quad \mu - a.e$ or $\bar{u} \leq u \quad \mu - a.\ e.$ Therefore we have shown that $\bar{u} = ess \inf_U u$, and 3.5 is proved.
End of the proof of Theorem 3.1. We obviously have the inequality

$$I_{ess \inf_X f(x)} \leq \inf_X I_f(x).$$

If $\inf_X I_f(x) = -\infty$, then the two terms coincide. Consequently, let us suppose $-\infty < \inf_X I_f(x) \leq \inf_U \int_\Omega u\, d\mu < \infty$. In this case, we obtain by Lemma 3.4 and Lemma 3.5

$$\begin{aligned} \inf_X I_f(x) \leq \inf_U \int_\Omega u\, d\mu &= \int_\Omega ess \inf_U u\, d\mu \\ &= \int_\Omega ess \inf_X f(x)\, d\mu. \end{aligned}$$

This proves the previous equality and ends the proof of 3.1.

Theorem 3.1 gives rise to the question of the calculation of the essential infimum of the range by an integrand of a decomposable set.

Definition 3.6. An increasing sequence $(\Omega_n)_n$ of measurable sets of finite measure is said to be a σ-finite covering of Ω if $\mu(\Omega \setminus \cup_n \Omega_n) = 0$.

Given a multifunction M defined on Ω with values in E and with measurable graph. We denote by S_M the decomposable set of measurable selections of M, i.e,

$$S_M = \{x \in L_0(\Omega, E),\ \forall\, \omega \in \Omega,\ x(\omega) \in M(\omega)\}.$$

Definition 3.7. Let X and Y be two decomposable sets of $L_0(\Omega, E)$. X is said to be rich in Y if X is a subset of Y and if for any y in Y, there exists a σ-finite covering $(\Omega_n)_n$ of Ω and a sequence (x_n) of elements of X verifying, $y 1_{\Omega_n} = x_n 1_{\Omega_n}$. for all $n \in \mathbb{N}$

Let us give an example. Let M be a multifunction defined on Ω with values in E and with a measurable graph. If $L_p(\Omega, E) \cap S_M$ is non empty, then it is a decomposable set which is rich in S_M.

Theorem 3.8. *Suppose the tribe τ μ-complete. Let M be a multifunction with non empty values and with $\tau \otimes \mathcal{B}(E)$ measurable graph, and $f : \Omega \times E \longrightarrow \bar{\mathbb{R}}$ a measurable integrand. If X is a decomposable set which is rich in S_M, then*

$$ess \inf_X f(x)(\omega) = \inf_{e \in M(\omega)} f(\omega, e) \qquad \mu - a.\ e.$$

Proof. Let $u(\omega) = \inf\limits_{e \in M(\omega)} f(\omega, e)$. By virtue of [7] III.39, u is τ-measurable and therefore satisfies $u \leq ess \inf\limits_X f(x) \qquad \mu - a.e$

Let us show the inverse inequality. For all $n \in \mathbb{N}$, we consider the multifunction N_n defined by :

$$N_n(\omega) = \begin{cases} \{e \in M(\omega),\ f(\omega, e) \leq u(\omega) + \frac{1}{n}\} & \text{if } \omega \in u^{-1}(\mathbb{R}) \\ M(\omega) & \text{if } \omega \in u^{-1}(\{\infty\}) \\ \{e \in M(\omega),\ f(\omega, e) \leq -n\} & \text{if } \omega \in u^{-1}(\{-\infty\}) \end{cases}$$

We check that N_n has an $\tau \otimes \mathcal{B}(E)$ measurable graph and it is with non empty values. Using [7] III.22, we get that N_n admits a measurable selection $x_n \in S_M$. Since X is rich in S_M, let $(\Omega_p^n)_p$ be a σ-finite covering of Ω and $(x_p^n)_p$ a sequence of elements of X such that for any $p \in \mathbb{N}$

$$x_n 1_{\Omega_p^n} = x_p^n 1_{\Omega_p^n} \qquad \mu - a.\ e.$$

For all $p \in \mathbb{N}$, we have

$$ess \inf_X f(x) 1_{\Omega_p^n} \leq f(x_p^n) 1_{\Omega_p^n}\ \mu - a.\ e.$$

Taking the limit in $p \in \mathbb{N}$, we obtain for all $n \in \mathbb{N}$

$$ess \inf_X f(x) \leq f(x_n) \qquad \mu - a.\ e.$$

Hence

$$ess \inf_X f(x) \leq \inf_{n \in \mathbb{N}} f(x_n) = u \qquad \mu - a.\ e.$$

Which proves 3.8.

