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Metric Bornologies and Kuratowski-Painlevé Convergence to the Empty Set

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Given a sequence $\langle T_n \rangle$ of nonempty closed sets Kuratowski-Painlevé convergent to the empty set in a noncompact metrizable space X, we show not only that there exists an admissible unbounded metric such that $\langle T_n \rangle$ converges to infinity in distance, but also that there must exist another such metric for which this is not the case. For such a sequence, let $\mathcal{A}$ consist of all subsets A of X whose closure hits T_n for at most finitely many indices n. We give necessary and sufficient conditions for $\mathcal{A}$ to be the family of bounded sets induced by some admissible metric for X, and show that all possible nontrivial metric bornologies for X arise in this manner if and only if the derived set of X is compact.

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1. Introduction

Let $\langle X, \tau \rangle$ be a metrizable topological space. If X is noncompact, then there exists an unbounded metric compatible with the topology for X. Many years ago, S.-T. Hu [8, 9] discovered a characteristic set of properties for a family of subsets $\mathcal{A}$ of X to be the family of bounded subsets with respect to some unbounded metric d compatible with τ. These properties are the following:

(a) $\mathcal{A}$ is an ideal of subsets of X, i.e., $\mathcal{A}$ is hereditarily closed and is closed under finite unions;

(b) $\bigcup \mathcal{A} = \mathcal{X}$, or equivalently by (a), $\{x\} \in \mathcal{A}$ for every $x \in X$;

(c) for each $A \in \mathcal{A}$, $X \setminus A \neq \emptyset$, i.e., $\mathcal{A}$ is a proper ideal;

(d) for each $A \in \mathcal{A}$, there exists B in $\mathcal{A}$ such that $\mathrm{cl}\, A \subset \mathrm{int}\, B$;

(e) $\mathcal{A}$ contains a countable base, i.e., a countable cofinal subfamily with respect to set inclusion.

A family $\mathcal{A}$ that satisfies conditions (a) and (b) is often called a *bornology* in the literature [7], and so it makes sense to call a family satisfying (a) through (e) a *nontrivial metric bornology* (the trivial metric bornology, corresponding to a bounded metric compatible

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with the topology, consists of the power set of X). Simple examples show that these conditions are minimal. In particular, condition (d) cannot be replaced by the weaker condition $A \in \mathcal{A} \Rightarrow \mathrm{cl}\, \mathcal{A} \in \mathcal{A}$, for the family of finite subsets of the rationals viewed as a subspace of the real line satisfies this weakened condition and all the others but does not satisfy (d). To see that condition (e) does not follow from the others, consider $\mathbb{Z}^+$ with the discrete topology and let $\mathcal{A} = \{\mathcal{A} \subset \mathbb{Z}^+ \ : \ \mathcal{A}$ contains at most finitely many powers of each prime$\}$.

In [4], the author introduced a notion dual to that of nontrivial metric bornology – the notion of metric mode of convergence to infinity – and argued for its use as the primitive notion in developing a theory of metric boundedness. In a noncompact metrizable space consider all sequences $\langle F_n \rangle$ of nonempty closed subsets such that for each positive integer n, $F_{n+1} \subset \mathrm{int}\, F_n$ and $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Define an equivalence relation $\equiv$ on the set of such sequences as follows: $\langle F_n \rangle \equiv \langle E_n \rangle$ provided for each positive integer k there exist positive integers n and j such that $E_k \supset F_j$ and $F_k \supset E_n$. By a *metric mode of convergence to infinity* we mean an equivalence class for this equivalence relation. As in [4] we freely engage in the abuse of identifying an equivalence class under $\equiv$ with a particular representative of that class.

The connection between these objects and nontrivial metric bornologies is as follows. Given a metric mode of convergence to infinity $\langle F_n \rangle$ define a subset A of X to be *bounded* with respect to $\langle F_n \rangle$ provided for some n we have $A \cap F_n = \emptyset$. We denote the bounded subsets of X with respect to $\langle F_n \rangle$ by $\mathcal{B}(\langle \mathcal{F}_\backslash \rangle)$. One can show that $\mathcal{B}(\langle \mathcal{F}_\backslash \rangle) = \mathcal{B}(\langle \mathcal{E}_\backslash \rangle)$ if and only if $\langle F_n \rangle \equiv \langle E_n \rangle$ and, furthermore, that each family of the form $\mathcal{B}(\langle \mathcal{F}_\backslash \rangle)$ satisfies Hu's axioms for a nontrivial metric bornology. Conversely, if $\mathcal{A}$ is a family of subsets of X satisfying Hu's axioms and d is a compatible metric whose bounded sets coincide with $\mathcal{A}$, then it is easy to show that $\mathcal{A} = \mathcal{B}(\langle \{\S : \lceil(\S, \S_\prime) \geq \backslash\} \rangle)$ where $x_0 \in X$ is an arbitrary fixed point.

