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Semiconvex Hulls of Quasiconformal Sets

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We make some remarks concerning the p-semiconvex hulls of the quasiconformal sets, using a recent significant observation of T. Iwaniec in the paper [7] on the important relation between the regularity of quasiregular mappings in the theory of geometric functions and the notion of Morrey's quasiconvexity in the calculus of variations. We also point out several partial results on a conjecture in that paper.

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1. Introduction

The notion of quasiconvexity was first introduced by C. B. Morrey [11] in the study of variational integrals of the form

$$I(u) = \int_\Omega f(Du(x))\, dx, \tag{1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain, $u\colon \Omega \to \mathbf{R}^m$ is a map with gradient matrix $Du(x) = (\partial u^i/\partial x_j)$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, and $f\colon \mathbf{M}^{m\times n} \to \mathbf{R}$ is a given function defined on the space $\mathbf{M}^{m\times n}$ of all real $m \times n$ matrices. In the sense of Morrey, function f is said to be *quasiconvex* on $\mathbf{M}^{m\times n}$ if

$$\int_D f(\xi + D\varphi(x)) - f(\xi)\, dx \geq 0$$

for all $\xi \in \mathbf{M}^{m\times n}$, bounded domains $D \subset \mathbf{R}^n$, and smooth maps $\varphi\colon \Omega \to \mathbf{R}^m$ with compact support in Ω. This condition is in general difficult to verify and hence there have been many attempts in replacing it by other easier conditions; see, e.g., J. M. Ball [3] and B. Dacorogna [5]. Recall that we say f is *rank-one convex* if for any given matrices ξ and η with $\operatorname{rank}\eta = 1$ the function $g(t) = f(\xi + t\eta)$ is a convex function of $t \in \mathbf{R}$, and f is *polyconvex* if $f(\xi)$ can be represented as a convex function of sub-determinants of ξ. It is well-known that ([3, 5, 11]) for continuous functions on $\mathbf{M}^{m\times n}$ one has

$$\text{polyconvexity} \Longrightarrow \text{quasiconvexity} \Longrightarrow \text{rank-one convexity} \tag{2}$$

and the converse of each of these implications is known to be false for $m \geq 2$, $n \geq 2$ with the exception of that of the second implication when $m = 2$ and $n \geq 2$; in fact, whether any rank-one convex function on $\mathbf{M}^{2\times 2}$ must be quasiconvex remains one of the most challenging open problems in the vectorial calculus of variations. We refer to Alibert

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and Dacorogna [1] and V. Šverák [15] for some counterexamples and refer to S. Müller's lecture notes [12] for some recent related developments.

To proceed, we let $C_0^\infty(\Omega; \mathbf{R}^m)$ be the family of all smooth maps with compact support in Ω and let $W^{1,p}(\Omega; \mathbf{R}^m)$, $W_0^{1,p}(\Omega; \mathbf{R}^m)$ be the usual Sobolev spaces of mappings from Ω to $\mathbf{R}^m$. We also define two matrix norms on $\mathbf{M}^{m\times n}$ by

$$|\xi| = \max_{|h|=1} |\xi h|, \quad \|\xi\| = (\xi \colon \xi)^{1/2} = (\operatorname{tr}(\xi^T \xi))^{1/2}. \tag{3}$$

In the following, we assume $m = n \geq 2$. It is easy to see that $|\det \xi| \leq n^{-n/2} \|\xi\|^n \leq |\xi|^n$ for $\xi \in \mathbf{M}^{n\times n}$. For each $K \geq 1$, using the norm $|\xi|$, we define the *K-quasiconformal set* $\mathcal{S}_K$ to be the closed cone

$$\mathcal{S}_K = \{\xi \in \mathbf{M}^{n\times n} \mid |\xi|^n \leq K \det \xi\}. \tag{4}$$

From Hadamard's inequality [8], we easily have $\mathcal{S}_1 = \{\lambda Q \mid \lambda \geq 0,\ Q^T Q = I,\ \det Q = 1\}$, where $I = \operatorname{diag}(1, \cdots, 1) \in \mathbf{M}^{n\times n}$ is identity matrix; the set $\mathcal{S}_1$ is thus called the n-dimensional *conformal set*. A map $u \in W^{1,p}(\Omega; \mathbf{R}^n)$, $p \geq 1$, is said to be (*weakly* if $p < n$) *K-quasiregular* in Ω if

$$Du(x) \in \mathcal{S}_K \quad a.e.\ x \in \Omega, \tag{5}$$

and *K-quasiconformal* maps are those K-quasiregular maps in $W^{1,n}(\Omega; \mathbf{R}^n)$ that are homeomorphisms; see [6, 7, 9, 14].