Corollary 3.9. *Suppose that the tribe τ is μ-complete. Let M be a $\tau \otimes \mathcal{B}(E)$ measurable multifuction with non empty values and $f : \Omega \times E \longrightarrow \bar{\mathbb{R}}$ a measurable integrand. If X is a decomposable subset which is rich in S_M, we have*

$$\inf_{x \in X} I_f(x) = I \inf_{e \in M(.)} f(., e),$$

provided the left hand side is distinct from ∞.

Proof. This follows from the theorems 3.1 and 3.8.

A result in relation with the Corollary 3.9 is the Theorem 2.2 of [29] by F. Hiai and H.

Umegaki. In this last theorem X has been replaced by $L_p(\Omega, E) \cap S_M$ and the integrand is supposed to be normal. When the tribe is μ-complete, Corollary 3.9 is a natural extension of the result of Hiai and Umegaki.

Examples of rich decomposable sets which will be useful in the sequel.
Let $f : \Omega \times E \longrightarrow \bar{\mathbb{R}}$ and $g : \Omega \times E \longrightarrow \bar{\mathbb{R}}^n$ be two measurable integrands. The multifunctions defined by $\forall \omega \in \Omega, domf(\omega) = domf(\omega, .)$ and $Domg(\omega) = Domg(\omega, .)$ have a measurable graph. We set

$$L_f = \{x \in L_0(\Omega, E),\ I_f(x) < \infty\}.$$

$$L_g^1 = \{x \in L_0(\Omega, E),\ I_g(x) \in \mathbb{R}^n\}.$$

Lemma 3.10.

(i) *If L_f is non empty, then it is a decomposable subset which is rich in S_{domf}.*
(ii) *If L_g^1 is non empty, then it is a decomposable subset which is rich in S_{Domg}.*

Proof. We prove (i). Let us first show that L_f is decomposable. Let x, y be two elements of L_f, then there exists u, v in $L_1(\Omega, \mathbb{R})$ such that

$$f(x) \le u \qquad \mu - a.e \text{ and } f(y) \le v \qquad \mu - a.\ e.$$

If A is an element of τ, then

$$f(x1_A + y1_{\Omega\setminus A}) = f(x)1_A + f(y)1_{\Omega\setminus A} \le u1_A + v1_{\Omega\setminus A} \qquad \mu - a.e,$$

which shows that $x1_A + y1_{\Omega\setminus A}$ is in L_f. Therefore L_f is decomposable. We now prove the richness of L_f in S_{domf}. Let y be in L_f and x in S_{domf}. If $(\Omega_n)_n$ is a σ-finite covering of Ω, then the sequence $S_n = \{\omega \in \Omega_n\ f(\omega, x(\omega)) \le n\}$ is also a σ-finite covering of Ω. For all $n \in \mathbb{N}, x_n = x1_{S_n} + y1_{\Omega\setminus S_n}$ is in L_f and we have

$$x1_{S_n} = x_n 1_{S_n}.$$

Which proves the first assertion. The decomposability of L_g^1 is verified thanks to the classical Lemma:

Lemma 3.11. *Let x be in $L_0(\Omega, E)$. $I_g(x)$ is in $\mathbb{R}^n$ if and only if the map $g(x)$ is integrable.*

By 3.11, if x, y are in L_g^1 and A is in τ, we have

$$g(x1_A + y1_{\Omega\setminus A}) = g(x)1_A + g(y)1_{\Omega\setminus A} \in L_1(\Omega, \mathbb{R}^n).$$

Consequently $x1_A + y1_{\Omega\setminus A}$ is in L_g^1. Let us show the richness of L_g^1 in S_{Domg}. Let y be in $L_g^1, x \in S_{Domg}$ and let $(\Omega_n)_n$ be a σ-finite covering of Ω. Set $S_n = \{\omega \in \Omega_n,\ |g(\omega, x(\omega))| \le n\}$, where $|.|$ is a norm of $\mathbb{R}^n$. The sequence $(S_n)_n$ is a σ-finite covering of Ω. The sequence $x_n = x1_{S_n} + y1_{\Omega\setminus S_n}$ is in L_g^1 and satisfies $x1_{S_n} = x_n 1_{S_n}$ $\mu - a.e.$ Therefore L_g^1 is rich in S_{Domg}.