There are advantages to this dual approach. First, examples of metric modes of convergence to infinity are natural and exist in abundance, e.g., in the context of a normed linear space, convergence to infinity in a particular direction, and positive convergence to infinity as determined by a closed convex cone with nonempty interior fall within this rubric. Second, these objects are simpler and are more tractible than the nontrivial metric bornologies to which they correspond, and proofs about boundedness structures are invariably simpler when approached dually. Finally, they provide the link between metric bornologies and one-point regular extensions of the space with a countable base at the ideal point. Such extensions are automatically metrizable [5], and provide an alternate generalization in the framework of noncompact metrizable spaces *rather than in the framework of noncompact locally compact Hausdorff spaces* of the familiar construction of the Riemann sphere as a one-point extension of the metrizable Euclidean plane.

A metric mode of convergence to infinity $\langle F_n \rangle$ is a special case of a sequence of closed sets that converges to the empty set in the sense of Kuratowski-Painlevé. It is the purpose of this article to clarify the relationship between Kuratowski-Painlevé convergence to the empty set for a general sequence of nonempty closed sets in a metric space and nontrivial metric bornologies. Our analysis leads to another axiomatic approach to metric bornologies that is valid only in those metrizable spaces $\langle X, \tau \rangle$ whose derived set X' is compact. Such metrizable spaces have been well-studied, as they are the carriers of UC metrics.

2. Notation and terminology

Let $\langle X, \tau \rangle$ be a metrizable space. We denote the interior, closure, boundary, and set of accumulation points of a subset A of X by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, and A'. A metric d defined on X is called *admissible* if it is compatible with the topology τ. Given an admissible metric d, the symbol $d(x, A)$ will represent $\inf\{d(x, a) : a \in A\}$, the usual distance from a point x in X to a nonempty subset A of X. Given $x \in X$ and $\delta > 0$, we denote the open ball of radius δ and center x by $S_\delta^d(x)$. More generally, if A is a subset of X, we denote the *enlargement* $\bigcup_{x \in A} S_\delta^d(x)$ of A of radius δ by $S_\delta^d(A)$. For A nonempty, $S_\delta^d(A) = \{x : d(x, A) < \delta\}$. Each enlargement of the empty set is again empty. A subset A of X is called *d-bounded* if for some $x \in X$ and $\delta > 0$, we have $A \subset S_\delta^d(x)$, which means that $\operatorname{diam}_d(A) = \sup\{d(a_1, a_2) : a_1 \in A,\ a_2 \in A\} < \infty$. We denote the family of d-bounded subsets of X by $\mathcal{B}(\lceil)$.

In an introductory exposure to metric space topology, one becomes aware of the following results valid in a compact metric space $\langle X, d \rangle$:

(1) Each continuous function f from $\langle X, d \rangle$ to an arbitrary metric space $\langle Y, \rho \rangle$ is uniformly continuous;

(2) Each open cover of X has a Lebesgue number λ, i.e., if A is a subset of X with $\operatorname{diam}_d(A) < \lambda$, then A lies within some single member of the cover;

(3) Whenever A and B are disjoint closed subsets of X, there exists $\delta > 0$ such that $S_\delta^d(A) \cap B = \emptyset$.

Although these properties are equivalent for a metric d [1, 3, 13], it cannot be shown that these properties characterize compactness, for indeed they do not. For example, each holds in a zero-one discrete metric space. Compact metric spaces and zero-one metric spaces are extreme cases of the class of metric spaces in which these properties are valid. Such metric spaces, called *UC spaces* or *Atsuji spaces* in the literature, are perhaps best characterized visually as follows by the conjunction of two conditions [10]:

(1) X' is compact (but is perhaps empty), and

(2) for each $\delta > 0$, $(S_\delta^d(X'))^c$ is *uniformly discrete*, i.e., there exists $\varepsilon > 0$ such that whenever $x, y \in (S_\delta^d(X'))^c$, $x \neq y$, then $d(x, y) > \varepsilon$.

For a proof that compactness of the derived set X' for a metrizable space $\langle X, \tau \rangle$ is sufficient for the existence of an admissible UC metric compatible with the topology, the reader may consult [2, 11, 13].

3. On Kuratowski-Painlevé convergence to the empty set

We begin this section with a review of some basic facts about Kuratowski-Painlevé convergence. All of the information given below is either explicit or implicit in Section 5.2 of the monograph of the author [3] and is well-known.

Let $\langle T_n \rangle$ be a sequence of nonempty closed sets in a metrizable space $\langle X, \tau \rangle$. A point $x \in X$ is called a *limit point* (resp. a *cluster point*) of $\langle T_n \rangle$ provided each neighborhood V of x intersects T_n for all but finitely many (resp. infinite many) indices n. We denote the set of limit points (resp. cluster points) of the sequence T_n by $\operatorname{Li} T_n$ (resp. $\operatorname{Ls} T_n$). Clearly, $\operatorname{Li} T_n \subset \operatorname{Ls} T_n$ and it is not hard to show that both sets are closed. In particular, closedness of $\mathrm{Ls}\, T_n$ follows from the formula

$$\mathrm{Ls}\, T_n = \bigcap_{n=1}^{\infty} (\mathrm{cl} \bigcup_{j=n}^{\infty} T_j).$$

We declare $\langle T_n \rangle$ *Kuratowski-Painlevé convergent* to a (closed) set T provided $T = \mathrm{Li}\, T_n = \mathrm{Ls}\, T_n$, and in this case we write $T = \text{K-lim}\, T_n$. Notice that when $\langle T_n \rangle$ is a decreasing sequence, we have $\text{K-lim}\, T_n = \bigcap_{n=1}^{\infty} T_n$. We will be interested in convergence to the empty set $\emptyset$. We list some criteria for this to occur in the following lemma whose proof is left as an exercise for the reader.