One could also define K-quasiregular mappings by using the cone

$$\mathcal{C}_K = \{\xi \in \mathbf{M}^{n\times n} \mid \|\xi\|^n \leq n^{n/2} K \det \xi\}. \tag{6}$$

Note that $\mathcal{S}_K \neq \mathcal{C}_K$ unless $K = 1$. However, as we shall see later, it is of great advantage to use the set $\mathcal{S}_K$ instead of $\mathcal{C}_K$.

In connection with the set $\mathcal{S}_K$ defined above, we consider a function introduced by T. Iwaniec in [7], using the norm $|\cdot|$; namely,

$$h_p(\xi) = |1 - \frac{n}{p}|\, |\xi|^p - |\xi|^{p-n} \det \xi. \tag{7}$$

Theorem 1.1 (Iwaniec [7]). *h_p is rank-one convex for all $p \geq n/2$.*

Based on this theorem and other properties of function h_p, it has been conjectured in [7] that $h_p(\xi)$ is quasiconvex. It is this conjecture that signifies the relations between the notion of semiconvexity in the calculus of variations and the regularity of quasiregular mappings in the theory of geometric functions, and the confirmation of this conjecture, especially in two dimensional case ($n = 2$), can make important impact on some open problems in both areas. For a recent discussion on this conjecture, see also an article by K. Astala [2].

The purpose of the present paper is using the important observations of Iwaniec [7] to make some remarks concerning several *p-semiconvex hulls* of the quasiconformal sets $\mathcal{S}_K$. The p-semiconvex hulls of a set $\mathcal{S}$ are the certain generalization of the closed convex hull of $\mathcal{S}$ depending on a power $p \geq 1$, and we shall briefly review in Section 2 the definition of these p-semiconvex hulls and refer to [16, 19, 20, 21] for their applications in some variational problems. The main remarks of this paper include a complete description of the p-rank-one convex hulls of $\mathcal{S}_K$ for all $p \geq 1$ and some partial results concerning the more important p-quasiconvex hulls of $\mathcal{S}_K$. In addition, we also make some remarks on Iwaniec's conjecture mentioned above.

2. Semiconvex hulls of quasiconformal sets

Given a closed subset $\mathcal{S}$ of $\mathbf{M}^{m\times n}$, for any power $p \geq 1$, let $C_p^+(\mathcal{S})$ be the class of continuous functions f on $\mathbf{M}^{m\times n}$ satisfying

$$0 \leq f(\xi) < c\,(|\xi|^p + 1), \quad f|_{\mathcal{S}} = 0, \tag{8}$$

where $c > 0$ is a constant depending on f. Let $f^{-1}(0)$ be the zero set of f. The semiconvex hulls of set $\mathcal{S}$ with power p are defined as follows; see, e.g., [16, 19, 20, 21] for their applications in some variational problems.

Definition 2.1. *p-quasiconvex hull*:

$$Q_p(\mathcal{S}) = \cap\{f^{-1}(0) \mid f \in C_p^+(\mathcal{S}), \text{ quasiconvex}\}. \tag{9}$$

p-rank-one convex hull:

$$R_p(\mathcal{S}) = \cap\{f^{-1}(0) \mid f \in C_p^+(\mathcal{S}), \text{ rank-one convex}\}. \tag{10}$$

p-polyconvex hull:

$$P_p(\mathcal{S}) = \cap\{f^{-1}(0) \mid f \in C_p^+(\mathcal{S}), \text{ polyconvex}\}. \tag{11}$$

With this definition, it is easily seen that the p-semiconvex hulls are closed sets and decreasing with respect to the power p; furthermore,

$$\mathcal{S} \subseteq R_p(\mathcal{S}) \subseteq Q_p(\mathcal{S}) \subseteq P_p(\mathcal{S}). \tag{12}$$

This paper concerns the p-semiconvex hulls of the K-quasiconformal set $\mathcal{S}_K$ defined in the introduction. The following theorem summarizes some known results proved in [13, 17, 18, 20].

