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# Partial Regularity for Minimizers of Degenerate Polyconvex Energies*

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We prove partial regularity of minimizers for a class of polyconvex integral functionals

$$\int_\Omega f(Du, \mathrm{Ad}\ Du, \det\ Du)\ dx,$$

where $f$ is degenerate convex. Our class includes the model case

$$\int_\Omega \left(\mid Du \mid^p + \mid \mathrm{Ad}\ Du \mid^p + \mid \det\ Du \mid^p\right)\ dx.$$

The method of proof involves a blow-up technique combined with a suitable asymptotic analysis of the degeneration nature of the first term $\int_\Omega \mid Du \mid^p\ dx$.

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## Contents



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## 1. Introduction

In this paper we investigate the partial regularity properties of minimizers of integral functionals of the Calculus of Variations

$$\int_\Omega f(Du)\ dx \qquad u : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^N. \tag{1}$$

This problem has been widely investigated in the case $f$ is a quasiconvex integrand (see the pioneering paper [12], and also [19, 25, 13, 3, 9, 5, 6]). Another interesting class of integral functionals, which naturally arise as variational models for problems in nonlinear elasticity, is the one of polyconvex functionals i.e. functionals in which the integrand is a convex function of the minors $\wedge_i Du$ of order $i$ of the matrix $Du$

$$f(Du) = g(Du, \wedge_1 Du, \wedge_2 Du, ..., \wedge_k Du), \tag{2}$$
$$g \text{ convex, } k = \min\{n, N\}. \tag{3}$$

The class of these functionals was introduced by J.Ball (see [7]). For a rather comphrensive introduction to polyconvexity we refer to [10].

In the last years a wide literature on existence of minimizers of polyconvex integrals and related semicontinuity problems, has appeared. About semicontinuity we basically have that if $u_s$ is a sequence of $W^{1,n}(\Omega;\mathbb{R}^N)$ functions such that $u_s$ weakly converges to $u$ in $W^{1,p}(\Omega;\mathbb{R}^N)$ then

$$\int_\Omega f(Du)\ dx \le \liminf_s \int_\Omega f(Du_s)\ dx, \tag{4}$$

provided $p \ge n-1$, a hypothesis that turns out to be relevant also in the regularity theory (see section 9, below). For a proof of (4) we refer to [31], [11] and [1], where results from the theory of cartesian currents of Giaquinta, Modica and Soucek have been employed in order to get the semicontinuity (see [27]). For a very elementary approach valid also in the borderline case we address the reader to [23]. A counterexample showing that the restriction $p \ge n-1$ is essential for the lower semicontinuity has been found in [30].

For the regularity theory of polyconvex integrals relatively little has been done. In [21] a rather large class of integral functionals of the form

$$\begin{aligned} \mathcal{F}(u) \quad &= \quad \int_\Omega F_1(Du) + \sum_{i=2}^{k} F_i(\wedge_i Du)\ dx, \\ & \qquad F_i \ \text{ is convex, } \ k = \min\{n, N\}, \end{aligned} \tag{5}$$

has been considered, where $\wedge_i Du$ is the vector of all $i-$minors of the matrix $Du$. In that paper the main hypotheses were $p > n-1$ and the nondegenerate convexity of the functional $\mathcal{F}$, i.e.

$$\begin{aligned} &D^2 F_1(z)\xi \otimes \xi \ge \nu(1+ \mid z\mid^2)^{\frac{p-2}{2}} \mid \xi \mid^2, \\ &F_1(z) \le \Lambda(1+ \mid z \mid^p). \end{aligned} \tag{6}$$

A model case for the class of functionals in [21] is, for $n = N = 3$,

$$\int_\Omega (\mid Du \mid^2 + \mid Du \mid^p + \mid \mathrm{Ad}\, Du \mid^p + \mid \det Du \mid^p)\ dx, \tag{7}$$

with $p > 2$.

For local minimizers of these functionals Fusco and Hutchinson proved the $C^{1,\alpha}$ partial regularity i.e., the existence of an open subset $\Omega_0 \subset \Omega$ such that

$$| \Omega - \Omega_0 |= 0, \qquad u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N).$$

In this paper we consider the same class of functionals in (5) and we drop the nondegeneration hypothesis (6). So we consider $F_1$ such that

$$D^2 F_1(z)\xi \otimes \xi \geq \nu \mid z \mid^{p-2} \mid \xi \mid^2 .$$

In this case we include in our regularity theory also the relevant model functional, ($n = N = 3$, $p > 2$)

$$\int_\Omega (\mid Du \mid^p + \mid \mathrm{Ad}\ Du \mid^p + \mid \det\ Du \mid^p)\ dx. \tag{8}$$

We note that some relevant (global) regularity results for a class of degenerate polyconvex integrals have already been obtained by M. Fuchs (see [16]) under more restrictive growth conditions. His model case is the functional

$$\int_\Omega (\mid Du \mid^p + \sum_{i=2}^{k} \mid \wedge_i Du \mid^{\alpha_i})\ dx,$$

where $\alpha_i \leq p/i$; this case, differently from the model cases (7) and (8), shows an integrand with $p$-growth in the gradient. For related results about degenerate integral functionals we quote [37, 19, 26, 29, 14].

The class of polyconvex integral functionals stems its importance from the fact that many of the typical energies of nonlinear elasticity turn out to be polyconvex. So polyconvexity is a natural concept from the point of view of the applications though the class of quasiconvex integrals is the largest one for which the Direct Methods of the Calculus of Variations work (see [2, 10, 15]) and is the natural one from the viewpoint of the Calculus of Variations. We will make some remarks on these aspects in section 11.

We now comment on some technical aspects of our work. In order to get our regularity result we prove a decay estimate for a quantity $U$, commonly called excess, that provides an integral measure of the oscillations of $Du$ in a ball. This is a rather standard tool in order to get partial regularity. In our case we put

$$\begin{aligned} U(x,r) = & ⨍_{B(x,r)} \mid (Du)_{x,r} \mid^{p-2} \mid Du - (Du)_{x,r} \mid^2 + \mid Du - (Du)_{x,r} \mid^p \ \ dx \\ & + ⨍_{B(x,r)} \sum_{i=2}^{k} \mid \wedge_i (Du)_{x,r} \mid^{p-2} \mid \wedge_i (Du - (Du)_{x,r}) \mid^2 + \mid \wedge_i (Du - (Du)_{x,r}) \mid^p \ \ dx, \end{aligned}$$

and $(Du)_{x.r}$ is the average of $Du$ in $B(x,r)$. The particular form of $U$ in our case reflects the degenerate nature of our variational problem and the growth and structure assumptions on the energy density $f$.

To obtain such an estimate (see sections 5-9) we argue by contradiction. So we consider a sequence of balls $B(x_m, r_m)$ in which the decay estimate does not hold. Rescaling both $u$

and $\mathcal{F}$ we obtain a sequence of minimizers $v_m$ of the rescaled functionals $\mathcal{F}_m$, all defined on $B_1$. Then we analyze the asymptotic behaviour of the Euler equations of $\mathcal{F}_m$ and we argue on an alternative, that constitutes the main new technical point of the paper. In the first case we have a linear limit behaviour and we compare $v_m$ with the solution of a linear elliptic system. In the second case we find a $p$-laplacian type limit behaviour and we compare $v_m$ with the solution of a $p$-laplacian system. In both cases we find a contradiction, proving the estimate for $U(x,r)$.

Finally, we mention another technical problem arising in our context. We set our minimum problem in the Sobolev class $\wedge_k W^{1,p}(\Omega;\mathbb{R}^N)$ that consists of functions from $W^{1,p}(\Omega;\mathbb{R}^N)$ satisfying a higher integrability assumption on the minors of $Du$. In this class there always exists a minimizer (see section 3). A main difficulty is that this class is not a linear subspace of $W^{1,p}$. Thus the set of test functions is not a linear space. This is essentially due to the non standard, $(p,q)$-growth of $\mathcal{F}$ i.e.:

$$| Du |^p \leq f(Du) \leq L(| Du |^q + 1),$$

with $q = pk > p$, while $u$ is only in $W^{1,p}$ (for results about integrals with nonstandard growth we refer to the fundamental papers of Marcellini [32]–[34] and the bibliography quoted there). For this reason it is not possible to test the minimality with affine combinations of functions involving our minimizer $u$, as usually done in these cases, and a different type of comparison functions must be used (see section 8).

## 2. Some multilinear algebra

In this section we recall some facts and definitions from multilinear algebra that will be needed later. For the rest of the paper $n, N$ and $k$ will be positive integers such that

$$n \geq 2, \ N \geq 2, \quad 1 \leq k \leq \min\{n, N\}. \tag{9}$$

Furthermore $(e_i)_{i\leq n}$ and $(\epsilon_i)_{i\leq N}$ will be orthonormal basis for $\mathbb{R}^n$ and $\mathbb{R}^N$ respectively and we will denote

$$M_k = \{(i_1, i_2, ..., i_k) \ | \ 1 \leq i_1 < i_2 < .... < i_k \leq \max\{n, N\}, i_j \in \mathbb{N}\},$$

the set of all strictly increasing $k-$multindexes bounded by $\max\{n, N\}$. We will need the vector space of the $k-$vectors

$$\wedge_k \mathbb{R}^n = \{v_1 \wedge v_2 \wedge ... \wedge v_k \ | \ v_i \in \mathbb{R}^n\},$$

where a scalar product is defined by

$$\langle v_1 \wedge v_2 \wedge ... \wedge v_k, w_1 \wedge w_2, \wedge ... \wedge w_k\rangle := \det(\langle v_i, w_j\rangle)_{i,j}. \tag{10}$$

Setting

$$I = (i_1, i_2, ..., i_k) \quad e_I = (e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}),$$

from (10) we have that

$$(e_I)_{I\in M_k}, \quad (\epsilon_I)_{I\in M_k}$$

are orthonormal basis for $\wedge_k \mathbb{R}^n$ and $\wedge_k \mathbb{R}^N$ respectively. Now let $L : \mathbb{R}^n \to \mathbb{R}^N$ be a linear map. $L$ naturally induces a linear map $\wedge_k L : \wedge_k \mathbb{R}^n \to \wedge_k \mathbb{R}^N$ defined by

$$\wedge_k L(v_1 \wedge v_2 \wedge ... \wedge v_k) \ := \ Lv_1 \wedge Lv_2 \wedge ... \wedge Lv_k :=$$

$$\langle \wedge_k L, v_1 \wedge ... \wedge v_k \rangle. \tag{11}$$

The components of $L$ with respect to the basis $(e_i)$, $(\epsilon_i)$ are $L_{i,j} = \langle Le_j, \epsilon_i \rangle$, while the components of $\wedge_k L$ are for any $\mu, \lambda \in M_k$

$$(\wedge_k L)_{\mu,\lambda} = (Le_{\lambda_1} \wedge ... \wedge Le_{\lambda_k}, \epsilon_{\mu_1} \wedge ... \wedge \epsilon_{\mu_k}) = \det(L_{\mu_i \lambda_j})_{i,j}. \tag{12}$$

For instance we have with $k = 2$ and $\mu = (\mu_1, \mu_2)$, $\lambda = (\lambda_1, \lambda_2)$

$$\begin{aligned} (\wedge_k L)_{\mu,\lambda} &= \begin{vmatrix} L_{\mu_1\lambda_1} & L_{\mu_1\lambda_2} \\ L_{\mu_2\lambda_1} & L_{\mu_2\lambda_2} \end{vmatrix} \\ &= L_{\mu_1\lambda_1} L_{\mu_2\lambda_2} - L_{\mu_1\lambda_2} L_{\mu_2\lambda_1} \end{aligned}$$

We note that with the previous definitions the inner product norm is defined by

$$| \wedge_k L |^2 = \sum_{\lambda,\mu \in M_k} | (\wedge_k L)_{\lambda\mu} |^2 .$$

