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Kernelled Quasidifferential for a Quasidifferentiable Function in Two-Dimensional Space

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For a quasidifferentiable function f defined on $\mathbf{R}^2$, it is proved, in the sense of Demyanov and Rubinov, that the following assertion

$$\left[\bigcap_{[\underline{\partial} f(x),\overline{\partial} f(x)]\in\mathcal{D}f(x)}(\underline{\partial} f(x)+\overline{\partial} f(x)),\quad \bigcap_{[\underline{\partial} f(x),\overline{\partial} f(x)]\in\mathcal{D}f(x)}(\overline{\partial} f(x)-\overline{\partial} f(x))\right]\in\mathcal{D}f(x)$$

in this paper, where $\mathcal{D}f(x)$ denotes the set of all quasidifferentials of f at x. It is shown that this way can be viewed as an approach to determining or choosing a representative of the equivalent class of quasidifferentials of f at x, in the two-dimensional case.

Keywords: Quasidifferential calculus, kernelled quasidifferential, minimal quasidifferential, nonsmooth analysis

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1. Introduction

A function f defined on $\mathbf{R}^n$ is called quasidifferentiable at x, in the sense of Demyanov and Rubinov, if it is directionally differentiable at x and there exists a pair of convex compact sets $\underline{\partial} f(x), \overline{\partial} f(x) \subset R^n$ such that its directional derivative can be expressed as follows

$$\begin{aligned} f'(x;d) &= \lim_{t\to 0^+}(f(x+td)-f(x))/t \\ &= \max_{v\in\underline{\partial} f(x)} v^T d + \min_{w\in\overline{\partial} f(x)} w^T d,\ d\in\mathbf{R}^n. \end{aligned} \tag{1}$$

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The pair of sets $[\underline{\partial}f(x), \overline{\partial}f(x)]$ is called a quasidifferential of f at x. The sets $\underline{\partial}f(x)$ and $\overline{\partial}f(x)$ are called a subdifferential and a superdifferential of f at x, respectively.

It is well known that the quasidifferential is not uniquely defined. Actually, suppose that $[U, V]$ is a quasidifferential of f at x, then for any convex compact set $S \subset \mathbf{R}^n$, $[U + S, V - S]$ is still a quasidifferential of f at x. Throughout this paper, $\mathcal{D}f(x)$ denotes the set of all quasidifferentials of f at x. For the purpose of practice, it is necessary to find a way or a rule by which a quasidifferential, as a representative of the equivalent class of quasidifferentials, can be determined automatically. Many authors considered this problem, see for instance [5, 6, 9, 10, 11]. The following pair of sets

$$\left[\bigcap_{[\underline{\partial}f(x), \overline{\partial}f(x)] \in \mathcal{D}f(x)} (\underline{\partial}f(x) + \overline{\partial}f(x)), \bigcap_{[\underline{\partial}f(x), \overline{\partial}f(x)] \in \mathcal{D}f(x)} (\overline{\partial}f(x) - \overline{\partial}f(x)) \right] \tag{2}$$

was investigated and was conjectured to be a quasidifferential of f at x. Some results were obtained for the subclass of quasidifferentiable functions, where (2) is a quasidifferential, see for instance [9, 10, 11]. If (2) is a quasidifferential, then it is called kernelled quasidifferential and could be taken as a representative of the equivalent class of quasidifferentials. In the one-dimensional case, a quasidifferential happens to be a pair of nonempty closed intervals. It was shown that the pair of sets presented in (2) is just a quasidifferential of f at x, see [3, 11]. In the n-dimensional case ($n \geq 2$), whether the pair of sets given in (2) is a quasidifferential of f at x is still an open problem. However, Deng *et al* [2] proved that the set

$$\bigcap_{[\underline{\partial}f(x), \overline{\partial}f(x)] \in \mathcal{D}f(x)} (\underline{\partial}f(x) + \overline{\partial}f(x))$$

is nonempty. Besides, the nonemptiness of the set

$$\bigcap_{[\underline{\partial}f(x), \overline{\partial}f(x)] \in \mathcal{D}f(x)} (\overline{\partial}f(x) - \overline{\partial}f(x))$$

is obvious since it contains zero.