Lemma 3.1. *Let $\langle X, \tau \rangle$ be a metrizable space and let $\langle T_n \rangle$ be a sequence of nonempty closed sets. The following conditions are equivalent:*

(1) $\emptyset = \text{K-lim}\, T_n$;

(2) $\mathrm{Ls}\, T = \emptyset$;

(3) *Whenever C is a compact subset of X, we have $C \cap T_n = \emptyset$ eventually;*

(4) *$\{T_n : n \in \mathbb{Z}^+\}$ is a locally finite family of closed sets and for each $n \in \mathbb{Z}^+$ there exist at most finitely many indices j such that $T_n = T_j$.*

Clearly, none of these conditions can occur if X is compact, and it will be understood in the sequel that the underlying space is noncompact. The following result gives a new characterization of Kuratowski-Painlevé convergence to the empty set.

Proposition 3.2. *Let $\langle T_n \rangle$ be a sequence of closed nonempty sets in a noncompact metrizable space $\langle X, \tau \rangle$. The following conditions are equivalent:*

(1) $\emptyset = \text{K-lim}\, T_n$;

(2) *There exists an unbounded admissible metric d for X such that for each $B \in \mathcal{B}(d)$, $T_n \cap B = \emptyset$ eventually;*

(3) *There exists an unbounded admissible metric d for X such that for some $x_0 \in X$ we have $\lim_{n\to\infty} d(x_0, T_n) = \infty$;*

(4) *There exists an unbounded admissible metric d for X such that for all $x \in X$ we have $\lim_{n\to\infty} d(x, T_n) = \infty$.*

Proof. As the proof of the equivalence of conditions (2), (3) and (4) is trivial, we will just prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). For (1) $\Rightarrow$ (2) let ρ be an admissible metric for X and for each $n \in \mathbb{Z}^+$ let E_n be the following closed set:

$$E_n = \left\{ x \in X : \rho(x, \bigcup_{j=n}^{\infty} T_j) \leq \frac{1}{n} \right\} .$$

Since $\langle T_n \rangle$ has no cluster points, it follows that $\bigcap_{n=1}^{\infty} E_n = \emptyset$, and for each $n \in \mathbb{Z}^+$, we clearly have $E_{n+1} \subset \mathrm{int}\, E_n$. Thus, $\langle E_n \rangle$ is a metric mode of convergence to infinity. Since for each n we have $\bigcup_{j=n}^{\infty} T_j \subset E_n$, for each $B \in \mathcal{B}(\langle E_n \rangle)$ we have $T_n \cap B = \emptyset$ for all but finitely many indices n. Finally, by Hu's Theorem, there exists an unbounded admissible metric d such that $\mathcal{B}(\langle E_n \rangle) = \mathcal{B}(d)$, and this finishes the argument. For (3) $\Rightarrow$ (1), let C be a compact subset of X and let x_0 and d be as described in (3). Take $\lambda > 0$ with $C \subset S^d_\lambda(x_0)$ and then take $n_0 \in \mathbb{Z}^+$ such that for all $n \geq n_0$ we have $d(x_0, T_n) > \lambda$. Then $C \cap T_n = \emptyset$ for all $n \geq n_0$, whence by Lemma 3.1, it follows that $\emptyset = \text{K-lim}\, T_n$. $\square$

It can never be the case that for *all* admissible d and $x \in X$ we have

$$\lim_{n\to\infty} d(x, T_n) = \infty$$

However, the situation is a little subtle. We begin with

Proposition 3.3. *Let $\langle T_n \rangle$ be a decreasing sequence of nonempty closed sets in a noncompact metrizable space $\langle X, \tau \rangle$ with $\emptyset = \text{K-lim}\, T_n$. Then there exists an unbounded admissible metric d such that for some $x_0 \in X$, we have $\limsup_{n\to\infty} d(x_0, T_n) < \infty$.*

Proof. Let ρ be a bounded admissible metric with $\text{diam}_\rho(X) < 1$. We consider two cases for the sequence $\langle T_n \rangle$: (1) for all $n \in \mathbb{Z}^+$, T_n^c has compact closure; (2) for some $n \in \mathbb{Z}^+$, T_n^c does not have compact closure. In the first case, let $\langle w_n \rangle$ be a sequence in X with distinct terms having no cluster point. By the Tietze extension theorem, there exists a nonnegative continuous function f mapping w_{2n} to n and w_{2n-1} to zero, for $n \in \mathbb{Z}^+$. Then the metric $d(x, y) = \rho(x, y) + |f(x) - f(y)|$ is admissible and unbounded, and $\text{cl}\, S^d_1(w_1)$ is not compact. As a result, $S^d_1(w_1)$ cannot be contained in T_n^c for any n, and so $\lim\sup_{n\to\infty} d(w_1, T_n) \leq 1$. In the second case, fix $n_0 \in \mathbb{Z}^+$ such that $T^c_{n_0}$ has noncompact closure. Let $\langle x_j \rangle$ be a sequence in $T^c_{n_0}$ with distinct terms having no cluster point. By the Tietze extension theorem we can find a continuous nonnegative real valued function g on X such that for each $j \in \mathbb{Z}^+$ we have $g(x_j) = j$ and $g(T_{n_0}) = 0$. Define $d : X \times X \to [0, \infty)$ by the formula

$$d(x, y) = \rho(x, y) + |g(x) - g(y)|.$$

Then d is an unbounded admissible metric, and if $x_0 \in T_{n_0}$ and $n \geq n_0$, we have $d(x_0, T_n) \leq 1$. In fact, all T_n for $n \geq n_0$ are contained in a common d-ball. $\square$