For the rest of the paper, we will always identify a linear map $L : \mathbb{R}^n \to \mathbb{R}^N$ with the tensor of its components and viceversa, so that, for any $A \in \mathrm{Hom}(\mathbb{R}^n; \mathbb{R}^N)$, the expression

$$\langle \wedge_k A, v_1 \wedge ... \wedge v_k \rangle,$$

will have sense in view of (11)–(12). Note that in this way, for any $A \in \mathrm{Hom}(\mathbb{R}^n; \mathbb{R}^N)$ we will have

$$\wedge_k A = ((\wedge_k A)_{\mu,\lambda})_{\mu,\lambda \in M_k} \equiv ((\wedge_k A)_{\mu,\lambda}),$$

with

$$(\wedge_k A)_{\mu\lambda} = \det(A_{\mu_i,\lambda_j})_{i,j}.$$

According to this convention we shall define:

$$\wedge_i \mathrm{H}_n^N := \left\{ \wedge_i A : A \in \mathrm{Hom}(\mathbb{R}^n; \mathbb{R}^N) \right\} .$$

We remark that once again we identify $\wedge_i A$ with the tensor of its components. Now let us consider $A, B \in \mathrm{Hom}(\mathbb{R}^k; \mathbb{R}^k)$. We have the following way of expanding the determinant of $A + B$

$$\det(A + B) = \det A + \sum_{i=1}^{k-1} \sum A^{(k-i)} B^i + \det B, \tag{13}$$

where $A^{k-i}$ ranges over all $(k - i) \times (k - i)$ minors from $A$ and $B^i$ is always the complementary $i \times i$ minor from $B$. Starting from (13) we will write, following [21], for any $E, F \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$

$$\wedge_k (E + F) = \wedge_k E + \sum_{i=1}^{k-1} \wedge_{k-i} E \odot \wedge_i F + \wedge_k F; \tag{14}$$

so each that component of $\wedge_k (E + F)$ is linearly expressed as sum of terms of $\wedge_k E$, $\wedge_k F$ and $\wedge_{k-i} E \odot \wedge_i F$, the last one being a sum of terms of the form $ef$ where $e$ and $f$ are components of $\wedge_{k-i}E$ and $\wedge_i F$ respectively. Usually we will not need to specify the exact form of $\odot$ and we will also write, instead of (14):

$$\wedge_k(E+F) = \sum_{i=0}^{k} \wedge_{k-i}E \odot \wedge_i F, \tag{15}$$

with the convention that $\wedge_0 E = \wedge_0 F = 1$.

An example of this situation is given, for $k=2$, by

$$\begin{aligned}(\wedge_2(E+F))_{(\mu_1,\mu_2)(\lambda_1,\lambda_2)} &= \begin{vmatrix} E_{\mu_1,\lambda_1}+F_{\mu_1,\lambda_1} & E_{\mu_1,\lambda_2}+F_{\mu_1,\lambda_2} \\ E_{\mu_2,\lambda_1}+F_{\mu_2,\lambda_1} & E_{\mu_2,\lambda_2}+F_{\mu_2,\lambda_2}\end{vmatrix} \\ &= (E_{\mu_1,\lambda_1}E_{\mu_2,\lambda_2} - E_{\mu_1,\lambda_2}E_{\mu_2,\lambda_1}) \\ &\quad +(E_{\mu_1,\lambda_1}F_{\mu_2,\lambda_2} + E_{\mu_2,\lambda_2}F_{\mu_1,\lambda_1} - E_{\mu_1,\lambda_2}F_{\mu_2,\lambda_1} \\ &\quad -E_{\mu_2,\lambda_1}F_{\mu_1,\lambda_2}) + (F_{\mu_1,\lambda_1}F_{\mu_2,\lambda_2} - F_{\mu_1,\lambda_2}F_{\mu_2,\lambda_1}).\end{aligned}$$

In (15) when $i=1$ we will also write

$$\wedge_{k-1}E \odot \wedge_1 F = \wedge_{k-1}E \tilde{\odot} F,$$

a notation that will be useful in the sequel.

We conclude this section with a lemma that will be used in the last sections and whose proof can be easily achieved on expanding the determinant and using the boundedness hypothesis.

**Lemma 2.1.** *Let* $\{A_m\} \subset \mathrm{Hom}\ (\mathbb{R}^n;\mathbb{R}^N)$ *such that* $|\ A_m\ | \leq M < \infty$ *for any* $m \in \mathbb{N}$, *then there exists a constant* $c \equiv c(M)$ *such that*

$$|\ \wedge_i A_m\ | \leq c\ |\ \wedge_j A_m\ |, \tag{16}$$

*for any* $1 \leq j \leq i \leq k$.

## 3. The class $\wedge_k W^{1,p}(\Omega)$ and the existence of minimizers

In this section we briefly describe the Sobolev class $\wedge_k W^{1,p}$ of functions in which we set our variational problem, and discuss some of its properties. Finally we recall some results about the existence of minimizers for polyconvex integrals.

**Definition 3.1.** Assume that $p \geq 2$ if $n=2$ and $p \geq n-1$ if $n>2$, $k \in \mathbb{N}$ such that $1 \leq k \leq \min\{n,N\}$ and let $\Omega$ be a smooth, bounded domain of $\mathbb{R}^n$. By $\wedge_k W^{1,p}(\Omega;\mathbb{R}^N) \equiv \wedge_k W^{1,p}$ we denote the class of functions $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that $\wedge_i Du \in L^p(\Omega)$ for any $1 \leq i \leq k$.

A first remark on the Sobolev class $\wedge_k W^{1,p}$ is that this is not a vector space. Indeed, in general if $u,v \in \wedge_k W^{1,p}$, then it may happen that $u+v \notin \wedge_k W^{1,p}$ while this happens, for example, when one of the two functions is smooth.

From now on we will denote by

$$[Du/D_i u^\alpha]$$

the matrix obtained from $Du$ by deleting the $\alpha-$th row and the $i-$th column.

Given a smooth function $u \in C^\infty(\Omega;\mathbb{R}^N)$, if $k = n = N$ then the following classical identity holds

$$\det Du = \sum_{i=1}^{k} (-1)^{i+\alpha} D_i(u^\alpha \det[Du/D_i u^\alpha]) \quad \alpha = 1, ...., n. \tag{17}$$

Assuming suitable integrability of the minors of $Du$ it is possible to extend the validity of (17) to more general $u$. Indeed in [21] (see lemma 2.2.1) it has been proved that if (again we are supposing $k = n = N$):

$$u \in W^{1,n-1}(\Omega;\mathbb{R}^n), \text{ for } k > 2, \text{ and } u \in W^{1,2}(\Omega;\mathbb{R}^n) \text{ for } k = 2,$$

and

$$\det[Du/D_i u^\alpha] \in L^{(k-1)/(k-2)}(\Omega) \quad \text{if } k > 2 \quad \forall\, i, \alpha$$

then

$$\det Du \in L^1(\Omega),$$

and furthermore the identity (17) is valid in the sense of distributions; moreover the pointwise and the distributional defintions of det $Du$ agree. As a consequence of this fact and of the definition 3.1, it follows from (12) that, given $u \in \wedge_k W^{1,p}(\Omega;\mathbb{R}^N)$ and $\mu, \lambda \in M_k$

$$\begin{aligned} (\wedge_k Du)_{\mu\lambda} &= \det[D_{\lambda_i} u^{\mu_j}]_{i,j} \\ &= \sum_{i=1}^{k} (-1)^{i+j} D_{\lambda_i}(u^{\mu_j} (\wedge_{k-1} Du)_{\hat{\mu}_j \hat{\lambda}_i}), \end{aligned} \tag{18}$$

where $\hat{\mu}_i = (\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_k)$, $\hat{\lambda}_i = (\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_k) \in M_{k-1}$. For example, if $k = 2$

$$\begin{aligned} (\wedge_2 Du)_{\mu\lambda} &= \begin{vmatrix} D_{\lambda_1} u^{\mu_1} & D_{\lambda_2} u^{\mu_1} \\ D_{\lambda_1} u^{\mu_2} & D_{\lambda_2} u^{\mu_2} \end{vmatrix} \\ &= D_{\lambda_1}(u^{\mu_1} D_{\lambda_2} u^{\mu_2}) - D_{\lambda_2}(u^{\mu_1} D_{\lambda_1} u^{\mu_2}) \end{aligned}$$

while, if $k = 3$ we have for example:

$$\begin{aligned} (\wedge_k Du)_{(\mu_1,\mu_2,\mu_3)(\lambda_1,\lambda_2,\lambda_3)} &= \begin{vmatrix} D_{\lambda_1} u^{\mu_1} & D_{\lambda_2} u^{\mu_1} & D_{\lambda_3} u^{\mu_1} \\ D_{\lambda_1} u^{\mu_2} & D_{\lambda_2} u^{\mu_2} & D_{\lambda_3} u^{\mu_2} \\ D_{\lambda_1} u^{\mu_3} & D_{\lambda_2} u^{\mu_3} & D_{\lambda_3} u^{\mu_3} \end{vmatrix} \\ &= D_{\lambda_1}(u^{\mu_1} (\wedge_2 Du)_{(\mu_2,\mu_3)(\lambda_2,\lambda_3)}) \\ &\quad - D_{\lambda_2}(u^{\mu_1} (\wedge_2 Du)_{(\mu_2,\mu_3)(\lambda_1,\lambda_3)}) \\ &\quad + D_{\lambda_3}(u^{\mu_1} (\wedge_2 Du)_{(\mu_2,\mu_3)(\lambda_1,\lambda_2)}) \end{aligned}$$

and so on. In other words by (18) it follows that the components of $\wedge_k Du$ are derivatives of products of the form $fg$ where $f$ is a component of $u$ and $g$ a component of $\wedge_{k-1} Du$. We will abbreviate

$$\wedge_k Du = \sum D(u \odot \wedge_{k-1} Du)$$

or even

$$\wedge_k Du = D(u \wedge_{k-1} Du)$$

where, again we stress, the exact form of $\odot$ will not be relevant for our purposes. We explicitely remark that the identities of the type above will be crucial in the regularity theory. This is one of the reasons why we set our minimum problem in the class $\wedge_k W^{1,p}$. Moreover we will restrict ourself to the case $p > n-1$ for reasons that will become clear in section 9 (see Remark 9.1 below). So for the rest of the paper we will keep this restriction.

According to the previous section the norm of $\wedge_i Du$ is:

$$|| \wedge_i Du||_{L^p} = \left( \int_\Omega \sum_{\lambda,\mu \in M_i} |(\wedge_i Du(x))_{\mu\lambda}|^p \, dx \right)^{\frac{1}{p}} .$$

In order to apply the direct methods of the Calculus of Variations it will be convenient to equip the class $\wedge_k W^{1,p}$ with a suitable (weak) convergence.

**Definition 3.2.** A sequence $\{u_j\} \subset \wedge_k W^{1,p}$ converges weakly in $\wedge_k W^{1,p}$ to $u \in W^{1,p}(\Omega)$ iff

$$\begin{aligned} &u_j \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega;\mathbb{R}^N), \\ &|| \wedge_i Du_j ||_p \leq M < +\infty \qquad \forall\, i \leq k,\ j \in \mathbb{N}. \end{aligned}$$

It is possible to prove (see [21], proposition 2.3.4) that if $u_j \to u$ weakly in $\wedge_k W^{1,p}$ then

$$u \in \wedge_k W^{1,p}, \tag{19}$$

and

$$\wedge_i Du_j \rightharpoonup \wedge_i Du \quad \text{weakly in } L^p. \tag{20}$$

Using (19) and (20) and well known results about the semicontinuity of convex integrals with respect to the weak convergence (see [28], chapter 4) it is possible to prove the existence in $\wedge_k W^{1,p}$ of minimizers of the functional:

$$\int_\Omega f(Du, \wedge_2 Du, ..., \wedge_k Du) \, dx, \tag{21}$$

in the class:

$$\left\{ u \in \wedge_k W^{1,p}(\Omega;\mathbb{R}^N) : u - u_0 \in W_0^{1,p}(\Omega;\mathbb{R}^N) \right\}$$

(with $u_0 \in \wedge_k W^{1,p}(\Omega;\mathbb{R}^N)$ being a fixed Dirichlet data) where the convex, continuous function $f : \mathbb{R}^{nN} \times \wedge_2 \mathrm{H}_n^N \times \ldots \times \wedge_k \mathrm{H}_n^N \to \mathbb{R}$ satisfies, for example, a coercivity assumption of the type

$$f(z) \geq \nu \mid z \mid^p \qquad \nu > 0.$$

We finally observe that recently, some optimal semicontinuity results for functionals (21) in $\wedge_k W^{1,p}$ have been obtained with respect to the following weaker convergence:

$$\begin{aligned} &u_k \to u \quad \text{in } L^1 \\ &\wedge_i Du_j \quad \text{is bounded in } L^1 \end{aligned} \tag{22}$$

under the condition $p \geq n-1$ (the same that we require, apart from the bordeline case $p = n-1$, which remains an open problem, for our regularity result). For these issues see [31, 11, 1, 23].