For the same purpose, Pallaschke *et al* [5] introduced an important notion, the minimal quasidifferential.

$$[\underline{\partial}^m f(x), \overline{\partial}^m f(x)] \in \mathcal{D}f(x)$$

is called minimal, provided that any pair $[\underline{\partial}f(x), \overline{\partial}f(x)] \in \mathcal{D}f(x)$ satisfying that the relation $\underline{\partial}f(x) \subset \underline{\partial}^m f(x), \overline{\partial}f(x) \subset \overline{\partial}^m f(x)$ implies $[\underline{\partial}f(x), \overline{\partial}f(x)] = [\underline{\partial}^m f(x), \overline{\partial}^m f(x)]$. Furthermore, Pallaschke *et al* [5] proved the existence of the minimal quasidifferentials, which reads: if $[\underline{\partial}f(x), \overline{\partial}f(x)] \in \mathcal{D}f(x)$, then there exists a minimal quasidifferential $[\underline{\partial}^m f(x), \overline{\partial}^m f(x)] \in \mathcal{D}f(x)$ such that $\underline{\partial}^m f(x) \subset \underline{\partial}f(x)$, $\overline{\partial}^m f(x) \subset \overline{\partial}f(x)$. However, the minimal quasidifferential is not uniquely defined either. Actually, any translation of a minimal quasidifferential is still minimal, say, if $[A, B]$ is a minimal quasidifferential, then for any singleton $\{c\}, [A + \{c\}, B - \{c\}]$, a translation of $[A, B]$, is still a minimal quasidifferential.

Grzybowski [4] and Scholtes [8] proved independently the fact that equivalent minimal quasidifferentials, in the two-dimensional case, are uniquely determined up to a translation, i.e., let f be quasidifferentiable on $\mathbf{R}^2$, given the two minimal quasidifferentials

$[\underline{\partial}_1^m f(x), \overline{\partial}_1^m f(x)]$ and $[\underline{\partial}_2^m f(x), \overline{\partial}_2^m f(x)]$, there exists $c \in \mathbf{R}^2$ (where c is depends on $[\underline{\partial}_1^m f(x), \overline{\partial}_1^m f(x)]$ and $[\underline{\partial}_2^m f(x), \overline{\partial}_2^m f(x)]$) satisfying

$$[\underline{\partial}_2^m f(x), \overline{\partial}_2^m f(x)] = [\underline{\partial}_1^m f(x) + \{c\}, \overline{\partial}_1^m f(x) - \{c\}]. \tag{3}$$

In this paper, it is to explore the kernelled quasidifferential in the two-dimensional case. The remainder of this paper is organized as follows: In the section 2, we prove that the pair of sets in (2) is a quasidifferential and give its expression by using of a minimal quasidifferential. In the section 3, we propose a method of determining whether a given quasidifferential is minimal.

2. Existence and Structure of Kernelled Quasidifferential

In this section, the existence and structure of the kernelled quasidifferential of a quasidifferentiable function are established.

Theorem 2.1. *Suppose that f is a quasidifferentiable function defined on $\mathbf{R}^2$ and $[\underline{\partial}_0^m f(x), \overline{\partial}_0^m f(x)]$ is a minimal quasidifferential of f at x. Then, the relations below hold*

$$\bigcap_{[\underline{\partial} f(x),\, \overline{\partial} f(x)] \in \mathcal{D}f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)) = \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x), \tag{4a}$$

$$\bigcap_{[\underline{\partial} f(x),\, \overline{\partial} f(x)] \in \mathcal{D}f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)) = \overline{\partial}_0^m f(x) - \overline{\partial}_0^m f(x). \tag{4b}$$