Proposition 3.4. *Let $\langle T_n \rangle$ be a sequence of nonempty closed sets in a noncompact metrizable space $\langle X, \tau \rangle$ with $\emptyset = \text{K-lim}\, T_n$. Then there exists an unbounded admissible metric d such that for some $x_0 \in X$, we have $\liminf_{n\to\infty} d(x_0, T_n) < \infty$.*

Proof. For each $n \in \mathbb{Z}^+$, write $F_n = \bigcup_{j=n}^\infty T_j$. By local finiteness of $\{T_n : n \in \mathbb{Z}^+\}$ each set F_n is closed, and since $\emptyset = \text{Ls}\, T_n = \bigcap_{n=1}^\infty \text{cl}(\bigcup_{j=n}^\infty T_j) = \bigcap_{n=1}^\infty \text{cl}\, F_n = \bigcap_{n=1}^\infty F_n$, we also have $\emptyset = \text{K-lim}\, F_n$. By Proposition 3.3,

$$\limsup_{n\to\infty} d(x_0, F_n) < \infty$$

for some x_0. Thus for some $\lambda > 0$ and all indices n, we have $d(x_0, F_n) < \lambda$. It now follows that $d(x_0, T_n) < \lambda$ for infinitely many indices n. $\square$

Example. In the statements of Propositions 3.3 and 3.4, "some" can be replaced by "all". However, in the last result, we cannot in general replace "lim inf" by "lim sup". For our metrizable space, consider $\mathbb{Z}^+$ with the discrete topology and for each $n \in \mathbb{Z}^+$

let $T_n = \{n\}$. If d is any unbounded admissible metric and $k_0 \in \mathbb{Z}^+$ and $\lambda > 0$ are arbitrary, then the complement of $S_\lambda^d(k_0)$ must consist of infinitely many points, else $\mathbb{Z}^+$ would be a bounded set. This means that $\limsup_{n\to\infty} d(k_0, T_n) \geq \lambda$, and so we obtain $\limsup_{n\to\infty} d(k_0, T_n) = \infty$.

4. Metric bornologies in spaces with compact derived set

Suppose $\langle T_n \rangle$ is a sequence of nonempty closed sets in a metrizable space $\langle X, \tau \rangle$ that is Kuratowski-Painlevé convergent to the empty set. Let us write $\mathcal{B}(\langle T_n \rangle)$ for the following family of subsets of X:

$$\mathcal{B}(\langle \mathcal{T}_\backslash \rangle) = \{\mathcal{A} : \mathrm{cl}\,\mathcal{A} \text{ intersects } T_n \text{ for at most finitely many indices } n\}.$$

Now if $\langle T_n \rangle$ were a metric mode of convergence to infinity, for $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ as just defined, we obtain the same class of bounded sets as indicated by the same notation in the introduction. Thus, each nontrivial metric bornology in an arbitrary metrizable space $\langle X, \tau \rangle$ arises in this way. But there are other natural ways to represent a given nontrivial metric bornology $\mathcal{A}$ as $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ where the sets T_n are not nested. To see this, let d be an unbounded admissible metric for which $\mathcal{A} = \mathcal{B}(\lceil)$. Fix $x_0 \in X$, and let $\langle k_n \rangle$ be a strictly increasing sequence in $\mathbb{Z}^+$ such that for each n, the closed annulus $T_n = \{x \in X : k_n \leq d(x, x_0) \leq k_{n+1}\}$ is nonempty. Then the sequence $\langle T_n \rangle$ converges to the empty set and $\mathcal{A} = \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$.

A basic question still remains: for a general sequence $\langle T_n \rangle$ of nonempty closed sets in a noncompact metrizable space with $\emptyset = \text{K-lim}\, T_n$, must $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ always satisfy Hu's axioms? It is clear that $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ contains only proper subsets of X, that $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ forms a cover of X, and that $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ is closed under finite unions and is hereditarily closed. Thus, it is always true that $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ forms a nontrivial bornology on X. Furthermore, Hu's axiom (d) is satisfied for $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$. To see this, suppose $A \in \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$, and choose $n \in \mathbb{Z}^+$ such that $\mathrm{cl}\, A \cap (\bigcup_{j=n}^\infty T_j) = \emptyset$. By local finiteness, $\bigcup_{j=n}^\infty T_j$ is a closed subset of X, whence by normality there exists $B \in \tau$ such that

$$\mathrm{cl}\, A \subset B \subset \mathrm{cl}\, B \subset (\bigcup_{j=n}^\infty T_j)^c.$$

As a result, $B \in \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ and $\mathrm{cl}\, A \subset \mathrm{int}\, B$, as required.