Theorem 2.2.
(a) *If $1 \leq p < \frac{nK}{K+1}$ then $R_p(\mathcal{S}_K) = \mathbf{M}^{n\times n}$.*
(b) *There exists an $\epsilon > 0$ such that $Q_p(\mathcal{S}_K) = \mathcal{S}_K$ for all $p > n - \epsilon$. Moreover, if n is even and $K = 1$ then $Q_p(\mathcal{S}_1) = \mathcal{S}_1$ for $p \geq n/2$.*
(c) *For $p \geq n$, $P_p(\mathcal{S}_K) = \mathcal{S}_K$. Moreover, if n is odd then $P_p(\mathcal{S}_K) = \mathbf{M}^{n\times n}$ for all $1 \leq p < n$; if n is even then $P_p(\mathcal{S}_K)$ contains the set of all matrices with* rank $\leq \frac{n}{2} - 1$ *for all $1 \leq p < n$.*

The following result and Theorem 2.2(a) give a complete description of p-rank-one convex hulls of $\mathcal{S}_K$ for all $p \geq 1$.

Proposition 2.3. *For $K \geq 1$ and $p \geq \frac{nK}{K+1}$, $R_p(\mathcal{S}_K) = \mathcal{S}_K$.*

Proof. Define
$$f_K(\xi) = \max\{0,\ K^{-1}|\xi|^p - |\xi|^{p-n}\det\xi\}. \tag{13}$$
It is easy to see that $f_K \in C_p^+(\mathcal{S}_K)$ and $f_K^{-1}(0) = \mathcal{S}_K$. Note that, by Iwaniec's theorem above (Theorem 1.1), the function

$$K^{-1}|\xi|^p - |\xi|^{p-n}\det\xi = h_p(\xi) + (K^{-1} - |1 - \frac{n}{p}|)|\xi|^p$$

is rank-one convex if $K^{-1} \geq |1 - \frac{n}{p}|$ or, equivalently, $\frac{nK}{K+1} \leq p \leq \frac{nK}{K-1}$; for such values of p, f_K is also rank-one convex. By definition, we have $R_p(\mathcal{S}_K) = \mathcal{S}_K$ for all $p \geq \frac{nK}{K+1}$. □

From Theorem 2.2(c), we have the following result.

Proposition 2.4. *Let $n \geq 3$. Then the function h_p is not polyconvex for $n/2 \leq p < n$. The conclusion also holds when $n = 2$ and $1 < p < 2$.*

Proof. Consider $f(\xi) = |\xi|^p - |\xi|^{p-n}\det\xi$. We have $f \in C_p^+(\mathcal{S}_1)$ and $f^{-1}(0) = \mathcal{S}_1$. For $n \geq 3$ and $n/2 \leq p < n$, it also follows that

$$f(\xi) = h_p(\xi) + \frac{2p-n}{p}|\xi|^p. \tag{14}$$

Since $2p - n \geq 0$, f would be polyconvex if h_p were polyconvex. However, for $n \geq 3$, by Theorem 2.2(c), f cannot be polyconvex; this shows that h_p is not polyconvex for all $n/2 \leq p < n$. When $n = 2$, since it is easy to see that every polyconvex function with *subquadratic* growth at infinity on $\mathbf{M}^{2\times 2}$ must be convex, and since it is easy to see $h_p(\delta + tI)$ is not convex function in $t > 0$, where $\delta = \operatorname{diag}(1, 0)$, we therefore also have proved h_p is not polyconvex for $n = 2$ and $1 < p < 2$. □

Remark. When $n = 2$ and $p = 1$ it turns out

$$h_1(\xi) = |\xi| - |\xi|^{-1}\det\xi = |\xi - (\operatorname{adj}\xi)^T|,$$

where $\operatorname{adj}\xi$ is the adjugate matrix given by $\xi\,(\operatorname{adj}\xi) = (\operatorname{adj}\xi)\,\xi = (\det\xi)\,I$. Since, when $n = 2$, $\operatorname{adj}\xi$ is linear in ξ and hence $h_1(\xi)$ is convex in this case.