Furthermore,

$$\left[\bigcap_{[\underline{\partial} f(x),\, \overline{\partial} f(x)] \in \mathcal{D}f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)), \bigcap_{[\underline{\partial} f(x),\, \overline{\partial} f(x)] \in \mathcal{D}f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)) \right] \in \mathcal{D}f(x). \tag{5}$$

Proof. Let $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D}f(x)$. From the existence of the minimal quasidifferentials, as mentioned in the last section, or see [Theorem, 5], it follows that there exists a minimal quasidifferential of f at x, denoted by $[\underline{\partial}^m f(x), \overline{\partial}^m f(x)]$, such that $\underline{\partial}^m f(x) \subset \underline{\partial} f(x)$, $\overline{\partial}^m f(x) \subset \overline{\partial} f(x)$. Consequently,

$$\underline{\partial}^m f(x) + \overline{\partial}^m f(x) \subset \underline{\partial} f(x) + \overline{\partial} f(x), \tag{6a}$$

$$\overline{\partial}^m f(x) - \overline{\partial}^m f(x) \subset \overline{\partial} f(x) - \overline{\partial} f(x). \tag{6b}$$

Note that both $[\underline{\partial}^m f(x), \overline{\partial}^m f(x)]$ and $[\underline{\partial}_0^m f(x), \overline{\partial}_0^m f(x)]$ are the minimal quasidifferentials of f at x. According to the translation property of the equivalent minimal quasidifferentials in the two-dimensional case, there exists $c \in \mathbf{R}^2$ such that the minimal quasidifferential $[\underline{\partial}^m f(x), \overline{\partial}^m f(x)]$ can be expressed as

$$[\underline{\partial}^m f(x), \overline{\partial}^m f(x)] = [\underline{\partial}_0^m f(x) + \{c\}, \overline{\partial}_0^m f(x) - \{c\}]. \tag{7}$$

This leads to

$$\underline{\partial}^m f(x) + \overline{\partial}^m f(x) = \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x), \tag{8a}$$

$$\overline{\partial}^m f(x) - \overline{\partial}^m f(x) = \overline{\partial}_0^m f(x) - \overline{\partial}_0^m f(x). \tag{8b}$$

It follows from (6a), (6b), (8a) and (8b) that

$$\underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x) \subset \underline{\partial} f(x) + \overline{\partial} f(x), \tag{9a}$$

$$\overline{\partial}_0^m f(x) - \overline{\partial}_0^m f(x) \subset \overline{\partial} f(x) - \overline{\partial} f(x). \tag{9b}$$

Taking the intersection on the right hands of (9a) and of (9b) for all quasidifferentials of f at x, we have that

$$\underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x) \subset \bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)), \tag{10a}$$

$$\overline{\partial}_0^m f(x) - \overline{\partial}_0^m f(x) \subset \bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)). \tag{10b}$$

On the other hand, $[\underline{\partial}_0^m f(x), \overline{\partial}_0^m f(x)] \in \mathcal{D} f(x)$ implies that

$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)) \subset \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x), \tag{11a}$$

$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)) \subset \overline{\partial}_0^m f(x) - \overline{\partial}_0^m f(x). \tag{11b}$$

The relations (10a), (10b), (11a) and (11b) lead to that

$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)) = \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x), \tag{12a}$$

$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)) = \overline{\partial}_0^m f(x) - \overline{\partial}_0^m f(x). \tag{12b}$$

Note that $[\underline{\partial}_0^m f(x), \overline{\partial}_0^m f(x)] \in \mathcal{D} f(x)$ and $\overline{\partial}_0^m f(x)$ is a convex compact set of $\mathbf{R}^2$. Hence,

$$[\underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x), \overline{\partial}_0^m f(x) - \overline{\partial}_0^m f(x)] \in \mathcal{D} f(x). \tag{13}$$

(12a), (12b) and (13) show that (5) holds. This completes the proof of the theorem. □