Lemma 3.12. *Let M and N be two multifunctions with non empty values and with an $\tau \otimes \mathcal{B}(E)$ measurable graph. Let X be a decomposable set which is rich in S_M and Y a decomposable set which is rich in S_N. If $X \cap Y$ is non empty, then it is a decomposable set wich is rich in $S_{M\cap N}$.*

Proof. Let y be in $X \cap Y$ and x be in $S_{M\cap N}$. Since X is rich in S_M, there exists a σ-finite covering $(\Omega_{1,n})_n$ and a sequence $(x_n)_n$ of elements of X verifying

$$x1_{\Omega_{1,n}} = x_n 1_{\Omega_{1,n}}\ \mu - a.\ e., \quad \text{for any } n \in \mathbb{N} \tag{1}$$

In the same way, there exists a σ-finite covering $(\Omega_{2,n})_n$ and a sequence $(y_n)_n$ of elements of Y verifying

$$x1_{\Omega_{2,n}} = y_n 1_{\Omega_{2,n}}\ \mu - a.\ e. \ \text{for any } n \in \mathbb{N} \tag{2}$$

The sequence $\Omega_n = \Omega_{1,n} \cap \Omega_{2,n}\ \mu - a.e$ is a σ-finite covering of Ω. Let $z_n = x1_{\Omega_n} + y1_{\Omega/\Omega_n}$. Since X and Y are decomposable, by using (1) and (2), we deduce that z_n is in $X \cap Y$. Besides, from (1) and (2), $z_n 1_{\Omega_n} = x1_{\Omega_n}$. Which proves that $X \cap Y$ is rich in $S_{M\cap N}$.

Proposition 3.13. *Let M be a multifunction with non empty values and with an $\tau \otimes \mathcal{B}$ measurable graph and X a rich decomposable subset in S_M. We consider the integral functionals I_f and I_g defined on X.*

(i) If $DomI_g$ is non empty, then it is decomposable set which is rich in $S_{M\cap Domg}$.
(ii) If $DomI_g \cap domI_f$ is non empty, then it is decomposable set which is rich in S_N with

$$N = M \cap Domg \cap domf.$$

Proof. For the first assertion, we notice the equality

$$DomI_g = X \cap L^1_g,$$

and we use 3.10 and 3.12. The second assertion can be proved in the same way, noting that $domI_f = X \cap L_f$.

4. Epi-convexity and other properties of the range of a vector integral functional

Throughout this section, the measure μ is assumed to be atomless and the tribe τ μ-complete.
Let $f : \Omega \times E \longrightarrow \bar{\mathbb{R}}$, $g : \Omega \times E \longrightarrow \bar{\mathbb{R}}^n$ be two measurable integrands, M be a multifunction with an $\tau \otimes \mathcal{B}(E)$ measurable graph and X be a decomposable subset which is rich in S_M. The functional (I_g, I_f) is defined on X.

Theorem 4.1. *The functional (I_g, I_f) is epi-convex.*

Proof. Let us prove that the strict epigraph

$$epi^+(I_g, I_f) = \{(y, r) \in \mathbb{R}^{n+1},\ \exists x \in X,\ y = I_g(x) \text{ and } I_f(x) < r\},$$

is convex. Given (y_i, r_i) in $epi^+(I_g, I_f)$, $i = 0, 1$, and t in the unit interval $]0, 1[$, let us verify that

$$t(y_1, r_1) + (1 - t)(y_0, r_0) \in epi^+(I_g, I_f).$$

There esists two elements $x_i, i = 0, 1$ of X such that

$$y_i = I_g(x_i) \text{ and } I_f(x_i) < r_i.$$

By the definition of I_f there exists two elements $u_i, i = 0, 1$ of $L_1(\Omega, \mathbb{R})$ satisfying

$$f(x_i) \leq u_i \; \mu - a.e \text{ and } \int_\Omega u_i \, d\mu \leq r_i, \; i = 0, 1. \tag{1}$$