## 4. Preliminary results and notation

We start fixing some notation. In the following $\Omega$ will denote a smooth bounded domain of $\mathbb{R}^n$, $B(x,R)$ will denote the open ball of $\mathbb{R}^n$ of center $x$ and radius $R$. When no confusion about the center will arise we will also put $B_R \equiv B(x,R)$. If $f$ is an integrable function in $B(x,R)$ we set

$$(f)_{x,R} = \fint_{B(x,R)} f(y)\, dy = \frac{1}{\omega_n R^n} \int_{B(x,R)} f(y)\, dy,$$

where $\omega_n$ is the Lebesgue measure of the unit ball of $\mathbb{R}^n$. In the following it will be useful to rescale functionals of the form

$$\mathcal{F}(u) = \int_\Omega F^1(Du) + \sum_{i=2}^k F^i(\wedge_i Du)\, dx,$$

where $F^i$ are $C^2$ functions, defined on $\wedge_i \mathrm{H}_n^N$, with the convention that if $V$ is a vector space then $\wedge_1 V \equiv V$. If $\{A_m\} \subset \mathbb{R}^{nN}$ and $\{\lambda_m\} \subset \mathbb{R}^+$ are two given sequences we let for any $P \in \mathbb{R}^{nN}$

$$\begin{aligned} F_m^1(P) &= F^1(A_m + \lambda_m P) - F^1(A_m) - DF^1(A_m)\lambda_m P \\ F_m^i(P) &= F^i(\wedge_i(A_m + \lambda_m P)) - F^i(\wedge_i A_m) \\ &\quad - DF^i(\wedge_i A_m)(\wedge_i(A_m + \lambda_m P) - \wedge_i A_m). \end{aligned}$$

Finally we put

$$\mathcal{F}_m(w) = \int_{B_1} F_m^1(Dw) + \sum_{i=2}^k F_m^i(\wedge_i Dw)\, dx, \tag{23}$$

where $w \in \wedge_k W^{1,p}(B_1)$.

The proof of our regularity theorem will be achieved by means of a blow-up argument and of some decay estimates for solutions of $p-$Laplacian systems

$$-\Delta_p u = 0. \tag{24}$$

Indeed we have the following regularity result due to K.Uhlenbeck [37] and, in the version below, to Fusco-Hutchinson [20] and Giaquinta-Modica [26].

**Theorem 4.1.** *Let $u \in W^{1,p}(B_1; \mathbb{R}^N)$ be a weak solution to the p-Laplacian system* (24) *in the ball $B_1$, i.e.*

$$\int_{B_1} |\, Du\,|^{p-2} \langle Du, D\varphi \rangle dx = 0,$$

*for any $\varphi \in C_c^\infty(B_1; \mathbb{R}^N)$. Then there exist $c_0 > 0$ and $\mu \in (0,2)$ such that*

$$\begin{aligned} &\fint_{B(x,\rho)} [|\, (Du)_\rho\,|^{p-2} |\, Du - (Du)_\rho\,|^2 + |\, Du - (Du)_\rho\,|^p]\, dx \\ &\le c_0 \rho^\mu \fint_{B_1} [|\, (Du)_1\,|^{p-2} |\, Du - (Du)_1\,|^2 + |\, Du - (Du)_1\,|^p]\, dx, \end{aligned}$$

*for any $0 < \rho < 1/2$.*

We recall that, using Campanato's integral characterization of Hölder continuity, the estimate of Theorem 4.1 implies that any solution to (24) is of class $C^{1,\alpha}$ for $\alpha = \frac{\mu}{p}$. We conclude this section with a technical, elementary lemma whose simple proof can be found in [21].

**Lemma 4.2.** *Let $\{u_m\} \subset L^1(B_1)$ such that $\| u_m \|_1 \leq const < \infty$. Then for a.e. $t \in (0,1)$ there exists a constant $M_t < \infty$ and a subsequence still denoted by $\{u_m\}$, depending on $t$, such that*

$$\int_{\partial B_t} | u_m | \; d\mathcal{H}^{n-1} \leq M_t,$$

*where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.*

## 5. Statement of the alternative

In this section we state our regularity result and begin its proof. First of all we precisely describe the hypotheses on the functionals covered by our results. We shall deal with polyconvex integrals of the type

$$\mathcal{F}(u) = \int_\Omega F^1(Du) + \sum_{i=2}^{k} F^i(\wedge_i Du) \; dx,$$

defined on $\wedge_k W^{1,p}(\Omega; \mathbb{R}^N)$.

From now on we will denote:

$$k = \min\{n, N\} \qquad u : \Omega \; (\subset \mathbb{R}^n) \to \mathbb{R}^N. \tag{25}$$

Let us recall the following definition of local minimizer:

**Definition 5.1.** A function $u \in \wedge_k W^{1,p}(\Omega; \mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$ iff $\mathcal{F}(u) < +\infty$ and moreover:

$$\mathcal{F}(u) \leq \mathcal{F}(u + \phi) \tag{26}$$

for any $\phi \in \wedge_k W^{1,p}(\Omega; \mathbb{R}^N)$ such that $\mathrm{spt}\phi \subset\subset \Omega$.

Our hypotheses are the following:

(H1) $F^i : \wedge_i \mathrm{H}_n^N \to \mathbb{R}^+$ is of class $C^2$, for all $1 \leq i \leq k$

(H2) $p > 2$ if $n = 2$, $p > n-1$ if $n > 2$

(H3) (degenerate ellipticity) $D_{z_{\mu\lambda}, z_{\bar{\mu}\bar{\lambda}}} F^i(z) \xi_{\mu\lambda} \otimes \xi_{\bar{\mu}\bar{\lambda}} \geq \nu \, | z |^{p-2} | \xi |^2$ for all $1 \leq i \leq k$, $z, \xi \in \wedge_i \mathrm{H}_n^N$ $(\mu, \lambda, \bar{\mu}, \bar{\lambda} \in M_i)$, where $\nu > 0$

(H4) (growth condition) $| D^2 F^i(z) | \leq \Lambda \, | z |^{p-2}$ for all $1 \leq i \leq k$, $z \in \wedge_i \mathrm{H}_n^N$, where $\Lambda < +\infty$

(H5) (Hölder continuity of second derivatives) $| D^2 F^i(\xi) - D^2 F^i(\eta) | \leq \Lambda(| \xi |^{p-2-\delta} + | \eta |^{p-2-\delta}) \, | \xi - \eta |^\delta$, for all $1 \leq i \leq k$, $z, \xi \in \wedge_i \mathrm{H}_n^N$, where $0 < \delta < \min\{1, p-2\}$

(H6) ($p$-laplacian type behaviour at zero) $\lim_{t \to 0} t^{1-p} DF^1(tz) = L \, | z |^{p-2} \, z$ for all $z \in \wedge_i \mathrm{H}_n^N$, where $0 < L < +\infty$.

Under the previous assumptions we prove our:

**Main Theorem.** *Let* $u \in \wedge_k W^{1,p}$ *be a minimizer of* $\mathcal{F}(u)$ *and let the hypotheses* (H1) $-$ (H6) *be satisfied. Then there exist an open set* $\Omega_0 \subset \Omega$ *and* $0 < \alpha < 1$ *such that*

$$| \Omega - \Omega_0 |= 0, \qquad u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N). \tag{27}$$

**Remark.** In the case $n = 2$, due to some technical simplifications and since we are mainly interested in the degenerate case, we shall restrict to the case $p > 2$. The (non degenerate) case $p = 2$ has been treated, for instance, in the paper [22].

The proof of the Main Theorem will be based on a blow-up argument combined with the analysis of the asymptotic behaviour of certain gradient averages described below. We start proving a decay estimate for a quantity $U$, usually called excess, that, roughly speaking, measures the oscillations of $Du$ and $\wedge_i Du$ in a ball $B_R$. Once we have proved Theorem 5.2 below we will get the Main Theorem by means of a more or less standard iteration argument. We put

$$U(x,r) = \fint_{B(x,r)} [| (Du)_{x,r} |^{p-2}| Du - (Du)_{x,r} |^2 + | Du - (Du)_{x,r} |^p \tag{28}$$

$$+ \sum_{i=2}^{k} | \wedge_i (Du)_{x,r} |^{p-2}| \wedge_i (Du - (Du)_{x,r}) |^2 + | \wedge_i (Du - (Du)_{x,r}) |^p ]\, dx.$$

**Theorem 5.2.** *Let* $M > 0$ *be fixed. For each* $0 < \tau < \frac{1}{2}$ *there exists* $\epsilon \equiv \epsilon(\tau, M)$ *such that for* $B(x,r) \subset \Omega$ *if*

$$| (Du)_{x,r} |\leq M, \qquad U(x,r) \leq \epsilon, \tag{29}$$

*then*

$$U(x, \tau r) \leq C_M \tau^{\mu} U(x,r), \tag{30}$$

*where* $\mu$ *is as in* Theorem 4.1.

**Proof.** The proof of this result will go on throughout sections 5-9 and will be divided in several steps. We argue by contradiction, i.e. we suppose there exists a sequence of balls $B(x_m, r_m) \subset \Omega$ such that

$$| (Du)_{x_m,r_m} |\leq M, \qquad U(x_m, r_m) \to 0, \tag{31}$$

and

$$U(x_m, \tau r_m) > C(M)\tau^{\mu} U(x_m, r_m), \tag{32}$$

where $C(M)$ will be specified later.

Without loss of generality we will suppose in the following that $\lambda_m > 0$ (otherwise $Du$ is constant in $B(x_m, R_m)$ and the contradiction to (32) follows immediately).

Now we define

$$\begin{aligned} a_m &:= (u)_{x_m,r_m}, \quad A_m := (Du)_{x_m,r_m}, \quad \lambda_m^p := U(x_m, r_m), \\ v_m(z) &:= (\lambda_m r_m)^{-1}[u(x_m + r_m z) - a_m - r_m A_m z], \\ w_m(z) &:= (| A_m | \lambda_m^{-1})^{\frac{p-2}{2}} v_m \end{aligned} \tag{33}$$

for any $z \in B(0,1) \equiv B_1$. We have $v_m, w_m \in \wedge_k W^{1,p}(B_1)$ and

$$\begin{aligned} &Dv_m(z) = \lambda_m^{-1}[Du(x_m + r_m z) - A_m],\\ &(Dv_m)_\tau = \lambda_m^{-1}((Du)_{x_m,\tau r_m} - A_m),\\ &(v_m)_1 = (Dv_m)_1 = 0. \end{aligned}$$

By (28) and the definition of $\lambda_m$ it follows that

$$\begin{aligned} \lambda_m^{-p} U(x_m, r_m) = 1 = &\fint_{B(x_m,r_m)} |\, A_m\,|^{p-2}\, \lambda_m^{2-p}\, |\, Dv_m\,|^2 + |\, Dv_m\,|^p \, dz \\ &+ \sum_{i=2}^{k} \fint_{B(x_m,r_m)} (|\wedge_i A_m\,|^{p-2}\, \lambda_m^{2i-p}\, |\wedge_i Dv_m\,|^2 + \lambda_m^{p(i-1)}\, |\wedge_i Dv_m\,|^p)\, dz\,. \end{aligned} \tag{34}$$

By (34) and standard weak compactness arguments we get, up to not relabelled subsequences, that

$$\begin{aligned} &A_m \to A \text{ in } \mathbb{R}^{nN},\\ &v_m \rightharpoonup v \text{ weakly in } W^{1,p}(B_1;\mathbb{R}^N),\\ &w_m \rightharpoonup w \text{ weakly in } W^{1,2}(B_1;\mathbb{R}^N),\\ &v_m \to v \text{ strongly in } L^p(B_1;\mathbb{R}^N),\\ &w_m \to w \text{ strongly in } L^2(B_1;\mathbb{R}^N),\\ &|\wedge_i A_m\,|^{\frac{p-2}{2}}\, \lambda_m^{\frac{2i-p}{2}} \wedge_i Dv_m \rightharpoonup 0 \text{ weakly in } L^2(B_1;\wedge_i \mathrm{H}_n^N),\ i \geq 2,\\ &\lambda_m^{i-1} \wedge_i Dv_m \rightharpoonup 0 \text{ weakly in } L^p(B_1;\wedge_i \mathrm{H}_n^N),\ i \geq 2. \end{aligned} \tag{35}$$