Since the operator norm $|\xi|$ is not differentiable and thus h_p is not a smooth function, one would expect to use the smooth norm $\|\xi\|$ and hence the set $\mathcal{C}_K$ instead of the set $\mathcal{S}_K$; however, this replacement would not work as we can easily prove the following result.

Proposition 2.5. *The function*

$$g_p(\xi) = |1 - \frac{n}{p}|\,\|\xi\|^p - n^{n/2}\|\xi\|^{p-n}\det\xi \tag{15}$$

is not rank-one convex for all $n/2 \leq p < n$.

Proof. Let $\xi = n^{-1/2}\operatorname{diag}(1, \cdots, 1, -1)$, $\eta = \operatorname{diag}(0, \cdots, 0, 1)$ and consider function $g(t) = g_p(\xi + t\eta)$. A direct computation shows that $g''(0) = p - n < 0$, and hence $g(t)$ is not convex in t and g_p is not rank-one convex. □

3. Iwaniec's conjectures

Note that the function h_p defined by (7) is rank-one convex for all $p \geq n/2$. The following conjectures have been made by T. Iwaniec in [7].

Conjecture 3.1. *For each $p \geq n/2$, the function h_p defined by* (7) *is quasiconvex; that is, for all $\xi \in \mathbf{M}^{n\times n}$ and $\varphi \in C_0^\infty(\Omega;\mathbf{R}^n)$, we have*

$$\int_\Omega h_p(\xi + D\varphi(x)) - h_p(\xi)\, dx \geq 0. \tag{16}$$

In particular, for $\xi = 0$, this suggests a much weaker conjecture.

Conjecture 3.2. *For each $p > n/2$ and all $\varphi \in W_0^{1,p}(\Omega;\mathbf{R}^n)$, we have*

$$\int_\Omega |D\varphi(x)|^{p-n} \det D\varphi(x)\, dx \leq |1 - \frac{n}{p}| \int_\Omega |D\varphi(x)|^p\, dx. \tag{17}$$

There exist other explicit examples of rank-one convex functions which we don't know whether are quasiconvex; see for instance [1]. However, Iwaniec's functions h_p defined above, especially when $n = 2$, relate directly to an important conjecture concerning the norm of the so-called *Beurling-Ahlfors* transform.

Remark. Let $n = 2$. Consider point $(x, y) \in \mathbf{R}^2$ as a complex number $z = x+iy \in \mathbf{C}$ and map $\varphi(x, y)$ as a complex function $\varphi(z)$ on $z \in \mathbf{C}$. Let $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$ be the complex Cauchy-Riemann operators; then $h_p(D\varphi)$ can be expressed in terms of $\varphi_z = \partial_z\varphi$, $\varphi_{\bar z} = \partial_{\bar z}\varphi$ as

$$h_p(D\varphi) = (1 - |1 - \frac{2}{p}|)\left[(p^* - 1)|\varphi_{\bar z}| - |\varphi_z|\right](|\varphi_{\bar z}| + |\varphi_z|)^{p-1},$$

where $p^* - 1 = \max\{p - 1, (p-1)^{-1}\}$. The Beurling-Ahlfors transform is defined by

$$(Sf)(z) = -\frac{1}{2\pi i}\int_{\mathbf{C}} \frac{f(\zeta)\, d\zeta \wedge d\bar\zeta}{(z-\zeta)^2} \tag{18}$$

and has an important feature that $S(f_{\bar z}) = f_z$. Using the Burkholder-type inequalities [4], it has been shown in Iwaniec [7] that when $n = 2$ Conjecture 3.2 is equivalent to a long-standing conjecture that the L^p-operator norm of the Beurling-Ahlfors transform $\|S\|_p = p^* - 1$ for all $1 < p < \infty$.

Consequently, in the case of dimension $n = 2$, the truth of Conjecture 3.1 would confirm the norm conjecture of the Beurling-Ahlfors transform, while the failure of it would provide a counterexample of a rank-one convex function on $\mathbf{M}^{2\times 2}$ which is not quasiconvex, settling another long-standing open problem in the calculus of variations.

We would complete the computation of the quasiconvex hulls of the quasiconformal set from Iwaniec's conjectures.