By Lemma 3.11, the vector measure ν defined on τ by

$$\nu(A) = (\int_A (g(x_1) - g(x_0)) \, d\mu, \int_A (u_1 - u_0) \, d\mu)$$

is well defined and atomless. Using Lyapunov's theorem [8], we obtain the convexity of the range of ν. Therefore, there exists a measurable set $A \in \tau$ such that

$$\nu(A) = t\nu(\Omega) + (1 - t)\nu(\emptyset) = t\nu(\Omega). \tag{2}$$

Let $x = x_1 1_A + x_0 1_{\Omega \backslash A}$, then $x \in X$ and we have

$$\begin{aligned} I_g(x) &= \int_A g(x_1) \, d\mu + \int_{\Omega\backslash A} g(x_0) \, d\mu = \int_A (g(x_1) - g(x_0)) \, d\mu + \int_\Omega g(x_0) \, d\mu \\ &= t \int_\Omega g(x_1) - g(x_0) \, d\mu + \int_\Omega g(x_0) \, d\mu \qquad \text{(see (2))} \\ &= tI_g(x_1) + (1 - t)I_g(x_0) = ty_1 + (1 - t)y_0. \end{aligned}$$

Besides

$$\begin{aligned} I_f(x) &= \int_A^\star f(x_1) \, d\mu + \int_{\Omega\backslash A}^\star f(x_0) \, d\mu \\ &\leq \int_A u_1 \, d\mu + \int_{\Omega\backslash A} u_0 \, d\mu \qquad \text{(see (1))} \\ &= \int_A (u_1 - u_0) \, d\mu + \int_\Omega u_0 \, d\mu = t \int_\Omega (u_1 - u_0) \, d\mu + \int_\Omega u_0 \, d\mu \qquad \text{(see (2))} \\ &= t \int_\Omega u_1 \, d\mu + (1 - t) \int_\Omega u_0 \, d\mu < tr_1 + (1 - t)r_0. \qquad \text{(see (1))}. \end{aligned}$$

Hence $(ty_1 + (1 - t)y_0, tr_1 + (1 - t)r_0) \in epi^+(I_g, I_f)$. This completes the proof of 4.1.

By Lemma 2.18, the set $r_{I_f}(I_g) = \{I_g(x), \; x \in DomI_g \cap domI_f\}$ is convex. We will give some criteria that will allow to attest that the interior or the relative interior of $r_{I_f}(I_g)$ are non empty. C^{int} denotes the interior of a subset C of $\mathbb{R}^n$ and $\bar{C}$ its closure.

Proposition 4.2. *Let x_0 be in $DomI_g \cap domI_f$ and $N(\omega) = (M \cap Domg \cap domf)(\omega)$. If $A_{x_0} = \{\omega \in \Omega, \; g(\omega, x_0(\omega)) \in \overline{g(\omega, N(\omega))}^{int}\}$ is of positive measure, then $I_g(x_0)$ is an element of $r_{I_f}(I_g)^{int}$.*

Proof. Let us prove firstly the following lemma.

Lemma 4.3. *The multifunction $L(\omega) = \overline{g(\omega, N(\omega))}^{int}$ has an $\tau \otimes \mathcal{B}(E)$ measurable graph. Its domain is measurable and, as a consequence, A_{x_0} is measurable.*

Proof of 4.3. The multifunction N has a measurable graph. If $F(\omega) = \overline{g(\omega, N(\omega))}$, then F has non empty values. Moreover for any $y \in \mathbb{R}^n$, the function $d(y, F(\omega)) = \inf\{|y - g(\omega, e)|,\ e \in N(\omega)\}$ is measurable by virtue of [7] III.39. Therefore $F$is measurable in the sense of [7] III.30. If $G(\omega) = \mathbb{R}^n \backslash F(\omega)$ then G has a measurable graph and by virtue of [7] III.23, the domain Ω_0 of G is τ measurable. The multifunction G has non empty values on Ω_0 and a measurable graph, hence as a consequence of [7] III.40, $\bar{G}$ has a measurable graph. Finally $L(\omega) = \mathbb{R}^n \backslash \bar{G}(\omega)$ has a measurable graph. By virtue of [7] III.23, the domain of L is measurable.
Let us prove 4.2. If $I_g(x_0)$ is not in $r_{I_f}(I_g)^{int}$ then $I_g(x_0)$ is on the boundary of this convex set. By [4] 2.3.7, there exists $y^\star$ in $\mathbb{R}^n \backslash \{0\}$ such that for all $y \in r_{I_f}(I_g)$,