Indeed $(35)_1$–$(35)_5$ easily follow by weak compactness, the others can be proved by induction. In order to prove $(35)_6$ we preliminarily remark that if $f_m \to f$ strongly in $L^p$, $p \geq 2$ and $g_m \rightharpoonup g$ weakly in $L^2$ then $f_m g_m \to fg$ in the sense of distributions. Furthermore, without loss of generality we may suppose here that $\wedge_i A_m \neq 0\ \forall\ i \leq k$ and $m \in \mathbb{N}$, otherwise $(35)_6$ would trivially follow. Now for $i = 2$ we write by (18)

$$|\wedge_2 A_m\,|^{\frac{p-2}{2}}\, \lambda_m^{\frac{4-p}{2}} \wedge_2 Dv_m = (|\wedge_2 A_m\,||\, A_m\,|^{-1})^{\frac{p-2}{2}} \lambda_m \sum D(w_m \tilde{\odot} Dv_m).$$

This quantity converges to zero in the sense of distributions, by the previous remark and Lemma 2.1. It also weakly converges in $L^2$ by the fact is bounded in $L^2$ (again up to not relabelled subsequences). Similarly, if $i > 2$, by induction one has that

$$\begin{aligned} &|\wedge_i A_m\,|^{\frac{p}{2}-1}\, \lambda_m^{\frac{2i-p}{2}} \wedge_i Dv_m \\ &= (|\wedge_i A_m\,||\wedge_{i-1} A_m\,|^{-1})^{\frac{p}{2}-1} \lambda_m \sum D(v_m \tilde{\odot}\, |\wedge_{i-1} A_m\,|^{\frac{p}{2}-1}\, \lambda_m^{\frac{2(i-1)-p}{2}} \wedge_{i-1} Dv_m) \end{aligned}$$

converges to zero in the sense of distributions, and hence weakly in $L^2$. A similar, actually simpler, inductive argument using (34), may be derived to prove $(35)_7$.

Having introduced rescaled functions $v_m$ it will be useful to introduce rescaled functionals

$$\mathcal{F}_m^t(w) = \int_{B_t} F_m^1(Dw) + \sum_{i=2}^{k} F_m^i(\wedge_i Dw)\ dz,$$

for $0 < t < 1$, where $F_m^i$ has been introduced in section 4 and $w \in \wedge_k W^{1,p}(B_1)$. By lemma 5.4 from [21] it follows that the rescaled function $v_m$ is actually a local minimizer of the rescaled functional $\mathcal{F}_m^t$, $\forall\ t > 0$, i.e.

$$\mathcal{F}_m^t(v_m) \leq \mathcal{F}_m^t(w), \tag{36}$$

for any $w \in \wedge_k W^{1,p}(B_1)$ such that $w = v_m$ outside some compact subset $K \subset\subset B_t$. In the sequel we will put $\mathcal{F}_m = \mathcal{F}_m^1$.

Our next step is to write the Euler equation for the functional $\mathcal{F}_m^t$. By (14) it follows that, for any $\phi \in C_c^\infty(B_t;\mathbb{R}^N)$,

$$\frac{d}{dt} \wedge_i (A_m + \lambda_m(Dv_m + tD\phi))\ |_{t=0} = \wedge_{i-1}(A_m + \lambda_m Dv_m)\tilde{\odot}\lambda_m D\phi.$$

We have in this way a sequence of Euler equations (actually systems) relative to the functionals $\mathcal{F}_m^t$:

$$\begin{aligned} &\int_{B_t} [DF^1(A_m + \lambda_m Dv_m) - DF^1(A_m)]D\phi\ dz \\ &\qquad + \int_{B_t} \sum_{i=2}^{k} [DF^i(\wedge_i(A_m + \lambda_m Dv_m)) - DF^i(\wedge_i A_m)] \cdot \\ &\qquad\qquad \cdot [\wedge_{i-1}(A_m + \lambda_m Dv_m)\tilde{\odot} D\phi]\ dz = 0, \end{aligned} \tag{37}$$

for any $\phi \in C_c^\infty(B_t;\mathbb{R}^N)$.

Now we are ready to state the fundamental alternative. Up to a not relabelled subsequence we have either:

**The First Case:**

$$|\ A_m\ |\ \lambda_m^{-1} \to +\infty, \tag{38}$$

or:

**The Second Case:**

$$|\ A_m\ |\ \lambda_m^{-1} \to l < +\infty. \tag{39}$$

More precisely, if $|A_m|\lambda_m^{-1} \to +\infty$ then we are in the first case otherwise, up to not relabelled subsequences, we reduce to the second one. In a similar manner, again up to subsequences, in the first case we introduce an integer that will be useful below

$$\bar{k} = \max\{i \in \mathbb{N} : 1 \leq i \leq k,\ |\ \wedge_i A_m\ |\ \lambda_m^{-1} \to +\infty\}; \tag{40}$$

by its definition we immediately have that

$$|\ \wedge_j A_m\ ||\ \wedge_i A_m\ |^{-1} \to 0 \tag{41}$$

whenever $1 \le i \le \bar{k} < j \le k$; note that, without loss of generality and always up to a subsequence, we supposed that

$$| \wedge_i A_m | \neq 0 \ \forall \ i \le \bar{k}, \ m \in \mathbb{N} \,. \tag{42}$$

It is possible to do this by the very definition of $\bar{k}$.

From now on, the rest of the proof of Theorem 5.2 will be split in two parallel parts, according to the case coming into the play. In the first case the "degeneration speed" with which $A_m$ may approach to 0 is not so large (with respect to $\lambda_m$) so in the blow-up procedure we will find a linear asymptotic behaviour of the systems (37) and the sequence $v_m$ will be compared with the (smooth) solution of a linear system to get the desired contradiction. In the second case, we will have a true degeneration of the systems (37) that, suitably rescaled, will have an asymptotic $p$-laplacian type behaviour by hypothesis (H6); this time the sequence $v_m$ will be compared with the solution of a system like the one in (24), that is still regular by Theorem 4.1 (see [26, 20, 37]), and the contradiction will follow once again.

## 6. Decay estimate in the first case

In this section we prove Theorem 5.2 in the First Case. We will use the fact that weak convergences stated in section 5 are actually strong; the proof of this fact will be given in section 8. For the rest of the section $f^1_m, \ f^2_m, \ g_m$ etc. will be auxiliary function that may vary through the estimates of the various terms involved in the proof and whose exact form will not be relevant in the context.

Before starting with the estimates, we recall the reader that we shall very often use the following elementary fact, a consequence of lemma 2.1 and of (31):

$$| \wedge_i A_m | \le c \, | A_m | \le c \equiv c(M).$$

From (H4) and the previous bound, up to a (not relabelled) subsequence, we have that

$$\begin{aligned} &| \wedge_i A_m |^{2-p} \, D^2 F^i (\wedge_i A_m) \longrightarrow C_i, \\ &| A_m |^{2-p} \, D^2 F^i (\wedge_i A_m) \longrightarrow \tilde{C}_i. \end{aligned} \tag{43}$$

There is actually a problem for $(43)_1$. Indeed, by (42), it is possible to divide by $| \wedge_i A_m |$ only in the case $i \le \bar{k}$. Anyway, according to the following developments and to (H4), in the case $| \wedge_i A_m | = 0 \ \forall m \in \mathbb{N}$, we shall put $C_i = 0$ (see remark 6.1 below).

The system in (37) can be written as (the second variation of $\mathcal{F}_m$):

$$\begin{aligned} &\lambda_m \int_{B_1} [ \int_0^1 D^2 F^1 (A_m + \tau \lambda_m D v_m) \, d\tau \, ] D v_m D\phi \, dz \\ &\quad + \int_{B_1} \sum_{i=2}^{k} [ \int_0^1 D^2 F^i (\wedge_i A_m + \tau (\wedge_i (A_m + \lambda_m D v_m) - \wedge_i A_m)) \, d\tau ] \cdot \\ &\qquad \cdot [\wedge_i (A_m + \lambda_m D v_m) - \wedge_i A_m] [\wedge_{i-1} (A_m + \lambda_m D v_m) \tilde{\odot} D\phi] \, dz \\ &= \lambda_m \int_{B_1} [ \int_0^1 D^2 F^1 (A_m + \tau \lambda_m D v_m) \, d\tau \, ] D v_m D\phi \, dz \end{aligned} \tag{44}$$

$$
\begin{aligned}
&+\lambda_m \int_{B_1} \sum_{i=2}^{k} [\int_0^1 D^2F^i(\wedge_i A_m + \tau(\wedge_i(A_m+\lambda_m Dv_m) - \wedge_i A_m))\, d\tau] \cdot \\
&\qquad \cdot[\wedge_{i-1}A_m \tilde{\odot} Dv_m + \sum_{j=2}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m] \cdot \\
&\qquad \cdot[\wedge_{i-1}A_m \tilde{\odot} D\phi + \sum_{j=1}^{i-1} \lambda_m^{j} \wedge_{i-j-1} A_m \odot \wedge_j Dv_m \tilde{\odot} D\phi]\, dz \\
&= \lambda_m I_m + \lambda_m II_m = 0.
\end{aligned}
$$

We get rid of $\lambda_m$ in (44) and divide the previous equation by the quantity (in view of (42))

$$S_m =| A_m |^{p-2} (| A_m | \lambda_m^{-1})^{\frac{2-p}{2}}$$

and go on evaluating $S_m^{-1} I_m$ and $S_m^{-1} II_m$.

We write

$$S_m^{-1} I_m = \int_{B_1} (f_m^1 + f_m^2) D\phi \; dz, \tag{45}$$

$$
\begin{aligned}
f_m^1 &= | A_m |^{2-p} \int_0^1 [D^2F^1(A_m + \tau\lambda_m Dv_m) - D^2F^1(A_m)] Dw_m \; d\tau, \\
f_m^2 &= | A_m |^{2-p} \; D^2F^1(A_m) Dw_m.
\end{aligned}
$$

Using (H5), a routine computation gives:

$$| f_m^1 |\leq c(| A_m | \lambda_m^{-1})^{\delta(\frac{2-p}{2}-1)} \mid Dw_m \mid^{1+\delta} + c(| A_m | \lambda_m^{-1})^{\frac{2-p}{2}} \mid Dv_m \mid^{p-1},$$

and the first term in (45) disappears as $m \to +\infty$ by $(35)_2$, $(35)_3$ and the assumption $| A_m | \lambda_m^{-1} \to +\infty$ as $m \to +\infty$. From (43) and $(35)_3$ we conclude that

$$S_m^{-1} I_m \to C_1 \int_{B_1} Dw \cdot D\phi \; dz, \tag{46}$$

for any $\phi \in C_c^\infty(B_1;\mathbb{R}^n)$. In order to estimate $S_m^{-1} II_m$ we write $II_m$ in a different way

$$
\begin{aligned}
II_m &= \sum_{i=2}^{k} \{ \int_{B_1} [\int_0^1 D^2F^i(\wedge_i A_m + \tau(\wedge_i(A_m+\lambda_m Dv_m) - \wedge_i A_m))\, d\tau] \cdot \\
&\qquad \cdot[(\wedge_{i-1}A_m \tilde{\odot} Dv_m)(\wedge_{i-1}A_m \tilde{\odot} D\phi) \\
&\quad +(\sum_{j=2}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m)(\wedge_{i-1}A_m \tilde{\odot} D\phi) \\
&\quad +(\sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m)(\sum_{j=1}^{i-1} \lambda_m^{j} \wedge_{i-j-1} A_m \odot \wedge_j Dv_m \tilde{\odot} D\phi)]\; dz\} \\
&= \sum_{i=2}^{k} A_m^i + B_m^i + C_m^i.
\end{aligned}
$$