Proposition 3.3. *Conjecture* 3.2 *implies $Q_p(\mathcal{S}_K) = \mathcal{S}_K$ for all $p > \frac{nK}{K+1}$; while Conjecture* 3.1 *implies $Q_p(\mathcal{S}_K) = \mathcal{S}_K$ for $p = \frac{nK}{K+1}$.*

Proof. We only prove the first implication; the second one is easy. Let f_K be defined by (13). Since $f_K(\xi) \geq h_p(\xi) + (K^{-1} - |1 - \frac{n}{p}|)|\xi|^p$, we easily see that Conjecture 3.2 would imply that f_K is L^p-mean coercive for $\frac{nK}{K+1} < p < \frac{nK}{K-1}$ in the sense that

$$\int_\Omega f_K(D\varphi(x))\,dx \geq \gamma \int_\Omega |D\varphi(x)|^p\,dx \tag{19}$$

holds for all $\varphi \in W_0^{1,p}(\Omega; \mathbf{R}^n)$, where $\gamma > 0$ is a constant; in this case, $\gamma = K^{-1} - |1 - \frac{n}{p}| > 0$. Therefore, by a theorem in Yan and Zhou [20], $Q_p(\mathcal{S}_K)$ will be constant for $\frac{nK}{K+1} < p < \frac{nK}{K-1}$. In particular, by Theorem 2.2(b) above, we have $Q_p(\mathcal{S}_K) = \mathcal{S}_K$ for all $p > \frac{nK}{K+1}$. $\square$

The following result could be derived from Conjecture 3.2 and a theorem in [20]; the conclusion of this result in the case dimension $n = 2$ has been proved by K. Astala regardless of Conjecture 3.2, using a different method of measurable Riemann mapping theorem (see the references given in [2]).

Proposition 3.4. *Conjecture* 3.2 *implies that any weakly K-quasiregular maps in $W^{1,p}(\Omega; \mathbf{R}^n)$ for some $p > \frac{nK}{K+1}$ must belong to $W^{1,q}_{loc}(\Omega; \mathbf{R}^n)$ for all $\frac{nK}{K+1} < q < \frac{nK}{K-1}$.*

In particular, if $K = 1$, this would imply that any weakly conformal mappings in $W^{1,p}(\Omega; \mathbf{R}^n)$ with some $p > n/2$ must be a restriction of a Möbius transform on Ω as in a classical Liouville theorem [14]. If dimension n is even, this generalized Liouville theorem has proved even true with $p = n/2$; see Iwaniec and Martin [9].

4. A special case and final remarks

In this final section, we make a few remarks concerning Conjecture 3.1. We will prove the estimate (16) in Conjecture 3.1 for a special class of radially symmetric test functions; the result is useful in both aspects: it makes the conjecture more convincing and, on the other hand, it helps in excluding the possible counterexamples.

Let B be the unit open ball in $\mathbf{R}^n$. A map $u\colon B \to \mathbf{R}^n$ is said to be *radial* if there exists a function $U(r)$, $0 < r < 1$, such that

$$u(x) = U(|x|)\,x \quad a.e.\ x \in B. \tag{20}$$

If U is smooth away from $r = 0$, then u is smooth on $B \setminus \{0\}$ and an easy calculation shows that

$$Du(x) = U(r)I + rU'(r)\,\omega \otimes \omega, \quad r = |x|,\ \omega = x/|x|. \tag{21}$$

Let $W(\xi)$ be any finite continuous rank-one convex function. We first assume W is smooth. Then it follows that (cf. [3])

$$W(\xi + a \otimes b) \geq W(\xi) + \sum_{i,j=1}^n \frac{\partial W(\xi)}{\partial \xi_{ij}}\,a_i b_j \tag{22}$$

for all $\xi \in \mathbf{M}^{n\times n}$ and $a, b \in \mathbf{R}^n$. Therefore, for any given ξ,

$$\begin{aligned} W(\xi + Du(x)) &= W(\xi + U(r)I + rU'(r)\omega \otimes \omega) \\ &\geq W(\xi + U(r)I) + rU'(r) \sum_{i,j=1}^n \frac{\partial W}{\partial \xi_{ij}}(\xi + U(r)I)\,\omega_i\omega_j. \end{aligned}$$

Integrating this over $B_\rho \setminus B_\epsilon$ ($0 < \epsilon < \rho < 1$) using spherical coordinates, we obtain (cf. [18])

$$\int_{B_\rho \setminus B_\epsilon} W(\xi + Du) \geq |B| \left(\rho^n W(\xi + U(\rho)I) - \epsilon^n W(\xi + U(\epsilon)I)\right). \tag{23}$$

A smoothing argument shows that this inequality holds for all finite continuous rank-one convex functions W.