$$< y^\star, I_g(x_0) > \leq < y^\star, y >$$

or equivalently,

$$0 = Min\{< y^\star, I_g(x) - I_g(x_0) >,\ x \in DomI_g \cap domI_f\}.$$

But $DomI_g \cap domI_f$ is a decomposable set which is rich in S_N (3.13(ii)). Using 3.9, this last equality is equivalent to

$$0 = Min\{< y^\star, g(\omega, e) - g(\omega, x_0(\omega)),\ e \in N(\omega)\} \text{ for } \mu - a.e\ \ \omega \in \Omega$$

which proves that for any $\omega \in A_{x_0}$, $y^\star$ attains its minimum on $\overline{g(\omega, N(\omega))}$ at $g(\omega, x_0(\omega))$. Since $g(\omega, x_0(\omega))$ is in $\overline{g(\omega, N(\omega))}^{int}$, we obtain the contradiction $y^\star = 0$. Therefore $I_g(x_0)$ is in $r_{I_f}(I_g)^{int}$.

Definition 4.4. Let C be a subset of $\mathbb{R}^n$. An element c_0 of C is in the relative interior of C if $\overline{\mathbb{R}_+(C - c_0)}$ is a vector space.

This generalization of the notion of relative interior is due to J.M.Borwein and A.S.Lewis in the convex case [27], here is justified by the following two results.

Lemma 4.5. *If a linear functional l assumes its minimum on C at a point c_0 of the relative interior of C, then it is constant on C.*

Proof. As a matter of fact, we have $l(\mathbb{R}_+(C - c_0)) \geq 0$. Therefore $l(\mathbb{R}_+(C - c_0)) = 0$. As a consequence l is identically equal to $l(c_0)$ on C.

Proposition 4.6. *Let x_0 be in $DomI_g \cap domI_f$. We set $N(\omega) = (M \cap Domg \cap domf)(\omega)$. If $g(x_0)$ is a selection of $\omega \mapsto ri(g(\omega, N(\omega)))$, then $I_g(x_0)$ is an element of $ri(r_{I_f}(I_g))$.*

Proof. Suppose that $I_g(x_0)$ is not in $ri(r_{I_f}(I_g))$. Then $I_g(x_0)$ is on the boundary of this convex set. By [21], Theorem 1.3, There exists $y^\star \in \mathbb{R}^n \backslash \{0\}$ such that

(1) for any y in $r_{I_f}(I_g) :< y^\star, I_g(x_0) > \leq < y^\star, y >$,
(2) there exists y_0 in $r_{I_f}(I_g)$ such that : $< y^\star, I_g(x_0) > < < y^\star, y_0 >$.

Assertion (1) is equivalent to

$$0 = Min\{< y^\star, I_g(x) - I_g(x_0) >,\ x \in DomI_g \cap domI_f\}.$$

By 3.9 and 3.13(ii), this last assertion is equivalent to

$$0 = Min\{< y^\star, g(\omega, e) - g(\omega, x_0(\omega)) >,\ e \in N(\omega)\} \text{ for } \mu - a.e\ \ \omega \in \Omega.$$

Since $g(\omega, x_0(\omega))$ is in $ri(g(\omega, N(\omega)))$, we deduce that $y^\star$ is constant on $g(\omega, N(\omega))$, by virtue of 4.5. Assertion (2) is equivalent to the existence of an element x of $DomI_g \cap domI_f$ such that the set $A = \{\omega \in \Omega,\ < y^\star, g(\omega, x_0(\omega)) > < < y^\star, g(\omega, x(\omega)) >\}$ is of positive measure. As $x(\omega)$ is in $N(\omega)$, this shows that $y^\star$ is not constant on $g(\omega, N(\omega))$ and gives a contradiction. This completes the proof of 4.6.