Only Hu's axiom (e) – the requirement that the bornology have a countable base – is at issue. We intend to give necessary and sufficient conditions for this to occur. Given a sequence $\langle T_n \rangle$ of nonempty closed sets with $\emptyset = \text{K-lim}\, T_n$, let us adopt some notation introduced in the proof of Proposition 3.4: F_n for $\bigcup_{j=n}^\infty T_j$. Not only is it true that $\emptyset = \text{K-lim}\, F_n$, but also that $\mathcal{B}(\langle \mathcal{F}_\backslash \rangle) = \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$.

Theorem 4.1. *Let $\langle X, \tau \rangle$ be a noncompact metrizable space and let $\langle T_n \rangle$ be a sequence of nonempty closed sets with $\emptyset = \text{K-lim}\, T_n$. For each positive integer n let $F_n = \bigcup_{j=n}^\infty T_j$. The following conditions are equivalent:*

(1) *For each $k \in \mathbb{Z}^+$ there exists $n \in \mathbb{Z}^+$ with $\mathrm{int}\, F_k \supset F_n$;*

(2) *There exists a subsequence $\langle F_{n_k} \rangle$ of $\langle F_n \rangle$ that is a metric mode of convergence to infinity;*

(3) *For each $A \subset X$, $A \in \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ if and only if there exists $j \in \mathbb{Z}^+$ with $A \cap F_j = \emptyset$;*

(4) *the bornology $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ has a countable base with respect to set inclusion;*

(5) *$\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ is a nontrivial metric bornology.*

Proof. (1) $\Rightarrow$ (2). By (1) we can find a strictly increasing sequence of positive integers $\langle n_k \rangle$ such that for all $k \in \mathbb{Z}^+$ we have $\operatorname{int} F_{n_k} \supset F_{n_{k+1}}$. Each set F_{n_k} is closed and $\bigcap_{k=1}^{\infty} F_{n_k} = \emptyset$, because $\bigcap_{n=1}^{\infty} F_n = \emptyset$. By definition, $\langle F_{n_k} \rangle$ is a metric mode of convergence to infinity.

(2) $\Rightarrow$ (3). If for some $j \in \mathbb{Z}^+$ we have $A \cap F_j = \emptyset$, then choosing $n_k > j$ we have

$$\operatorname{cl} A \cap F_{n_{k+1}} \subset \operatorname{cl} A \cap \operatorname{int} F_{n_k} = \emptyset.$$

This shows that $A \in \mathcal{B}(\langle \mathcal{F}_\backslash \rangle) = \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$. The converse is immediate from the definition of $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ and holds without the validity of condition (2).

(3) $\Rightarrow$ (4). By (3), $\{F_n^c : n \in \mathbb{Z}^+\}$ is a countable base for $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle) = \mathcal{B}(\langle \mathcal{F}_\backslash \rangle)$.

(4) $\Rightarrow$ (5). By (4), $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ satisfies Hu's axiom (e) for a nontrivial metric bornology. The other axioms are automatically satisfied.

(5) $\Rightarrow$ (1). Suppose $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle) = \mathcal{B}(\lceil)$ for some unbounded admissible metric d yet (1) fails. Then for some $k \in \mathbb{Z}^+$ and all larger integers n we have $F_n \cap \operatorname{bd} F_k \neq \emptyset$. For each $n > k$ pick $a_n \in F_n \cap \operatorname{bd} F_k$. Since $\text{K-lim}\, F_n = \operatorname{Ls} F_n = \emptyset$, the sequence $a_{k+1}, a_{k+2}, a_{k+3}, \ldots$ can have no cluster point. Also, by passing to a subsequence, we can assume that all the terms of the sequence are distinct. By Hu's axioms (d) and (e), we may choose a countable base $\{B_i : i \in \mathbb{Z}^+\}$ for $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ consisting of closed sets. Starting with B_1 there exists $n_1 > k$ such that $B_1 \cap F_{n_1} = \emptyset$. As $a_{n_1} \in F_{n_1}$ there exists $\varepsilon(n_1) \in (0,1)$ such that $B_1 \cap S^d_{\varepsilon(n_1)}(a_{n_1}) = \emptyset$. Since $\langle F_n \rangle$ is a decreasing sequence of sets, there exists $n_2 > n_1$ such that $B_2 \cap F_{n_2} = \emptyset$. As before, there exists $\varepsilon(n_2) \in (0, 1/2)$ such that $B_2 \cap S^d_{\varepsilon(n_2)}(a_{n_2}) = \emptyset$. Continuing, we produce a strictly increasing sequence $\langle n_i \rangle$ of positive integers and an associated sequence $\langle \varepsilon(n_i) \rangle$ of positive reals such that for each $i \in \mathbb{Z}^+$ we have $\varepsilon(n_i) < 1/i$ and

$$B_i \cap S^d_{\varepsilon(n_i)}(a_{n_i}) = \emptyset.$$

Now each point a_{n_i} lies in $\operatorname{bd} F_k$ and so for each $i \in \mathbb{Z}^+$ there exists $c_i \notin F_k$ with $0 < d(c_i, a_{n_i}) < \varepsilon(n_i)$. The condition $\lim_{i\to\infty} \varepsilon(n_i) = 0$ guarantees that $\langle c_i \rangle$ has no cluster point because $\langle a_{n_i} \rangle$ has no cluster point. As a result, $C = \{c_i : i \in \mathbb{Z}^+\}$ is a closed set, and since $C \cap F_k = \emptyset$, we see that $C \in \mathcal{B}(\langle \mathcal{F}_\backslash \rangle) = \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$. But no set B_i contains C, which contradicts $\{B_i : i \in \mathbb{Z}^+\}$ serving as a base for $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$. As result, condition (1) follows from condition (5). □