In order to evaluate $S_m^{-1}A_m^i$ we put this time:

$$S_m^{-1}A_m^i = \int_{B_1} (f_m^i + g_m^i)\ dz,$$

where

$$f_m^i = \mid A_m \mid^{2-p} [\int_0^1 (D^2F^i(\wedge_i A_m + \tau(\wedge_i(A_m + \lambda_m Dv_m) - \wedge_i A_m)) \\ - D^2F^i(\wedge_i A_m))\ d\tau] \cdot (\wedge_{i-1}A_m \tilde{\odot} Dw_m)(\wedge_{i-1}A_m \tilde{\odot} D\phi),$$

and

$$g_m^i = \mid A_m \mid^{2-p} D^2F^i(\wedge_i A_m)(\wedge_{i-1}A_m \tilde{\odot} Dw_m)(\wedge_{i-1}A_m \tilde{\odot} D\phi).$$

Using (H5) and lemma 2.1 to estimate $| \wedge_j A_m| \leq c \equiv c(M)$, we have

$$\begin{aligned} \mid f_m^i \mid \ &\leq\ c \mid A_m \mid^{2-p} \mid \wedge_i(A_m + \lambda_m Dv_m) - \wedge_i A_m \mid^{p-2} \mid Dw_m \mid \\ &+ c \mid A_m \mid^{-\delta} \mid \wedge_i(A_m + \lambda_m Dv_m) - \wedge_i A_m \mid^{\delta} \mid Dw_m \mid = a_m^i + b_m^i, \end{aligned}$$

and using Young inequality (recall that $2\delta < p$), and formula (14)

$$\mid a_m^i \mid \leq c(\mid A_m \mid \lambda_m^{-1})^{\frac{2-p}{2}} (\mid Dv_m \mid^{\frac{p}{2}} + \mid \sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid\mid^p),$$

$$\mid b_m^i \mid \leq c(\mid A_m \mid \lambda_m^{-1})^{-\delta} (\mid Dw_m \mid^2 + \mid \sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid\mid^{2\delta}),$$

and by (35) and the fact that $\mid A_m \mid \lambda_m^{-1} \to \infty$ we have that $f_m^i \to 0$ in $L^1$; now, again by $(35)_3$ and (43) we find

$$S_m^{-1}A_m^i \to \int_{B_1} \tilde{C}_i(\wedge_{i-1}A \tilde{\odot} Dw)(\wedge_{i-1}A \tilde{\odot} D\phi)\ dz, \tag{47}$$

for any $\phi \in C_c^\infty(B_1; \mathbb{R}^n)$.

**Remark 6.1.** As mentioned at the beginning of the section a problem occurs when $| \wedge_i A_m| = 0$, $\forall m \in \mathbb{N}$. Anyway by (H4) it follows that $D^2F^i(\wedge_i A_m) = 0$ for every $m \in \mathbb{N}$ so that $g_m = 0$, $\forall m \in \mathbb{N}$, while we still have that $f_m \to 0$ and (47) holds also in this case.

In order to estimate $S_m^{-1}B_m^i$ we distinguish the case $i > \bar{k}$ from the case $i \leq \bar{k}$, where $\bar{k}$ is the integer defined in (40). If $i > \bar{k}$, then directly using growth conditions (H4), lemma 2.1 and formula (14) we get

$$\begin{aligned} \mid S_m^{-1}B_m^i \mid \ &\leq\ c(\mid \wedge_i A_m \mid\mid A_m \mid^{-1})^{\frac{p-2}{2}} \int_{B_1} \sum_{j=2}^{i} \mid \wedge_j A_m \mid^{\frac{p}{2}-1} \lambda_m^{j-\frac{p}{2}} \mid \wedge_j Dv_m \mid\ dz \\ &+ c(\mid A_m \mid \lambda_m^{-1})^{\frac{2-p}{2}} \int_{B_1} (\sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid)^{p-1}\ dz \to 0, \end{aligned}$$

by (35), (38) and (41).

If $i \leq \bar{k}$ we preliminarily set $T_m = |\wedge_i A_m|^{p-2}(|\wedge_i A_m|\lambda_m^{-1})^{\frac{2-p}{2}}$; this quantity is well defined in this case by (42). We note that by Lemma 2.1, $T_m^{-1} \geq cS_m^{-1}$, where $c$ depends only on M. Then we put

$$\begin{aligned}
f_m^1 &= (| \wedge_i A_m \mid \lambda_m^{-1})^{\frac{p-2}{2}}[\int_0^1 | \wedge_i A_m |^{2-p} \cdot \\
&\quad \cdot (D^2F^i(\wedge_i A_m + \tau(\wedge_i(A_m + \lambda_m Dv_m) - \wedge_i A_m)) - D^2F^i(\wedge_i A_m))\ d\tau] \cdot \\
&\quad \cdot [\sum_{j=2}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m][\wedge_{i-1} A_m \tilde{\odot} D\phi], \\
f_m^2 &= | \wedge_i A_m |^{2-p}\ D^2F^i(\wedge_i A_m) \cdot [\wedge_{i-1} A_m \tilde{\odot} D\phi], \\
g_m &= \sum_{j=2}^{i} \lambda_m^{j-\frac{p}{2}} | \wedge_j A_m |^{\frac{p}{2}-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m,
\end{aligned}$$

and, again estimating $| \wedge_i A_m| \leq c| \wedge_j A_m|$ by lemma 2.1, we have

$$| S_m^{-1} B_m^i | \leq c \mid T_m^{-1} B_m^i | \leq c \int_{B_1} | f_m^1 |\ \ dz + c \left| \int_{B_1} f_m^2 g_m\ dz \right| .$$

By (H5) and Young's inequality

$$\begin{aligned}
| f_m^1 | \ &\leq\ c(| \wedge_i A_m \mid \lambda_m^{-1})^{-\delta}[| \sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m ||^{2\delta} + | g_m |^2] \\
&\quad + c(| \wedge_i A_m \mid \lambda_m^{-1})^{\frac{2-p}{2}}(\sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m |)^{p-1} \to 0,
\end{aligned}$$

by (35) and (40). On the other hand, again by $(35)_6$, we have that $g_m \rightharpoonup 0$ in $L^2$ while (also using the fact that clearly $\wedge_{i-j} A_m \to \wedge_{i-j} A$) $f_m^2$ is a sequence of tensors that tends (in $L^\infty$) to $C_i \cdot [\wedge_{i-1} A \tilde{\odot} D\phi]$, so in any case $2 \leq i \leq k$, we have:

$$S_m^{-1} B_m^i \to 0. \tag{48}$$

We finally treat the remaining term $S_m^{-1} C_m^i$; to do this, we estimate, by (H4) and again by lemma 2.1 (used to estimate $| \wedge_i A_m| \leq c| \wedge_j A_m|$ when $j \leq i$)

$$\begin{aligned}
&| S_m^{-1} C_m^i | \leq c \mid A_m |^{2-p} (\mid A_m \mid \lambda_m^{-1})^{\frac{p-2}{2}} \cdot \\
&\cdot \int_{B_1} (| \wedge_i A_m |^{p-2} + \lambda_m^{p-2} | \sum_{j=1}^{i} \lambda_m^{j-1} \wedge_j Dv_m |^{p-2})(\sum_{j=1}^{i} \lambda_m^{2j-1} | \wedge_j Dv_m |^2) dz \\
&\leq c\lambda_m (\mid A_m \mid \lambda_m^{-1})^{\frac{2-p}{2}} \int_{B_1} (\sum_{j=1}^{i} \lambda_m^{2j-p} | \wedge_j A_m |^{p-2} | \wedge_j Dv_m |^2
\end{aligned}$$

$$+\sum_{j=1}^{i} \lambda_m^{p(j-1)} \mid \wedge_j Dv_m \mid^p)\ dz \to 0$$

and by (35), (38), finally

$$S_m^{-1} C_m^i \to 0. \tag{49}$$

Now from (44), (46), (47), (48), and (49) we deduce that $w$ solves a linear system with constant coefficients, that is

$$\int_{B_1} C_1 DwD\phi + \sum_{i=2}^{k} \tilde{C}_i(\wedge_{i-1} A\tilde{\odot} Dw)(\wedge_{i-1} A\tilde{\odot} D\phi)\ dz = 0. \tag{50}$$

for any $\phi \in C_c^\infty(B_1;\mathbb{R}^N)$.

This is an elliptic system with ellipticity bounds given by:

$$c^{-1}|\xi|^2 \le C_1 \xi \otimes \xi + \sum_{i=2}^{k} C_i(\wedge_{i-1} A\tilde{\odot}\xi) \otimes (\wedge_{i-1} A\tilde{\odot}\xi) \le c|\xi|^2$$

for any $\xi \in R^{nN}$ and where $0 < c \equiv c(M,\nu,\Lambda) < +\infty$, since the moduli of the constant tensors $C_i$ depend only on $\nu, \Lambda$ and $M$. From the theory of linear elliptic systems with constant coefficients (see [28], chapter 10) it follows that the function $w$ is smooth in $B_1$ and furthermore there exists a constant $c \equiv c(M,\nu,\Lambda)$ such that, for any $0 \le \tau < 1$

$$\fint_{B_\tau} \mid Dw - (Dw)_\tau \mid^2\ \ dz \le c\tau^2 \tag{51}$$

and

$$\sup_K |Dw| \le c \equiv c(K,M) < \infty, \qquad K \subset\subset B_1, \quad K \text{ compact}. \tag{52}$$

Let us observe that setting $\tilde{A}_m = (Du)_{x_m,\tau R_m}$, it follows

$$\begin{aligned}\lambda_m^{-p} U(x_m,\tau r_m) = \fint_{B_\tau} (\mid \tilde{A}_m \mid \lambda_m^{-1})^{p-2} \mid Dv_m - (Dv_m)_\tau \mid^2 \\ + \mid Dv_m - (Dv_m)_\tau \mid^p + [\sum_{i=2}^{k} \mid \wedge_i (A_m + \lambda_m (Dv_m)_\tau) \mid^{p-2} \lambda_m^{2i-p} \cdot \\ \cdot \mid \wedge_i (Dv_m - (Dv_m)_\tau) \mid^2 + \sum_{i=2}^{k} \lambda_m^{p(i-1)} \mid \wedge_i (Dv_m - (Dv_m)_\tau) \mid^p]\ dz.\end{aligned} \tag{53}$$

Now we claim that the convergences in (35) are actually strong, that is, up to a not relabelled subsequence, we have

$$\begin{aligned}&v_m \to 0 \ \text{ strongly in } \ W^{1,p}_{\rm loc}(B_1;\mathbb{R}^N),\\ &w_m \to w \ \text{ strongly in } \ W^{1,2}_{\rm loc}(B_1;\mathbb{R}^N),\\ &\mid \wedge_i A_m \mid^{\frac{p-2}{2}} \lambda_m^{\frac{2i-p}{2}} \wedge_i Dv_m \to 0 \ \text{ strongly in } L^2_{\rm loc}(B_1;\wedge_i \mathrm{H}_n^N),\ i \ge 2,\end{aligned} \tag{54}$$

$$\lambda_m^{i-1} \wedge_i Dv_m \to 0 \text{ strongly in } L^p_{\mathrm{loc}}(B_1; \wedge_i \mathrm{H}^N_n).$$

The proof of this claim will be given in section 9 - First case.