5. Kuhn-Tucker conditions and growth conditions

The same assumptions as in section 4 are made.
Let $h : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a convex function. Let us consider the problem

$$(\mathcal{P}) \qquad \inf\{I_f(x) + h(I_g(x)),\ x \in X,\ I_g(x) \in domh\}.$$

As in section 2, p stands the performance function of the problem $(\mathcal{P})$ and $\inf(\mathcal{P})$ is supposed to be finite.
The following Theorem extends a result of J.P.Aubin and I.Ekeland[3],[2] chapter 14.2.7.

Theorem 5.1. *Suppose that the origin is in $ri(domh - r_{I_f}(I_g))$. Then the problem $(\mathcal{P})$ is stable and we have:*

$$\inf(\mathcal{P}) = Max_{y^\star \in \mathbb{R}^n} I_{u_{y^\star}} - h^\star(y^\star)$$

where $u_{y^\star}(\omega) = \inf\{f(\omega, e)+ < y^\star, g(\omega, e) >,\ e \in M \cap Domg(\omega)\}$

Proof. By 4.1, (I_g, I_f) is epi-convex and the assumptions of 2.19 are verified. Therefore $(\mathcal{P})$ is stable. Using 2.6, we deduce

$$\inf(\mathcal{P}) = Max_{y^\star \in \mathbb{R}^n} \inf\{I_f(x)+ < y^\star, I_g(x) > -h^\star(y^\star),\ x \in DomI_g\}.$$

By using 3.9, 3.11 and 3.13(i), we have

$$\inf\{I_f(x)+ < y^\star, I_g(x) >,\ x \in DomI_g\} = I_{u_{y^\star}}.$$

Corollary 5.2. *Let C be a convex subset of $\mathbb{R}^n$. Suppose that the origin is in $ri(C - r_{I_f}(I_g))$. Then*

$$\inf\{I_f(x),\ x \in X,\ I_g(x) \in C\} = Max_{y^\star \in \mathbb{R}^n} I_{u_{y^\star}} - \sup\{< y^\star, c >, c \in C\}$$

where $u_{y^\star}(\omega) = \inf\{f(\omega, e)+ < y^\star, g(\omega, e) >,\ e \in M \cap Domg(\omega)\}$

Proof. We use 5.1 with $h = \psi_C$.

Corollary 5.3. *(Inclusion of a level line in a level set). Suppose that a is in $ri(r_{I_{-f}}(I_g))$. Given a real number r, the following assertions are equivalent*

(i) $I_g^{-1}(a) \subset I_f^{\leq r}$;
(ii) *there exists $y^\star \in \mathbb{R}^n$ such that the function*

$\omega \mapsto u(\omega) = \inf\{-f(\omega, e)+ < y^\star, g(\omega, e) >,\ e \in M \cap Domg(\omega)\}$ *is integrable and satisfies* $-r \leq \int_\Omega u\, d\mu- < y^\star, a >$.

Proof.

Lemma 5.4. *If $I_f(x) < \infty$ then $I_f(x) = -I_{-f}(x)$.*

Proof of 5.4. If $I_f(x) < \infty$ then $f^+(x) = \sup\{0, f(x)\}$ is integrable. Either $f(x)$ is integrable and in this case $I_f(x) = -I_{-f}(x)$; or $f(x)$ is not integrable. In the latter case $f^-(x) = \inf\{0, f(x)\}$ is not integrable and $I_f(x) = -\infty = -I_{-f}(x)$.

Proof of 5.3. By 5.4, if the assertion (i) is satisfied then we have

(i)' $I_g^{-1}(a) \subset (-I_{-f})^{\leq r}$;
which is equivalent to the next one

(i)" $-r \leq \inf\{I_{-f}(x), x \in X,\ I_g(x) = a\}$.