Our next result describes those metrizable spaces in which sequences of nonempty closed sets convergent to the empty set always determine nontrivial metric boundedness structures (see Theorem 2 of [13] for other characterizations of spaces with compact derived set).

Theorem 4.2. *Let $\langle X, \tau \rangle$ be a noncompact metrizable space. The following conditions are equivalent:*

(1) *Whenever $\langle T_n \rangle$ is a sequence of nonempty closed subsets of X with $\emptyset = \text{K-lim}\, T_n$, the bornology $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ is a nontrivial metric bornology;*

(2) *Whenever $\{T_n : n \in \mathbb{Z}^+\}$ is a discrete family of distinct closed subsets of X, $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ is a nontrivial metric bornology;*

(3) *X' is compact.*

Proof. (1) $\Rightarrow$ (2). By Lemma 3.1, this is trivial.

(2) $\Rightarrow$ (3). Suppose X' is noncompact. Then there exists a sequence $\langle x_n \rangle$ in X' with distinct terms that has no cluster point in X. For each $n \in \mathbb{Z}^+$, let $T_n = \{x_n\}$. Then $\{T_n : n \in \mathbb{Z}^+\}$ is a discrete family of closed sets. Now for each $k \in \mathbb{Z}^+$, we have $\text{int}\{x_n : n \geq k\} = \emptyset$. Thus with $F_k = \{x_n : n \geq k\}$, there cannot exist $n \in \mathbb{Z}^+$ such that $\text{int}\, F_k \supset F_n$. By Theorem 4.1, $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ cannot be a nontrivial metric bornology.

(3) $\Rightarrow$ (1). Assume X' is compact, and let d be an admissible UC-metric. Let $\langle T_n \rangle$ be a sequence of nonempty closed subsets of X with $\emptyset = \text{K-lim}\, T_n$. To show that $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ is a nontrivial metric bornology, we need only produce a countable base for $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$. For each $j \in \mathbb{Z}^+$, define B_j by the formula

$$B_j = \{x \in X : d(x, \bigcup_{n=j}^{\infty} T_n) \geq \frac{1}{j}\}.$$

We claim that $\{B_j : j \in \mathbb{Z}^+\}$ does the job. First note that each such B_j is a closed set that intersects no T_n for $n \geq j$ so that $B_j \in \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$. Now let $B \in \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$ be arbitrary. By definition, there exists $j \in \mathbb{Z}^+$ such that $\text{cl}\, B \cap (\bigcup_{n=j}^{\infty} T_n) = \emptyset$. Since d is a UC-metric and the union of any locally finite family of closed sets remains closed there exists $k > j$ with $\text{cl}\, B \cap S^d_{1/k}(\bigcup_{n=j}^{\infty} T_n) = \emptyset$. Since $\bigcup_{n=k}^{\infty} T_n \subset \bigcup_{n=j}^{\infty} T_n$, it follows that $B \subset \text{cl}\, B \subset B_k$ as required. $\square$

Our next result presents two alternative approaches to metric boundedness valid in spaces with compact derived set.

Theorem 4.3. *Let $\langle X, \tau \rangle$ be a noncompact metrizable space with X' compact, and let $\mathcal{A}$ be a family of subsets of X. The following conditions are equivalent:*

(1) *$\mathcal{A}$ is the family of bounded sets $\mathcal{B}(\lceil)$ corresponding to some unbounded admissible metric d for τ;*

(2) *There exists a sequence $\langle T_n \rangle$ of distinct closed subsets of X Kuratowski-Painlevé convergent to the empty set such that $\mathcal{A} = \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$;*

(3) *There exists a discrete family $\{E_n : n \in \mathbb{Z}^+\}$ of nonempty clopen subsets of X such that $\mathcal{A} = \{\mathcal{A} \subset \mathcal{X} : \mathcal{A} \cap \mathcal{E}_\backslash = \emptyset$ for all but finitely many $n\}$.*

Proof. The implication (3) $\Rightarrow$ (2) is trivial, for a set A fails to intersect a given open set if and only if its closure does. The implication (2) $\Rightarrow$ (1) is established by Theorem 4.2. To prove (1) $\Rightarrow$ (3), let d be an unbounded admissible metric with $\mathcal{A} = \mathcal{B}(\lceil)$. Fix $x_0 \in X$ and for each $n \in \mathbb{Z}^+$ let $F_n = (S^d_n(x_0))^c$. Then $\langle F_n \rangle$ is a representative of the metric mode of convergence to infinity whose bounded sets are $\mathcal{A}$. Since $\langle F_n \rangle$ is equivalent to each of its subsequences, without loss of generality we may assume that for each n,