Assuming (54) we first prove that the quantity in (53) inside the square brackets tends to 0 in $L^1$. Indeed the general $i$-term in the first sum in (53) is controlled by

$$c \int_{B_\tau} (| \wedge_i A_m |^{p-2} + \lambda_m^{p-2}) \lambda_m^{2i-p} (| \wedge_i (Dv_m)_\tau |^2 + \sum_{j=1}^{i} | \wedge_j Dv_m |^2) \, dz \tag{55}$$

$$\leq c \int_{B_\tau} \lambda_m^{2(i-1)} [1 + | Dw_m |^2] \, dz + c \int_{B_\tau} \sum_{j=2}^{i} | \wedge_j A_m |^{p-2} \lambda_m^{2j-p} | \wedge_j Dv_m |^2 \, dz$$

$$+ c (\int_{B_\tau} \sum_{j=1}^{i} \lambda_m^{p(j-1)} | \wedge_j Dv_m |^p) \, dz)^{\frac{2}{p}} \to 0.$$

Here we have also used the fact that, by weak convergence, the sequence $\{(Dv_m)_\tau\}_{m \in \mathbb{N}}$ is bounded, the fact that $i \geq 2$ and Lemma 2.1. For the reader's convenience we remark that the first integral in the previous sum appears also via the following estimate (obtained using once again lemma 2.1):

$$\begin{aligned}
(| \wedge_i A_m|^{p-2} + \lambda_m^{p-2}) \lambda_m^{2i-p} | \wedge_i (Dv_m)_\tau|^2 \\
&\leq c(M, \tau)(|A_m|^{p-2} + \lambda_m^{p-2}) \lambda_m^{2i-p} |(Dv_m)_\tau|^2 \\
&\leq c(M, \tau) \lambda_m^{2(i-1)} + c |A_m|^{p-2} \lambda_m^{2i-p} \fint_{B_\tau} |Dv_m|^2 \, dz \\
&\leq c(M, \tau) \lambda_m^{2(i-1)} \left[ 1 + \int_{B_\tau} |Dw_m|^2 \, dz \right].
\end{aligned}$$

The last terms on the right hand side of (53) may be controlled by

$$c \sum_{i=2}^{k} \int_{B_\tau} \lambda_m^{p(i-1)} (| \wedge_i Dv_m |^p + 1) \, dz \to 0. \tag{56}$$

where we used $(54)_4$.

Coming to the first terms in (53), we observe that

$$\begin{aligned}
(| \tilde{A}_m | \lambda_m^{-1})^{p-2} &\leq c[(| A_m | \lambda_m^{-1})^{p-2} + (| A_m - \tilde{A}_m | \lambda_m^{-1})^{p-2}] \\
&\leq c(| A_m | \lambda_m^{-1})^{p-2} + c(\fint_{B_\tau} | Dv_m |^p \, dz)^{\frac{p-2}{p}}.
\end{aligned}$$

Using previous estimate, by Hölder inequality

$$\fint_{B_\tau} (| \tilde{A}_m | \lambda_m^{-1})^{p-2} | Dv_m - (Dv_m)_\tau |^2 \, dz \tag{57}$$

$$\leq C(p) \fint_{B_\tau} | Dw_m - (Dw_m)_\tau |^2 \, dz + C(p) \fint_{B_\tau} | Dv_m |^p \, dz.$$

Taking into account (51)–(57), using strong convergences stated in (54) we have

$$\begin{aligned} C_M \tau^{\mu} &\leq \limsup_m \lambda_m^{-p} U(x_m, \tau r_m) \\ &\leq C(p) -\!\!\!\!\!\!\int_{B_\tau} \mid Dw - (Dw)_\tau \mid^2 \; dz \\ &\leq C(p) C(M) \tau^2 \\ &\leq C(p) C(M) \tau^{\mu} \end{aligned}$$

contradicting (32) if we choose $C_M$ such that $C_M > C(p)C(M)$.

## 7. Decay estimate in the second case

In this section we end the proof of Theorem 1 showing the decay estimate in the second case. Also this time the proof will be based on the fact that the convergences stated in (35) are actually strong; we remind the reader that this fact will be proved subsequently. Up to a (not relabelled) subsequence we may suppose this time (recall (39))

$$\lambda_m^{-1} A_m \to l\bar{A}, \tag{58}$$

where $l \in \mathbb{R}^+$, $\bar{A} \in \mathbb{R}^{nN}$, $\mid \bar{A} \mid= 1$.

By (58) and the very definition of $\wedge_i A_m$ we have also that:

$$\mid \wedge_i A_m \mid \leq c |A_m|^i \leq c\lambda_m^i \leq c\lambda_m \, . \tag{59}$$

We divide the Euler equation (37) by $\lambda_m^{p-1}$ to get

$$\begin{aligned} &\int_{B_1} \lambda_m^{1-p} DF^1(A_m + \lambda_m Dv_m) D\phi \; dz \\ &\qquad + \int_{B_1} \sum_{i=2}^{k} \lambda_m^{1-p} [DF^i(\wedge_i(A_m + \lambda_m Dv_m)) - DF^i(\wedge_i A_m)] \cdot \\ &\qquad\qquad \cdot [\wedge_{i-1}(A_m + \lambda_m Dv_m) \tilde{\odot} D\phi] \; dz = 0. \end{aligned} \tag{60}$$

Now we preliminarily show that the terms indexed with $i \geq 2$ in (60) are converging to 0. In order to do this we jump back to (44); using this formula the general $i$-term in (60) can be controlled by

$$\begin{aligned} &c\lambda_m^{2-p} \int_{B_1} [\mid \wedge_i A_m|^{p-2} + |\sum_{j=1}^{i} \wedge_{i-j} A_m \odot \lambda_m^j \wedge_j Dv_m|^{p-2}] \\ &\qquad [|\sum_{j=1}^{i} \wedge_{i-j} A_m \odot \lambda_m^{j-1} \wedge_j Dv_m|][|\sum_{j=0}^{i-1} \wedge_{i-1-j} A_m \odot \lambda_m^j \wedge_j Dv_m|] \; dz \\ &\leq c\lambda_m \int_{B_1} [(|A_m|\lambda_m^{-1})^{p-2} + (\sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m|)^{p-2}][1 + (\sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m|)^2] \; dz \end{aligned}$$

$$\leq c\lambda_m \int_{B_1} [1 + (\sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m|)^p] \, dz \to 0\,,$$

by (35) and (58)–(59).

As in the previous section we claim that the weak convergences in (35) are strong, that is, always up to a (not relabelled) subsequence:

$$\begin{aligned} &v_m \to v && \text{strongly in } W^{1,p}_{\mathrm{loc}}(B_1;\mathbb{R}^N), \\ &\lambda_m^{i-1} \wedge_i Dv_m \to 0 && \text{strongly in } L^p_{\mathrm{loc}}(B_1;\wedge_i \mathrm{H}^N_n),\ i \geq 2\,. \end{aligned} \tag{61}$$

**Remark 7.1.** In the follwing we do not need explicitely the convergence in $(35)_6$ to be strong. This is also because the strong convergence of the terms in $(35)_6$ follows from the one in $(61)_2$. Indeed, assuming $(61)_2$ and using (59) together with Hölder inequality, we have:

$$\int_{B_\tau} | \wedge_i A_m |^{p-2} \lambda_m^{2i-p} | \wedge_i Dv_m |^2 \ \, dz \leq c \left( \int_{B_\tau} \lambda_m^{p(i-1)} | \wedge_i Dv_m |^p \ \, dz \right)^{\frac{2}{p}} \to 0\,,$$

an estimate that will be useful in the sequel.

Using (H6), (58) and $(61)_1$, we have that

$$\int_{B_1} \lambda_m^{1-p} |DF^1(A_m + \lambda_m Dv_m) - DF^1(\lambda_m l\bar{A} + \lambda_m Dv)| \, dz \to 0\,. \tag{62}$$

To prove this, we argue as follows: by Egorov theorem, fixed $\epsilon, \delta > 0$ it is possible to determine $A \subset B_{1-\epsilon}$ such that $|B_1 \setminus A| \leq \delta$ and $Dv_m \to Dv$ uniformly in $A$; then the previous integral can be controlled by:

$$\begin{aligned} c\lambda_m^{2-p} &\int_{(B_{1-\epsilon}\setminus A)\cup(B_1\setminus B_{1-\epsilon})} \int_0^1 |D^2F^1(A_m + \lambda_m Dv_m + \tau(\lambda_m(Dv - Dv_m) \\ &\qquad\qquad + \lambda_m l\bar{A} - A_m))| \cdot |[(Dv - Dv_m) + l\bar{A} - \lambda_m^{-1} A_m]| \, d\tau \, dz \\ &+ \int_A \lambda_m^{1-p} |DF^1(A_m + \lambda_m Dv_m) - DF^1(\lambda_m l\bar{A} + \lambda_m Dv)| \, dz \\ \leq c &\int_{(B_{1-\epsilon}\setminus A)\cup(B_1\setminus B_{1-\epsilon})} (1 + |Dv|^{p-1} + |Dv_m|^{p-1}) \, dz \\ &+ \int_A \lambda_m^{1-p} |DF^1(A_m + \lambda_m Dv_m) - DF^1(\lambda_m l\bar{A} + \lambda_m Dv)| \, dz\,. \end{aligned}$$

The second integral clearly converges to zero as $m \to +\infty$, by (H6), since the uniform convergence on $A$. The first one can be made arbitrarily small, letting $\epsilon, \delta \to 0$, since the sequence $\{|Dv_m|^{p-1}\}_{m\in\mathbb{N}}$ is equiintegrable, by $(35)_2$.

By using a similar argument, (H6), (59) and the previous estimates, we infer:

$$\int_{B_1} | l\bar{A} + Dv |^{p-2} < (l\bar{A} + Dv), D\phi > \ \, dz = 0 \tag{63}$$

for any $\phi \in C_c^\infty(B_1;\mathbb{R}^N)$.
So if we put

$$\hat{v}(z) = v(z) + l\bar{A}z, \tag{64}$$

then

$$-\Delta_p \hat{v} = 0, \tag{65}$$

and by Theorem 4.1, (64)–(65), it follows that

$$\begin{aligned} \fint_{B_\tau} \mid (D\hat{v})_\tau \mid^{p-2}\mid D\hat{v} - (D\hat{v})_\tau \mid^2 + \mid D\hat{v} - (D\hat{v})_\tau \mid^p \ dz \\ \le c_0\tau^\mu \fint_{B_1} [l^{p-2} \mid Dv \mid^2 + \mid Dv \mid^p] \ dz\,. \\ \le c_0\tau^\mu\,. \end{aligned} \tag{66}$$

This is the analogous of estimate (51) for the first case, apart from the exponent $\mu$ instead of 2. Here we remark that the constant $c_0$ in (66) is independent of $l$; in fact we used the weak convergence of $Dv_m$ and (34) to obtain, via lower semicontinuity:

$$\int_{B_1} [l^{p-2} \mid Dv \mid^2 + \mid Dv \mid^p] \ dz \le \liminf_m \int_{B_1} (\frac{|A_m|}{\lambda_m})^{p-2}|Dv_m|^2 + |Dv_m|^p \ dz \le 1$$

and the fact that $(Dv_m)_1 = 0$ for each $m \in \mathbb{N}$.

Now we proceed as in section 6 and recall (53). The $i$-term in square brackets from (53) is estimated via (59) and (61) (also look at (55) and remark 7.1) by:

$$c\left(\int_{B_\tau} \sum_{j=2}^{i} \lambda_m^{p(j-1)} \mid \wedge_j Dv_m \mid^p \ dz\right)^{\frac{2}{p}} + c\left(\int_{B_\tau} \lambda_m^{p(i-1)}(1+ \mid Dv_m \mid^p) \ dz\right)^{\frac{2}{p}} \to 0$$

where we used (59), the fact that $i \ge 2$ and that $\{\mid \wedge_i(Dv_m)_\tau \mid\}_{m\in\mathbb{N}}$ is bounded, by lemma 2.1. So, as in section 6 we finally get, using (64)

$$\begin{aligned} C_M\tau^\mu &\le \limsup_m \lambda_m^{-p} U(x_m, \tau r_m) \\ &= \fint_{B_\tau} [\mid l\bar{A} + (Dv)_\tau \mid^{p-2}\mid Dv - (Dv)_\tau \mid^2 + \mid Dv - (Dv)_\tau \mid^p] \ dz \\ &\le c_0\tau^\mu, \end{aligned}$$

and the desired contradiction follows, as in the previous section, choosing $C_M > c_0$.

## 8. Preliminary estimates for strong convergences

The aim of this section, and of the following one, is to prove that the weak convergences in (35) are actually strong, thus also proving the claims of section 6-7, that is (54) and (61); once done, the proof of theorem 5.2 will be really complete. We will derive here some preliminary estimates that will be used both in the First and in the Second case.