By 5.2, (i)" is equivalent to (ii). the reciprocal is immediate.
For $e \in M \cap Domg(\omega)$, we get

$$f(\omega, e) \leq < y^\star, g(\omega, e) > -u(\omega).$$

If x is an element of X satisfying $I_g(x) = a$, we obtain

$$I_f(x) \leq < y^\star, I_g(x) > - \int_\Omega u\, d\mu = < y^\star, a > - \int_\Omega u\, d\mu \leq r.$$

Corollary 5.5. *(Inclusion of a level line in a level line). Suppose that a is an element of $ri(r_{I_{-f}}(I_g)) \cap ri(r_{I_f}(I_g))$. Given a real number r, the following assertions are equivalent*

$$\begin{array}{ll} (i) & I_g^{-1}(a) \subset I_f^{-1}(r) \\ (ii) & \text{there exists } y^\star, z^\star \text{ such that the functions} \end{array}$$

$$u(\omega) = \inf\{-f(\omega, e) + < y^\star, g(\omega, e),\ e \in M \cap Domg(\omega)\}$$

and $$v(\omega) = \inf\{f(\omega, e) + < y^\star, g(\omega, e) >,\ e \in M \cap Domg(\omega)\}$$

are integrable and satisfies

$$-r \leq \int_\Omega u\, d\mu - < y^\star, a > \ \text{ and } r \leq \int_\Omega v\, d\mu - < z^\star, a > .$$

Proof. The assertion (i) is equivalent to the two following conditions $I_g^{-1}(a) \subset I_f^{\leq r}$ and $r \leq \inf\{I_f(x),\ x \in X,\ I_g(x) = a\}$. Then we use 5.3 and 5.2.

Corollary 5.6. *We consider a family $(f_i)_{0 \leq i \leq n}$ of measurable scalar integrands. We suppose that there exists real numbers $c_i, 1 \leq i \leq n$, and x of X such that*

$$(H) \text{ for every } i = 1, ..., n, -\infty < I_{f_i}(x) < c_i \text{ and } I_{f_0}(x) < \infty.$$

Then if $r = \inf\{I_{f_0}(y),\ y \in X,\ -\infty < I_{f_i}(y) \leq c_i,\ i = 1, ..., n\}$ is finite, we have

$$r = Max_{y^\star \in (\mathbb{R}_+)^n} I_{u_{y^\star}} - < y^\star, c >$$

where $c = (c_i)_{1 \leq i \leq n}$, and
$u_{y^\star}(\omega) = \inf\{f_0(\omega, e) + \sum_{i=1}^n y_i^\star f_i(\omega, e),\ e \in M(\omega) \cap (\bigcap_{i \geq 1} Domf_i(\omega, .))\}$

Proof. Take $C = c - (\mathbb{R}_+)^n$, and $g = (f_i)_{1\leq i\leq n}$.
The assumption (H) ensures that the origin is in the interior of $C - r_{I_{f_0}}(I_g)$. For $y^\star \in \mathbb{R}^n$ we have $\sup\{< y^\star, c' >,\ c' \in C\} =< y^\star, c >$ if $y^\star \in \mathbb{R}_+{}^n$ and $+\infty$ if not.
Moreover, we have $Domg(\omega, .) = \bigcap_{i\geq 1} Domf_i(\omega, .)$, and therefore 5.6 is a consequence of 5.2.

The following result generalizes and completes some key theorems of [9], [10] and [6].

Corollary 5.7. *(Inclusion of a level set in a level set). Consider a family* $(f_i)_{0\leq i\leq n}$ *of measurable scalar integrands. We suppose that for all* $1 \leq i \leq n$, f_i *is proper. Besides, we suppose that there exists real numbers* $(c_i)_{0\leq i\leq n}$ *and an element* x *of* X *such that*

$$(H') \qquad \text{for every } i = 1, ..., n, -\infty < I_{f_i}(x) < c_i \text{ and } I_{-f_0}(x) < \infty.$$

The following assertions are equivalent

(i) $\bigcap_{i\geq 1} I_{f_i}^{\leq c_i} \subset I_{f_0}^{\leq c_0}$;

(ii) *there exists* $y^\star$ *in* $(\mathbb{R}_+)^n$*such that the function* $u(\omega) = \inf\{-f_0(\omega, e)+\sum_{i=1}^n y_i^\star f_i(\omega, e),$ $e \in M(\omega) \cap (\bigcap_{i\geq 1} domf_i(\omega, .))\}$ *is integrable and verifies* $-c_0 \leq \int_\Omega u\, d\mu- < y^\star, c >$,

where $c = (c_i)_{1\leq i\leq n}$.