$F_n \setminus F_{n+1}$ is nonempty. Since X' is compact, there exists $k \in \mathbb{Z}^+$ such that $F_k \cap X' = \emptyset$, else $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Also by the compactness of X', there exists $\delta > 0$ such that $S_\delta^d(X') \cap F_k = \emptyset$, and as a result, each subset of F_k is clopen. For each $n \in \mathbb{Z}^+$ set $E_n = F_{k+n-1} \setminus F_{k+n}$. Evidently, each set E_n is a nonempty clopen subset of X. Also, the family $\{E_n : n \in \mathbb{Z}^+\}$ is discrete, for if $x \in X'$, then $S_\delta^d(x)$ meets no E_n and if $x \notin X'$ there is a ball with center x consisting of just x. If $A \in \mathcal{A}$ there exists $j \geq k$ such that $A \cap F_j = \emptyset$ and so $A \cap E_n = \emptyset$ whenever $n > j - k$. On the other hand, if $A \cap E_n = \emptyset$ for each $n \geq j$, then $A \cap F_{k+j-1} = \emptyset$ because

$$F_{k+j-1} = \bigcup_{n=j}^{\infty} (F_{k+n-1} \setminus F_{k+n}).$$

This proves that $\mathcal{A} = \{\mathcal{A} \subset \mathcal{X} : \mathcal{A} \cap \mathcal{E}_\setminus = \emptyset$ for all but finitely many $n\}$. □

The bornology on $\mathbb{Z}^+$ mentioned in the introduction is an example of a bornology on a space with compact derived set satisfying all of Hu's axioms save (e) without being a metric bornology. In particular, this bornology cannot be expressed in the form $\mathcal{B}(\langle \mathcal{T}_\setminus \rangle)$ where $\emptyset = \text{K-lim}\, T_n$.

Unless X is compact, a metrizable space $\langle X, \tau \rangle$ with compact derived set will always have an admissible metric that is not UC because UC-metrics are always complete [3, p. 55] and a space all of whose admissible metrics are complete must be compact [6, 12] (for a direct proof, see the solution to advanced problem #5850 proposed by R. Tamaki, Amer. Math. Monthly 80 (1973), p. 815). Nevertheless, if $\langle X, \tau \rangle$ is noncompact and metrizable with compact derived set, then each nontrivial metric bornology $\mathcal{A}$ may be represented as $\mathcal{B}(\lceil)$ for some admissible UC-metric d. To see this, let ρ be an admissible UC-metric for X and let d^* be an unbounded admissible metric with $\mathcal{A} = \mathcal{B}(\lceil^*)$. Fix $x_0 \in X$ and define $f : X \to [0, \infty)$ by $f(x) = d^*(x, x_0)$. As f is continuous, f is actually uniformly continuous with respect to the metric ρ. Finally, define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \min\{\rho(x, y), 1\} + |f(x) - f(y)|.$$

Then d and ρ are uniformly equivalent so that d is a UC-metric. Finally, a subset A of X is d-bounded if and only if $f|A$ is bounded, and this occurs if and only if $A \in \mathcal{B}(\lceil^*) = \mathcal{A}$.

5. Upper semicontinuous functions and convergence to the empty set

Let $f : X \to [0, \infty)$ be an unbounded function. For each $n \in \mathbb{Z}^+$, let $L_n^+(f) = \{x \in X : f(x) \geq n\}$. Local finiteness of the family $\{L_n^+(f) : n \in \mathbb{Z}^+\}$ is equivalent to local boundedness of f. In the case that f is in addition continuous, then $\langle L_n^+(f) \rangle$ is a metric mode of convergence to infinity and thus determines a nontrivial metric bornology, namely $\{A \subset X : f|A \text{ is bounded}\}$. Conversely, if $\mathcal{B}(\lceil)$ is a nontrivial metric bornology, fixing $x_0 \in X$, we have $\mathcal{B}(\lceil) = \mathcal{B}(\langle \mathcal{L}_\setminus^+(\{) \rangle)$ where $f : X \to [0, \infty)$ is the continuous function defined by $f(x) = d(x, x_0)$.

Recall that a real valued function f on X is called *upper semicontinuous* (u.s.c.) if for each scalar α the set $f^{-1}((-\infty, \alpha))$ is an open subset of X. Our next result speaks to the representation of $\mathcal{B}(\langle \mathcal{T}_\setminus \rangle)$ for a sequence $\langle T_n \rangle$ of nonempty closed sets with $\emptyset = \text{K-lim}\, T_n$ by a nonnegative unbounded upper semicontinuous function.