From the above inequality, we deduce the following result as a consequence of Theorem 1.1. Let h_p be the function defined by (7) above.

Proposition 4.1. *Let $p \geq n/2$. Then, for all $\xi \in \mathbf{M}^{n\times n}$ and radial maps $\varphi \in W_0^{1,p}(B;\mathbf{R}^n)$, we have*

$$\int_B h_p(\xi + D\varphi(x)) - h_p(\xi)\, dx \geq 0. \tag{24}$$

Proof. We first assume that the radial map $\varphi \in W_0^{1,p}(B;\mathbf{R}^n)$ has support in B_ρ for some $\rho < 1$. Let $\varphi(x) = \Phi(|x|)x$. Since $\varphi \in W^{1,p}(B;\mathbf{R}^n)$ we have

$$\int_0^1 |\Phi(r)|^p r^{n-1} + |\Phi'(r)|^p r^{p+n-1}\, dr < \infty. \tag{25}$$

We claim that there exists a decreasing sequence $\{\epsilon_j\}$ such that $\epsilon_j \to 0$ and $|\Phi(\epsilon_j)|^p \epsilon_j^n \to 0$ as $j \to \infty$. If this were not the case, we would have $|\Phi(r)|^p r^n \geq \nu$ for some constant $\nu > 0$ and all $0 < r < 1$. This would imply $|\Phi(r)|^p r^{n-1} \geq \nu r^{-1}$ and thus

$$\int_\epsilon^{2\epsilon} |\Phi(r)|^p r^{n-1}\, dr \geq \nu \ln 2 > 0 \tag{26}$$

for all $0 < \epsilon < \frac{1}{2}$, which violates (25); the claim is proved. Using (23) with $W = h_p$, $\epsilon = \epsilon_j$ and $\rho \to 1$, we have

$$\int_B h_p(\xi + D\varphi(x))\chi_j(x)\, dx \geq |B| \left[h_p(\xi) - \epsilon_j^n h_p(\xi + \Phi(\epsilon_j)I)\right], \tag{27}$$

where χ_j is the characteristic function of $B \setminus B_{\epsilon_j}$. Note that

$$\epsilon_j^n \left|h_p(\xi + \Phi(\epsilon_j)I)\right| \leq C_p\, \epsilon_j^n \left(1 + |\Phi(\epsilon_j)|^p\right) \to 0 \tag{28}$$

as $j \to \infty$. Hence, by the Lebesgue dominated convergence theorem, we have

$$\int_B h_p(\xi + D\varphi(x))\, dx \geq |B|\, h_p(\xi), \tag{29}$$

as desired. For general radial maps φ, this can be proved by a density argument. □

Remark. The equality in (29) can hold for many nontrivial maps φ at certain ξ, but for all these φ, $\xi + D\varphi(x)$ lies in the set where h_p is smooth and moreover the first variation of the integral vanishes at $\xi + D\varphi$. This supports the conjectures made above; see also [7].

As a special case of the function h_p defined above, we consider the case $p = n-1$. Let $H(\xi) = |\xi|^{n-2}\,\xi - (\operatorname{adj}\xi)^T$. Then, using Lemma 2.1 of [7], we see that

$$|H(\xi)| = |\xi|^{n-1} - |\xi|^{-1}\det\xi = h_{n-1}(\xi) + \frac{n-2}{n-1}\,|\xi|^{n-1}, \tag{30}$$

and hence the estimate (17) in Conjecture 3.2 is equivalent to

$$\|D\varphi\|^{n-1}_{L^{n-1}(B)} \le \frac{n-1}{n-2}\,\|H(D\varphi)\|_{L^1(B)} \tag{31}$$

for all $\varphi \in C_0^\infty(B;\mathbf{R}^n)$. Since $\operatorname{div}(\operatorname{adj} D\varphi)^T = 0$, we have

$$\operatorname{div}|D\varphi|^{n-2}D\varphi = \operatorname{div} h, \tag{32}$$

where $h = H(D\varphi(x))$; this system is not the usual n-Laplacian system since the norm $|D\varphi|$ is used. Note that (31) would be derived easily from the following conjecture.