**Preliminary construction.** We use a sequence of comparison functions, firstly introduced in [21]. For $t \in (\frac{1}{2}, 1)$ and $\delta \in (0, \frac{1}{4})$ we define $z_m \equiv z_m^{t,\delta}$ as follows, let $x \in B_1$ and $x = r\omega$ be its polar decomposition, i.e. $r = | x |$; $\omega = x \mid x |^{-1}$; we put

$$z_m(x) = \begin{cases} \phi(r\omega) & r < t - \delta \\ \phi([t-\delta+2(r-(t-\delta))]\omega) & t-\delta \leq r \leq t - \frac{\delta}{2} \\ \frac{t-r}{\delta/2}\phi(t\omega) + \frac{r-(t-\delta/2)}{\delta/2} v_m(t\omega) & t - \frac{\delta}{2} \leq r \leq t \\ v_m(r\omega) & t \leq r \leq 1 \end{cases} \tag{67}$$

with $\phi \in C^\infty(B_1; \mathbb{R}^N)$.

We derive some estimates for $\wedge_k Dz_m$ that will also show that $z_m \in \wedge_k W^{1,p}(B_1; \mathbb{R}^N)$, for a.e. $t \in (1/2, 1)$.

Let $(\tau_1, .., \tau_{n-1}, \nu)$ be an orthonormal basis where $\nu$ is a radial vector. Then we have on $B_t - B_{t-\frac{\delta}{2}}$

$$\begin{aligned} D_{\tau_i} z_m &= \frac{t-r}{\delta/2}\frac{t}{r} D_{\tau_i}\phi(t\omega) + \frac{r-(t-\delta/2)}{\delta/2}\frac{t}{r} D_{\tau_i} v_m(t\omega), \\ D_\nu z_m(r\omega) &= 2\delta^{-1}(v_m(t\omega) - \phi(t\omega)). \end{aligned}$$

We have, keeping into account that $t/r \leq 2$, $(t-r)(\delta/2)^{-1} \leq 1$ and $(r-(t-\delta/2))(\delta/2)^{-1} \leq 1$,

$$\begin{aligned} &| D_{\tau_i} z_m(r\omega) | \leq c(| D\phi | + | Dv_m |)(t\omega), \\ &| D_\nu z_m(r\omega) | \leq c\delta^{-1} | v_m(t\omega) - \phi(t\omega) | . \end{aligned}$$

When $2 \leq i \leq k$ a straightforward computation gives (look at section 2):

$$| \langle \wedge_i Dz_m(r\omega), \tau_{j_1} \wedge ... \wedge \tau_{j_i} \rangle | \leq c(|| D\phi ||^i + \sum_{j=1}^{i} || D\phi ||^{i-j} | \wedge_j Dv_m(t\omega) |),$$

and

$$\begin{aligned} &| \langle \wedge_i Dz_m(r\omega), \nu \wedge \tau_{j_1} \wedge ... \wedge \tau_{j_{i-1}} \rangle | \\ &\quad \leq c(\delta^{-1} | v_m(t\omega) - \phi(t\omega) |)(|| D\phi ||^{i-1} + \sum_{j=1}^{i-1} || D\phi ||^{i-j-1} | \wedge_j Dv_m(t\omega)) |), \end{aligned}$$

where $|| D\phi ||$ stands for $|| D\phi ||_{L^\infty(B_t)}$ . In this way we finally have

$$\begin{aligned} &| \wedge_i Dz_m(r\omega) | \leq c(|| D\phi || + \delta^{-1} | v_m(t\omega) - \phi(t\omega) |) \cdot \\ &\quad \cdot (|| D\phi ||^{i-1} + \sum_{j=1}^{i-1} || D\phi ||^{i-j-1} | \wedge_j Dv_m(t\omega) |) + c | \wedge_i Dv_m(t\omega) | . \end{aligned} \tag{68}$$

Note that all the functions involved in the left hand side of (68) are evaluated at a generic $x \in B_t - B_{t-\delta/2}$, $x = r\omega$, while in the right hand side $x \in \partial B_t$ i.e. $x = t\omega$ and $t$ is fixed.

We now derive some estimates for $\mathcal{F}_m$ evaluated at any $z \in \wedge_k W^{1,p}(B_1;\mathbb{R}^N)$. Using growth conditions on second derivatives

$$\lambda_m^{-p} \mid F_m^1(Dz) \mid \leq c\lambda_m^{2-p} \mid [\int_0^1 (1-\tau) D^2F^1(A_m + \tau\lambda_m Dz)\, d\tau](Dz \otimes Dz) \mid \quad (69)$$
$$\leq c((\mid A_m \mid \lambda_m^{-1})^{p-2} \mid Dz \mid^2 + \mid Dz \mid^p).$$

In the same way some computations involving (H4) and formula (14) give

$$\lambda_m^{-p} \mid F_m^i(Dz) \mid \;\leq\; c\sum_{j=1}^{i} \mid \wedge_j A_m \mid^{p-2} \lambda_m^{2j-p} \mid \wedge_j Dz \mid^2 \quad (70)$$
$$+c\sum_{j=1}^{i} \lambda_m^{p(j-1)} \mid \wedge_j Dz \mid^p .$$

Putting (68) in (69) and (70) for $z = z_m$, on $B_t - B_{t-\frac{\delta}{2}}$ we have, for $2 \leq i \leq k$:

$$\lambda_m^{-p} \mid F_m^1(Dz_m) \mid \leq c(\mid A_m \mid \lambda_m^{-1})^{p-2}(\delta^{-2} \mid v_m - \phi \mid^2 + \mid D\phi \mid^2 + \mid Dv_m \mid^2) \quad (71)$$
$$+c(\delta^{-p} \mid v_m - \phi \mid^p + \mid D\phi \mid^p + \mid Dv_m \mid^p),$$

$$\lambda_m^{-p} \mid F_m^i(Dz_m) \mid \leq c(\mid A_m \mid \lambda_m^{-1})^{p-2}(\delta^{-2} \mid v_m - \phi \mid^2 + \mid D\phi \mid^2 + \mid Dv_m \mid^2)$$
$$+c(\delta^{-p} \mid v_m - \phi \mid^p + \mid D\phi \mid^p + \mid Dv_m \mid^p)$$
$$+c\sum_{j=2}^{i} \mid \wedge_j A_m \mid^{p-2} \lambda_m^{2j-p} \cdot [(\| D\phi \|^2 + \delta^{-2} \mid v_m - \phi \mid^2)(\| D\phi \|^{2(j-1)}$$
$$+\sum_{l=1}^{j-1} \| D\phi \|^{2(j-l-1)} \mid \wedge_l Dv_m \mid^2) + \mid \wedge_j Dv_m \mid^2] \quad (72)$$
$$+c\sum_{j=2}^{i} \lambda_m^{p(j-1)}[(\| D\phi \|^p + \delta^{-p} \mid v_m - \phi \mid^p) \cdot$$
$$\cdot(\| D\phi \|^{p(j-1)} + \sum_{l=1}^{j-1} \| D\phi \|^{p(j-l-1)} \mid \wedge_l Dv_m \mid^p) + \mid \wedge_j Dv_m \mid^p].$$

From now on, for all the rest of the proof, $c$ will denote a constant possibly depending on all the parameters of the proof: $n, N, p, \tau, M$ but independent of $\delta$ and $\| D\phi \|$. Instead $\tilde{c}$ will denote another kind of constant that will depend on the parameters mentioned above and also on $\| D\phi \|_\infty$, but not on $\delta$. The reasons for this distinction are technical and will be clear in section 9 - Second case.

Connecting (71) and (72), using the area formula and rearranging we get (with $\mathcal{H}^{n-1}$ denoting the $n-1$ dimensional Hausdorff measure):

$$\lambda_m^{-p} \int_{B_t - B_{t-\delta/2}} \sum_{i=1}^{k} F_m^i(Dz_m)\, dz \quad (73)$$

$$
\begin{aligned}
&\leq c\sum_{i=1}^{k}[\delta\int_{\partial B_t}(\mid A_m\mid\lambda_m^{-1})^{p-2}(\delta^{-2}\mid v_m-\phi\mid^2+\mid D\phi\mid^2+\mid Dv_m\mid^2)\ d\mathcal{H}^{n-1}\\
&+c\delta\int_{\partial B_t}(\delta^{-p}\mid v_m-\phi\mid^p+\mid D\phi\mid^p+\mid Dv_m\mid^p)\ d\mathcal{H}^{n-1}\\
&+\tilde{c}\lambda_m^2\delta(\mid A_m\mid\lambda_m^{-1})^{p-2}\left(\mid\mid D\phi\mid\mid^2+\delta^{-2}\int_{\partial B_t}\mid v_m-\phi\mid^2\ \ d\mathcal{H}^{n-1}\right)\\
&+\tilde{c}\lambda_m^2\delta(1+\sup_{\partial B_t}\mid v_m-\phi\mid^2\delta^{-2})\int_{\partial B_t}\sum_{j=2}^{i}\sum_{l=1}^{j-1}\lambda_m^{2l-p}\mid\wedge_l A_m\mid^{p-2}\mid\wedge_l Dv_m\mid^2\ d\mathcal{H}^{n-1}\\
&+c\delta\int_{\partial B_t}\sum_{j=1}^{i}\mid\wedge_j A_m\mid^{p-2}\lambda_m^{2j-p}\mid\wedge_j Dv_m\mid^2\ d\mathcal{H}^{n-1}\\
&+\tilde{c}\lambda_m^p\delta(1+\sup_{\partial B_t}\mid v_m-\phi\mid^p\delta^{-p})\\
&+\tilde{c}\lambda_m^p\delta(1+\sup_{\partial B_t}\mid v_m-\phi\mid^p\delta^{-p})\int_{\partial B_t}\sum_{j=2}^{i}\sum_{l=1}^{j-1}\lambda_m^{p(l-1)}\mid\wedge_l Dv_m\mid^p\ \ d\mathcal{H}^{n-1}\\
&+c\delta\int_{\partial B_t}\sum_{j=1}^{i}\lambda_m^{p(j-1)}\mid\wedge_j Dv_m\mid^p\ \ d\mathcal{H}^{n-1}]:=\sum_{i=1}^{k}X_i^m,
\end{aligned}
$$

where we have also used Lemma 2.1 to estimate $\mid\wedge_j A_m\mid\leq c\mid\wedge_l A_m\mid$, $l\leq j$.
On the other hand, by (69)–(70) and the definition of $z_m$, it follows

$$
\begin{aligned}
&\lambda_m^{-p}\sum_{i=1}^{k}\left[\int_{B_t-B_{t-\delta}}\mid F_m^i(D\phi)\mid\ \ dz+\int_{B_{t-\delta/2}-B_{t-\delta}}\mid F_m^i(Dz_m)\mid\ \ dz\right] \qquad (74)\\
&\leq c\int_{B_t-B_{t-\delta}}(\mid A_m\mid\lambda_m^{-1})^{p-2}\mid D\phi\mid^2+\mid D\phi\mid^p\ dz\\
&\quad+c\sum_{i=2}^{k}\int_{B_t-B_{t-\delta}}\sum_{j=1}^{i}(\mid\wedge_j A_m\mid^{p-2}\lambda_m^{2j-p}\mid\wedge_j D\phi\mid^2+\lambda_m^{p(j-1)}\mid\wedge_j D\phi\mid^p)\ dz\\
&:=\sum_{i=1}^{k}Y_i^m.
\end{aligned}
$$

Using the minimality of $v_m$, the definition of $z_m$ and formulas (73) and (74) we finally have

$$
\begin{aligned}
&\lambda_m^{-p}(\mathcal{F}_m^t(v_m)-\mathcal{F}_m^t(\phi))\leq\lambda_m^{-p}(\mathcal{F}_m^t(z_m)-\mathcal{F}_m^t(\phi)) \qquad (75)\\
&\leq\lambda_m^{-p}\int_{B_t-B_{t-\delta}}\sum_{i=1}^{k}\mid F_m^i(Dz_m)-F_m^i(D\phi)\mid\ \ dz\leq c\sum_{i=1}^{k}X_i^m+Y_i^m.
\end{aligned}
$$

**Lower bound.** Now we want to find a lower bound for the quantity $\lambda_m^{-p}(\mathcal{F}_m^t(v_m)-\mathcal{F}_m^t(\phi))$.