Theorem 5.1. *Let $\langle X, \tau \rangle$ be a noncompact metrizable space. If $f : X \to [0, \infty)$ is unbounded and upper semicontinuous, then $\langle L_n^+(f) \rangle$ is a sequence of nonempty closed sets Kuratowski- Painlevé convergent to the empty set. Conversely, if $\langle T_n \rangle$ is a sequence of nonempty closed sets with $\emptyset = \text{K-lim}\, T_n$, then there exists an unbounded nonnegative upper semicontinuous function f on X with $\mathcal{B}(\langle \mathcal{T}_\backslash \rangle) = \mathcal{B}(\langle \mathcal{L}^+_\backslash(\{\}) \rangle)$.*

Proof. Let $f : X \to [0, \infty)$ be unbounded and upper semicontinuous. Since f is unbounded, each superlevel set $L_n^+(f)$ is nonempty, and by upper semicontinuity, each is closed. Clearly, $\langle L_n^+(f) \rangle$ converges to the empty set.

Conversely, let $\langle T_n \rangle$ be a sequence of nonempty closed sets with $\emptyset = \text{K-lim}\, T_n$. As usual, for each n, let $F_n = \cup_{j=n}^{\infty} T_j$ and let g_n be the characteristic function of F_n. Since each F_n is closed, each g_n is upper semicontinuous. Given $x \in X$ all but finitely many g_n vanish off some neighborhood V of x whence

$$f = g_1 + g_2 + g_3 + \dots$$

is unbounded, finite valued, and upper semicontinuous. Since $f(x)$ is the largest n for which $x \in F_n$, we get $F_n = L_n^+(f)$, and so $\mathcal{B}(\langle \mathcal{L}^+_\backslash(\{\}) \rangle) = \mathcal{B}(\langle \mathcal{F}_\backslash \rangle) = \mathcal{B}(\langle \mathcal{T}_\backslash \rangle)$. □

Let $\langle T_n \rangle$ be a sequence of nonempty closed sets in a noncompact metrizable space $\langle X, \tau \rangle$ with $\emptyset = \text{K-lim}\, T_n$. If we equip X with the discrete topology τ_0, then by Theorem 4.3, $\mathcal{B}_{\tau\prime}(\langle \mathcal{T}_\backslash \rangle)$ is a nontrivial metric bornology for the discrete topology.

Proposition 5.1 leads us to a coarser metrizable topology $\tau^* \supset \tau$ which accomplishes the same goal. Starting with a bounded metric d compatible with τ and the upper semicontinuous function f constructed in the proof of Proposition 5.1, the unbounded metric d^* on X defined by $d^*(x, w) = d(x, w) + |f(x) - f(w)|$ defines a topology τ^* on X finer than τ. The precise relationship between τ^* and τ can be conveniently described in terms of convergent sequences in X as follows: $\langle x_k \rangle \to x$ in τ^* if and only if $\langle x_k \rangle \to x$ in τ and whenever $x \in F_n = \cup_{j=n}^{\infty} T_j$, we have $x_k \in F_n$ for all sufficiently large k. It follows that for an arbitrary subset A of X we have

$$\{n \in \mathbb{Z}^+ : (\text{cl}_{\tau^*} A) \cap F_n \neq \emptyset\} = \{n \in \mathbb{Z}^+ : A \cap F_n \neq \emptyset\},$$

whence by condition (3) of Theorem 4.1, $\mathcal{B}_{\tau^*}(\langle \mathcal{T}_\backslash \rangle)$ is a nontrivial metric bornology.

We conclude with an open question. Given a sequence $\langle T_n \rangle$ of nonempty closed sets in a metrizable space $\langle X, \tau \rangle$ with $\emptyset = \text{K-lim}\, T_n$, must there always exist a coarsest topology τ_1 on X finer than τ such that $\mathcal{B}_{\tau_\infty}(\langle \mathcal{T}_\backslash \rangle)$ is a nontrivial metric bornology?

References

[1] M. Atsuji: Uniform continuity of continuous functions of metric spaces, Pacific J. Math 8 (1958) 11-16.

[2] G. Beer: UC spaces revisited, Amer. Math. Monthly 95 (1988) 737-39.

[3] G. Beer: Topologies on Closed and Closed Convex Sets, Kluwer Academic Publishers, Dordrecht, Holland, 1993.

[4] G. Beer: On metric boundedness structures, Set-Valued Anal. 7 (1999) 195-208.

[5] G. Beer: On convergence to infinity, Monatshefte Math. 129 (2000) 267–280.

[6] R. Engelking: General Topology, Revised Edition, Heldermann Verlag, Berlin-Lemgo, 1989.

[7] H. Hogbe-Nlend: Bornologies and Functional Analysis, Mathematical Studies 26, North-Holland, Amsterdam, 1977.

[8] S.-T. Hu: Boundedness in a topological space, J. Math Pures Appl. 28 (1949) 287-320.

[9] S.-T. Hu: Introduction to General Topology, Holden-Day, San Francisco, 1966.

[10] H. Hueber: On uniform continuity and compactness in metric spaces, Amer. Math. Monthly 88 (1981) 204-205.

[11] S. Mrowka: On normal metrics, Amer. Math. Monthly 72 (1965) 998-1001.

[12] V. Niemytzki and A. Tychonoff: Beweis des Satzes, dass ein metrisierbarer Raum dann und nur dann kompakt ist, wenn er in jeder Metrik vollständig ist, Fund. Math. 12 (1928) 118-120.

[13] J. Rainwater, Spaces whose finest uniformity is metric, Pacific J. Math. 9 (1959) 567-570.