Conjecture 4.2. *Let $n \ge 3$. For any weak solution $\varphi \in W_0^{1,n-1}(B;\mathbf{R}^n)$ of system (32) with any $h \in L^1(B;\mathbf{M}^{n\times n})$, one has the estimate*

$$\int_B |D\varphi(x)|^{n-1}\,dx \le \frac{n-1}{n-2}\int_B |h(x)|\,dx. \tag{33}$$

Finally, the following result has been proved in [7] in partial support of Conjecture 3.2.

Theorem 4.3. *For each $n \ge 2$ there exists a minimal power $p_0(n) \in [\frac{n}{2}, n)$ such that for all $p > p_0(n)$ one has a positive number $\lambda_p(n) < 1$ for which*

$$\int_\Omega |D\varphi(x)|^{p-n}\det D\varphi(x)\,dx \le \lambda_p(n)\int_\Omega |D\varphi(x)|^p\,dx$$

holds for all $\varphi \in C_0^\infty(\Omega;\mathbf{R}^n)$.

Notice that Conjecture 3.2 is equivalent to the assertion that $p_0(n) = n/2$ and $\lambda_p(n) = |1-\frac{n}{p}|$ in Theorem 4.3. The minimal power $p_0(n)$ is also related to the L^p-mean coercivity of the conformal set $\mathcal{S}_1$ studied in [20] (see also [8]). It is easy to see that $p_0(n)$ is the infimum of powers p such that for all $q \ge p$ and $\varphi \in C_0^\infty(B;\mathbf{R}^n)$

$$\int_B g(D\varphi(x))\,dx \ge \gamma_g\int_B |D\varphi(x)|^q\,dx, \tag{34}$$

where g is any q-homogeneous nonnegative function vanishing exactly on $\mathcal{S}_1$, $\gamma_g > 0$ is a constant, and $|D\varphi|$ can be in any norm of $\mathbf{M}^{n\times n}$.

Remark. It has been proved that $p_0(n) = n/2$ for even dimensions n; see [7, 9, 13]. For odd dimensions n, we can relate this power $p_0(n)$ to an L^p-type estimate for q-harmonic systems. Consider the function

$$F_{l,\delta}(\xi) = \delta\,|\wedge^l(\xi)|^{\frac{q}{l}} - \binom{n}{l}^{\frac{n}{2l}}|\wedge^l(\xi)|^{\frac{q-n}{l}}\det\xi, \tag{35}$$

where $1 \leq l \leq n$ is an integer and $\wedge^l(\xi)$ is the l-th exterior power of ξ as a linear operator on $\wedge^l(\mathbf{R}^n)$ defined by

$$\wedge^l(\xi)(e_I) = \wedge^l(\xi)(e_{i_1} \wedge \cdots \wedge e_{i_l}) = \xi_{i_1} \wedge \cdots \wedge \xi_{i_l}$$

and $| \wedge^l (\xi)|^2 = \sum_I | \wedge^l (\xi)(e_I)|^2$, where $I = \{1 \leq i_1 < i_2 < \cdots < i_l \leq n\}$ is taken with all increasing indices and ξ_i is the i-th row of matrix ξ.

From a conjecture made in [8] after Theorem 7.1 (which would follow from the estimate of [8, Theorem 8.1] with all $r > \max\{\frac{1}{q}, \frac{1}{q'}\}$), we would obtain with $l = [\frac{n}{2}]$ being the largest integer less than or equal to $\frac{n}{2}$ that

$$\int_B F_{l,\delta}(D\varphi(x))\, dx \geq 0, \quad \forall\, \varphi \in C_0^\infty(B; \mathbf{R}^n) \tag{36}$$

for all $q > n - [\frac{n}{2}]$ and some $|1 - \frac{n}{q}| \leq \delta < 1$; hence a similar argument of [7, Section 11] would show that $p_0(n) \leq n - [\frac{n}{2}]$. However, it seems unclear whether the conjecture in [8] mentioned above would imply $p_0(n) = n/2$.

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