Theorem 2.1 is established for quasidifferentiable functions defined on $\mathbf{R}^2$. This is due to the fact that minimal quasidifferentials in the two dimensional case be translated to each other. In the case $\mathbf{R}^n (n \geq 3)$, the conclusion of Theorem 2.1 is not true in general, but we have the results:

$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)) = \bigcap_{[\underline{\partial}^m f(x), \overline{\partial}^m f(x)] \in \mathcal{D}^m f(x)} (\underline{\partial}^m f(x) + \overline{\partial}^m f(x)), \tag{14a}$$

$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)) = \bigcap_{[\underline{\partial}^m f(x), \overline{\partial}^m f(x)] \in \mathcal{D}^m f(x)} (\overline{\partial}^m f(x) - \overline{\partial}^m f(x)), \tag{14b}$$

where $\mathcal{D}^m f(x)$ denotes the set of all minimal quasidifferentials of f at x. It goes without saying that the conclusion of Theorem 2.1 is also valid for a class of quasidifferentiable functions defined on $\mathbf{R}^n (n \geq 3)$ satisfying the translation property among the equivalent minimal quasidifferentials.

3. A Method of Determining the Minimal Quasidifferential

Pallaschke *et al* [4] proposed a criteria for finding the minimal quasidifferential by using a exposed point. In general, this criteria is not easy to be implemented. In this section, we consider a particular case where a quasidifferential is a convex hull of a finite number of points. We first review some of relevant concepts.

Let U be a convex compact set in $\mathbf{R}^n$. Given a point $y \in \mathbf{R}^n$. The set

$$G_y(U) = \{u \in U \mid < u, y >= \max_{u \in U} < u, y >\}. \tag{15}$$

is called the max-face of U generated by y. A point $u \in U$ is called exposed, provided that there exists $y \in \mathbf{R}^n$ such that the max-face $G_y(U)$ is a singleton coinciding with u.

Suppose S is a convex hull of a finite number of points, denoted by $S = \mathrm{co}\{s_i \in \mathbf{R}^n \mid i \in I\}$, where I is a finite index set. It is not hard to see that only each $s_i (i \in I)$ may be an exposed point of S. Furthermore, for a fixed index $i \in I$, s_i is an exposed point of S if and only if the following system of linear inequalities

$$s_j^T y < s_i^T y, \quad y \in \mathbf{R}^n, \quad \forall j \in I \setminus \{i\} \tag{16}$$

is consistent. We can also say that s_i is an exposed point of S if and only if $s_i \notin \mathrm{co}\{s_j \mid j \in I \setminus \{i\}\}$. Evidently, (16) is a system of linear inequalities with $\mathrm{card}\, I - 1$ inequalities and n variables.

Now we present a criteria for minimal quasidifferential which is originally proposed by Pallaschke *et al* in [6], here we refer to an equivalent version, see [Th. 8, 1].

Theorem 3.1. *Let f be a quasidifferentiable function defined on $\mathbf{R}^n$ and $[U, V] \in \mathcal{D}f(x)$. Assume that for every exposed point $w = u - v$ of $U - V$, where $u \in U$, $v \in V$ are exposed points of U and of V, respectively, at least one of the following two conditions holds:*

(a) *there exists an exposed point v_1 of V such that*
$u + v_1$ is an exposed point of $U + V$
$u - v_1$ is an exposed point of $U - V$

(b) *there exists an exposed point u_1 of U such that*
$u_1 + v$ is an exposed point of $U + V$
$u_1 - v$ is an exposed point of $U - V$

Then $[U, V]$ is minimal.

By virtue of Theorem 3.1, the following theorem can be obtained immediately.