Proof. If the assertion (ii) is satisfied, then for any $e \in M(\omega)$, we have

$$f_0(\omega, e) \leq \Sigma_{i=1}^n y_i^\star f_i(\omega, e) - u(\omega).$$

Consequently, for $z \in X$ verifying $I_{f_i}(z) \leq c_i, i = 1, ..., n$, by integration, we obtain

$$I_{f_0}(z) \leq \Sigma_{i=1}^n y_i^\star I_{f_i}(z) - \int_\Omega u\, d\mu \leq < y^\star, c > +c_0- < y^\star, c >= c_0.$$

With the convention : $0 \times (-\infty) = 0$.
Conversely, by 5.4, if assertion (i) is true, then the following holds

$$\bigcap_{i\geq 1} I_{f_i}^{\leq c_i} \subset -I_{-f_0}^{\leq c_o}.$$

This assertion is equivalent to the following one

$$-c_0 \leq \inf\{I_{-f_0}(y),\ y \in X,\ I_{f_i}(y) \leq c_i,\ i = 1, ..., n\}.$$

therefore we get

$$-c_0 \leq \inf\{I_{-f_0}(y),\ y \in X,\ -\infty < I_{f_i}(y) \leq c_i,\ i = 1, ..., n\} = r.$$

By (H') and corollary 5.6, there exists $y^\star \in (\mathbb{R}_+)^n$ such that $-c_0 \leq r = I_{u_{y^\star}} - < y^\star, c >$, where we have $u_{y^\star}(\omega) = \inf\{-f_0(\omega, e) + \sum_{i=1}^n y_i^\star f_i(\omega, e),\ e \in M(\omega) \cap (\bigcap_{i\geq 1} Domf_i(\omega, .))\}$.
Since the $f_i, i = 1, ..., n$, are proper, we deduce $Domf_i = domf_i$, moreover, since $I_{u_{y^\star}}$ is finite, $u_{y^\star}$ is integrable and satisfies the second assertion of Corollary 5.7. This completes the proof.

Proposition 5.8. *We consider the problem* $(\mathcal{P})$. *Suppose that the origin is an element of* $ri(domh - r_{I_f}(I_g))$. *The following assertions are equivalent* *(i)* x_0 *is a minimizer of* $(\mathcal{P})$*;*

(ii) *there exists* $y^\star$ *in* $\partial h(I_g(x_0))$ *such that the function*
$v(\omega) = Min\{f(\omega,e) + < y^\star, g(\omega,e) > -(f(\omega), x_0(\omega)) + < y^\star, g(\omega, x_0(\omega)) >,$
$e \in M \cap Domg(\omega)\}$
is the null mapping almost everywhere.

Moreover, the set of all $y^\star$ *verifying (ii) is exactly* $\partial p(0)$*.*

Proof. By 4.1 and 2.19, the problem $(\mathcal{P})$ is stable. By virtue of 2.7, x_0 is a minimizer of $(\mathcal{P})$ if and only if there exists $y^\star$ in $\partial h(I_g(x_0))$ such that

$$I_f(x_0) + < y^\star, I_g(x_0) >= Min\{I_f(x) + < y^\star, I_g(x) >,\ x \in DomI_g\}.$$

Using 3.9, 3.13(i) and 3.11, this is equivalent to $\displaystyle\int_\Omega v\, d\mu = 0$, with
$v(\omega) = Min\{f(\omega,e) + < y^\star, g(\omega,e) > -(f(\omega), x_0(\omega)) + < y^\star, g(\omega, x_0(\omega)) >,$
$e \in M \cap Domg(\omega)\}$
Since v is non positive, we deduce that v is null almost everywhere. Therefore the assertions are equivalent.Besides, the set of all $y^\star$ satisfying (ii) is exactly $\partial p(0), by$ 2.7.

When $h = \psi_P$, where P is a cone of $\mathbb{R}^n$ of the following type

$$P = \{0\}^k \times (\mathbb{R}_+)^l, k + l = n,$$

we complete some results of [1].

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