Theorem 3.2. *Let f be a quasidifferentiable function defined on $\mathbf{R}^n$ and $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D}f(x)$ with $\underline{\partial} f(x) = \mathrm{co}\{u_i \mid i \in I\}$, $\overline{\partial} f(x) = \mathrm{co}\{v_j \mid j \in J\}$, where I and J are finite index sets and all of u_i and of v_i are exposed points of $\underline{\partial} f(x)$ and of $\overline{\partial} f(x)$, respectively. Assume that for every $u_i - v_j$ such that the following system of linear inequalities*

$$(u_{i'} - v_{j'})^T y < (u_i - v_j)^T y, \quad y \in \mathbf{R}^n, \quad \forall i' \in I \setminus \{i\}, \quad j' \in J \setminus \{j\} \tag{17}$$

is consistent implies that at least one of the following two conditions holds

(1) *there exists $v_{j(i)}$ such that the two systems*

$$(u_{i'} + v_{j'})^T y < (u_i + v_{j(i)})^T y, \quad y \in \mathbf{R}^n, \quad \forall i' \in I \setminus \{i\}, \quad j' \in J \setminus \{j(i)\} \tag{18}$$

and

$$(u_{i'} - v_{j'})^T y < (u_i - v_{j(i)})^T y, \quad y \in \mathbf{R}^n, \quad \forall i' \in I \setminus \{i\}, \quad j' \in J \setminus \{j(i)\} \tag{19}$$

are consistent.

(2) *there exists $u_{i(j)}$ such that the following two systems*

$$(u_{i'} + v_{j'})^T y < (u_{i(j)} + v_j)^T y, \quad y \in \mathbf{R}^n, \quad \forall i' \in I \setminus \{i(j)\}, \quad j' \in J \setminus \{j\} \tag{20}$$

and

$$(u_{i'} - v_{j'})^T y < (u_{i(j)} - v_j)^T y, \quad y \in \mathbf{R}^n, \quad \forall i' \in I \setminus \{i(j)\}, \quad j' \in J \setminus \{j\} \tag{21}$$

are consistent. Then $[\underline{\partial} f(x), \overline{\partial} f(x)]$ is minimal.

Corollary 3.3. *Let f be a quasidifferentiable function defined on $\mathbf{R}^n$ and $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D}f(x)$ with $\underline{\partial} f(x) = \mathrm{co}\{u_i \mid i \in I\}$ and $\overline{\partial} f(x) = \mathrm{co}\{v_j \mid j \in J\}$, where I and J are finite index sets and all of u_i and of v_i are exposed points of $\underline{\partial} f(x)$ and of $\overline{\partial} f(x)$, respectively. If every $u_i + v_j$ and $u_i - v_j$ are exposed points of $\mathrm{co}\{u_i + v_j \mid i \in I, j \in J\}$ and of $\mathrm{co}\{u_i - v_j \mid i \in I, j \in J\}$, respectively, then the quasidifferential $[\underline{\partial} f(x), \overline{\partial} f(x)]$ is minimal. In other words, for any pair of fixed indices $i \in I$, $j \in J$, the two systems of linear inequalities*

$$(u_{i'} + v_{j'})^T y < (u_i + v_j)^T y, \quad y \in \mathbf{R}^n, \quad \forall\, i' \in I \setminus \{i\}, \quad j' \in J \in J \setminus \{j\}, \tag{22}$$

$$(u_{i'} - v_{j'})^T y < (u_i - v_j)^T y, \quad y \in \mathbf{R}^n, \quad \forall\, i' \in I \setminus \{i\}, \quad j' \in J \in J \setminus \{j\} \tag{23}$$

are consistent, then $[\underline{\partial} f(x), \overline{\partial} f(x)]$ is minimal.

Therefore, in terms of Theorem 3.2 or Corollary 3.3, these can be transformed into solving auxiliary linear programming problems based on the next proposition.

Proposition 3.4. *Let A be an $m \times n$ matrix. Then the linear system $Ay < 0, y \in \mathbf{R}^n$ is consistent if and only if the minimum of the objective function of the following linear programming*

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} z_j \\ s.t. \quad & A^T p + (z_1, \cdots, z_n)^T \geq 0 \\ & \sum_{i=1}^{m} p_i = 1 \\ & p_i \geq 0, i = 1, \cdots, m, z_j \geq 0, j = 1, \cdots, n \end{aligned} \tag{P}$$

is non-zero, where p_i is the $i-th$ component of p.

Proof. It follows from the Gordan theorem that the linear system $Ay < 0, y \in \mathbf{R}^n$ is consistent if and only if the system of linear inequalities

$$A^T p = 0, \quad \sum_{i=1}^{m} p_i = 1, p_i \geq 0, \quad i = 1, \cdots, m$$

is inconsistent. Note that the latter is inconsistent if and only if the minimum of the objective function of (P) is non-zero. □

Example 3.5. Let
$$f(x) = \max_{i \in I} f_i(x) + \min_{j \in J} g_j(x)$$
where $f_i, i \in I$, $g_j, j \in J$ are continuously differentiable functions defined on $\mathbf{R}^2$ and I and J are finite index sets. Denote
$$I(x) = \{i \in I \mid f_i(x) = \max_{i \in I} f_i(x)\}$$
$$J(x) = \{j \in J \mid g_j(x) = \max_{j \in J} g_j(x)\}$$

Evidently, the pair of sets $\underline{\partial} f(x) = \mathrm{co}\{\nabla f_i(x) \mid i \in I(x)\}$ and $\overline{\partial} f(x) = \mathrm{co}\{\nabla g_j(x) \mid j \in J(x)\}$ is a quasidifferential of f at x. If for every $\nabla f_i(x) + \nabla g_j(x)$ and $\nabla f_i(x) - \nabla g_j(x)$ are exposed points, respectively, of the sets $\mathrm{co}\{\nabla f_i(x) + \nabla g_j(x) \mid i \in I(x), j \in J(x)\}$ and of $\mathrm{co}\{\nabla f_i(x) - \nabla g_j(x) \mid i \in I(x), j \in J(x)\}$, then $\mathrm{card}\, I(x) = 1$ or $\mathrm{card}\, J(x) = 1$ or $\mathrm{card}\, I(x) = \mathrm{card}\, J(x) = 2$). We denote
$$\underline{\partial}^m f(x) = \mathrm{co}\{\nabla f_i(x) \mid i \in I(x)\},$$
$$\overline{\partial}^m f(x) = \mathrm{co}\{\nabla g_i(x) \mid i \in J(x)\}.$$

It follows from Corollary 3.3 that $[\underline{\partial}^m f(x), \overline{\partial}^m f(x)]$ is a minimal quasidifferential of f at x. According to Theorem 2.1, one has that
$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)) = \mathrm{co}\{\nabla f_i(x) + \nabla g_j(x) \mid i \in I(x), j \in J(x)\},$$
$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)) = \mathrm{co}\{\nabla g_j(x) - \nabla g_k(x) \mid j, k \in J(x)\}.$$

It is easy to see that the set-valued mappings
$$x \longmapsto \bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x))$$
and
$$x \longmapsto \bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x))$$
are upper-semicontinuous.

Example 3.6. Let f be a quasidifferentiable function defined on $\mathbf{R}^2$ and $\underline{\partial} f(x) = \mathrm{co}\{u_i \mid i = 1, 2\}$ and $\overline{\partial} f(x) = \mathrm{co}\{v_i \mid i = 1, 2\}$. If $u_1 - u_2 \neq \alpha(v_1 - v_2)$, $\forall \alpha \in \mathbf{R}^1$, then $[\underline{\partial} f(x), \overline{\partial} f(x)]$ is a minimal quasidifferential. Thus,
$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)) = \mathrm{co}\{u_i + v_j \mid i, j = 1, 2\},$$
$$\bigcap_{[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathcal{D} f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)) = \mathrm{co}\{u_i - v_j \mid i, j = 1, 2\}.$$

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