We write

$$\lambda_m^{-p}(\mathcal{F}_m^t(v_m)-\mathcal{F}_m^t(\phi)) = \lambda_m^{-p}\int_{B_t}(F_m^1(Dv_m)-F_m^1(D\phi))\;dz + \lambda_m^{-p}\sum_{i=2}^k\int_{B_t}(F_m^i(Dv_m)-F_m^i(D\phi))\;dz\,. \tag{76}$$

We estimate from below the first integral appearing on the right hand side of (76)

$$\begin{aligned}
&\lambda_m^{-p}\int_{B_t}(F_m^1(Dv_m)-F_m^1(D\phi))\;dz = \lambda_m^{-p}\int_{B_t}DF_m^1(D\phi)(Dv_m-D\phi)\;dz\\
&\quad+\lambda_m^{-p}\int_{B_t}[\int_0^1(1-\tau)D^2F_m(D\phi+\tau(Dv_m-D\phi))\;d\tau]\cdot\\
&\qquad\cdot(Dv_m-D\phi)\otimes(Dv_m-D\phi)\;dz \qquad\qquad (77)\\
&\geq \lambda_m^{-p}c\int_{B_t}DF_m^1(D\phi)(Dv_m-D\phi)\;dz\\
&\quad+c\int_{B_t}\lambda_m^{2-p}|A_m+\lambda_mD\phi|^{p-2}|Dv_m-D\phi|^2+|Dv_m-D\phi|^p\;dz,
\end{aligned}$$

where we used the ellipticity of $D^2F^1$ stated in (H3) and lemma 8.1 from [12]. About the remaining terms in (76) we write (proceeding as in [21], page 1542), for $i\geq 2$:

$$\lambda_m^{-p}\int_{B_t}(F_m^i(Dv_m)-F_m^i(D\phi))\;dz = A_m^i+B_m^i+C_m^i,$$

where

$$\begin{aligned}
A_m^i &= \lambda_m^{-p}\int_{B_t}[\int_0^1(1-\tau)D^2F^i(\wedge_iA_m+\tau(\wedge_i(A_m+\lambda_mDv_m)-\wedge_iA_m))\;d\tau]\\
&[\wedge_i(A_m+\lambda_mDv_m)-\wedge_i(A_m+\lambda_mD\phi)]\cdot[\wedge_i(A_m+\lambda_mDv_m)-\wedge_i(A_m+\lambda_mD\phi)]\;dz,
\end{aligned}$$

$$\begin{aligned}
B_m^i &= 2\lambda_m^{-p}\int_{B_t}[\int_0^1(1-\tau)D^2F^i(\wedge_iA_m+\tau((\wedge_iA_m+\lambda_mDv_m)-\wedge_iA_m))\;d\tau]\\
&[\wedge_i(A_m+\lambda_mDv_m)-\wedge_i(A_m+\lambda_mD\phi)]\cdot[\wedge_i(A_m+\lambda_mD\phi)-\wedge_iA_m]\;dz,
\end{aligned}$$

$$\begin{aligned}
C_m^i &= \lambda_m^{-p}\int_{B_t}[\int_0^1(1-\tau)D^2F^i(\wedge_iA_m+\tau(\wedge_i(A_m+\lambda_mDv_m)-\wedge_iA_m))+\\
&\qquad -(1-\tau)D^2F^i(\wedge_iA_m+\tau(\wedge_i(A_m+\lambda_mD\phi)-\wedge_iA_m))\;d\tau]\\
&\qquad [\wedge_i(A_m+\lambda_mD\phi)-\wedge_iA_m]\cdot[\wedge_i(A_m+\lambda_mD\phi)-\wedge_iA_m]\;dz.
\end{aligned}$$

We now proceed to estimate from below the quantities $A, B$ and $C$ written above. Using (H4), the ellipticity of $D^2F^i$ and formula (14), a routine computation gives

$$A_m^i\geq c\int_{B_t}[(\mid\wedge_iA_m\mid\lambda_m^{-1})^{p-2}+\mid\sum_{j=1}^i\lambda_m^{j-1}\wedge_{i-j}A_m\odot\wedge_jDv_m\mid^{p-2}]\cdot \tag{78}$$

$$\cdot [| \sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot (\wedge_j Dv_m - \wedge_j D\phi) |^2] dz.$$

For future convenience we spread the quantities denoted by $B_m^i$ as follows

$$B_m^i = 2\lambda_m^{2-p} \int_{B_t} [\int_0^1 (1-\tau) D^2 F^i (\wedge_i A_m + \tau(\wedge_i (A_m + \lambda_m Dv_m) - \wedge_i A_m)) d\tau] \cdot$$

$$\cdot [(\wedge_{i-1} A_m \tilde{\odot} (Dv_m - D\phi)) + \sum_{j=2}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot (\wedge_j Dv_m - \wedge_j D\phi)] \cdot \qquad (79)$$

$$\cdot [\sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j D\phi] dz := 2D_m^i + 2E_m^i.$$

Finally, connecting (73)–(79), yields the next estimate from which we will later give the desired strong convergences in both cases:

$$\int_{B_t} \lambda_m^{2-p} |A_m + \lambda_m D\phi|^{p-2} |Dv_m - D\phi|^2 + |Dv_m - D\phi|^p \, dz \qquad (80)$$

$$+ \sum_{i=2}^{k} \int_{B_t} [(| \wedge_i A_m | \lambda_m^{-1})^{p-2} + | \sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m |^{p-2}] \cdot$$

$$\cdot | \sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot (\wedge_j Dv_m - \wedge_j D\phi) |^2 \, dz$$

$$\leq c\lambda_m^{-p} \int_{B_t} DF_m^1(D\phi)(D\phi - Dv_m) \, dz$$

$$+ c \sum_{i=1}^{k} (| C_m^i | + | D_m^i | + | E_m^i | + X_m^i + Y_m^i),$$

for any $0 \leq t < 1$ and $\phi \in C^\infty(B_1; \mathbb{R}^N)$.

## 9. Weak convergences turn into strong convergences

In this section we prove that the weak convergences stated in (35) are, up to not relabelled subsequences, actually strong as stated in (54) and (61), thus proving the claims of sections 6, 7. The proof of this fact will be achieved, once again, by distinguishing the two cases.

**(1) First case.**

We first prove that the right hand side of (80) tends to 0 for a suitable choice of the test function $\phi \equiv \phi_m$. Applying lemma 4.2 we have that for a.e. $t \in (1/2, 1)$ there exists $M_t < \infty$ and a not relabelled subsequence (also depending on $t$), such that

$$\int_{\partial B_t} [| Dw_m |^2 + | Dv_m |^p + \sum_{j=2}^{i} | \wedge_j A_m |^{p-2} \lambda_m^{2j-p} | \wedge_j Dv_m |^2 \qquad (81)$$

$$+\sum_{j=2}^{i} \lambda_m^{p(j-1)} \mid \wedge_j Dv_m \mid^p] \; d\mathcal{H}^{n-1} \leq M_t.$$

Now we put in (80) (recall that $w$ is a solution to a linear elliptic system with constant coefficients and hence is smooth inside $B_t$):

$$\phi \equiv \phi_m = (\mid A_m \mid \lambda_m^{-1})^{\frac{2-p}{2}} w.$$

We observe that, up to a not relabelled subsequence, we may suppose that $D(v_m - \phi_m) \rightharpoonup 0$ in $L^p(B_1)$. Indeed note that this sequence is bounded in $L^p(B_1)$ and by standard weak compactness arguments, up to a subsequence, $D(v_m - \phi_m) \rightharpoonup y$ in $L^p(B_1)$; then observe that $\mid v_m - \phi_m \mid^2 = (\mid A_m \mid \lambda_m^{-1})^{2-p} \mid w_m - w \mid^2$ and so $v_m - \phi_m \to 0$ in $L^2(B_1)$ that implies $y \equiv 0$.

Let us fix $0 < \epsilon \leq 1/1000$ and we note that by the previous observation and $(5.11)_5$ we may assume without loss of generality that:

$$\int_{\partial B_t} \mid v_m - \phi_m \mid^p + \mid w_m - w \mid^2 \; d\mathcal{H}^{n-1} \to 0\,.$$

Moreover by using Sobolev embedding Theorem together with the fact that $p > n-1$ and a simple argument based on Fubini's Theorem and the use of a countable set of test functions we get that, up to not relabelled subsequences, for almost every $t \in (0, 1-\epsilon)$:

$$\lim_m (\sup_{\partial B_t} \mid v_m - \phi_m \mid^p) = \lim_m (\sup_{\partial B_t} \mid v_m - \phi_m \mid^2) = 0. \tag{82}$$

With these choices of $\phi \equiv \phi_m$, keeping into account (81) and (73), it follows that

$$\begin{aligned} \mid X_m^i \mid \;\; &\leq \;\; c\delta + c\delta^{-1} \sup_{\partial B_t} \mid v_m - \phi_m \mid^2 + c\delta^{1-p} \sup_{\partial B_t} \mid v_m - \phi_m \mid^p \\ &\quad + c\delta^{-1} \int_{\partial B_t} \mid w_m - w \mid^2 \; d\mathcal{H}^{n-1} \end{aligned}$$

for some $c$ depending also on $M_t$ (by (81)) and on $\| \; Dw \; \|_{L^\infty} \leq c(M)$ (by (52) with $K = \overline{B_{1-\epsilon}}$). Letting first $m \to \infty$ and then $\delta \to 0$ and recalling (82), we get

$$X_m^i \to 0. \tag{83}$$

**Remark 9.1.** It is clear how the previous reasoning fails (when using Sobolev embedding theorem) in the bordeline case $p = n-1$, which is therefore excluded in this setting.

The terms $Y_m^i$ are easier to estimate. Indeed

$$\mid Y_m^i \mid \leq c \int_{B_t - B_{t-\delta}} \mid Dw \mid^2 \; dz + c(\mid A_m \mid \lambda_m^{-1})^{\frac{2-p}{2}}$$

and also this time we let first $m \to \infty$ and then $\delta \to 0$ to get

$$Y_m^i \to 0. \tag{84}$$

We estimate now the remaining terms starting with $C_m^i$.

From formula (14) and the fact that $w$ is smooth it follows that

$$| \wedge_i(A_m + \lambda_m Dv_m) - \wedge_i(A_m + \lambda_m D\phi_m) | \leq c\lambda_m[1 + \sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m |], \quad (85)$$

$$| \wedge_i(A_m + \lambda_m D\phi) - \wedge_i A_m | \leq c\lambda_m(| A_m | \lambda_m^{-1})^{\frac{2-p}{2}}. \quad (86)$$

From (H5) and (85)–(86) we get

$$| C_m^i | \leq c \int_{B_t} (| A_m | \lambda_m^{-1})^{-\delta}[1 + \sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m |]^{p-2} \, dz \to 0, \quad (87)$$

by $(35)_7$ and (38).

Now, in order to estimate $D_m^i$ we put

$$\begin{aligned} f_m^1 &= \lambda_m^{2-p} \int_0^1 (1-\tau)[D^2F^i(\wedge_i A_m + \tau(\wedge_i(A_m + \lambda_m Dv_m) - \wedge_i A_m)) \quad (88) \\ &\qquad -D^2F^i(\wedge_i A_m)] \, d\tau, \\ f_m^2 &= \lambda_m^{2-p} D^2F^i(\wedge_i A_m), \\ g_m &= (\wedge_{i-1} A_m \tilde{\odot}(Dv_m - D\phi_m))(\sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j D\phi_m), \end{aligned}$$

and also this time note that

$$D_m^i = \int_{B_t} f_m^1 g_m \, dz + \int_{B_t} f_m^2 g_m \, dz.$$

Then by (H5), formula (14) and Young inequality

$$\begin{aligned} &\int_{B_t} | f_m^1 g_m | \, dz \\ &\leq c(| A_m | \lambda_m^{-1})^{\frac{2-p}{2}} \int_{B_t} (| \sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m ||^p + | Dv_m - D\phi_m |^{\frac{p}{2}}) \, dz \\ &\quad + c(| A_m | \lambda_m^{-1})^{-\delta} \int_{B_t} (| \sum_{j=1}^{i} \lambda_m^{j-1} | \wedge_j Dv_m ||^{2\delta} + | Dw_m - Dw |^2) \, dz \to 0, \end{aligned}$$

by (35) (observe that $2\delta \leq p$). Furthermore we note that (also look at Remark 6.1)

$$\begin{aligned} &\int_{B_t} f_m^2 g_m \, dz = (| \wedge_i A_m |^{-1} | A_m |)^{2-p} (| \wedge_i A_m |^{2-p} \, D^2F^i(\wedge_i A_m)) \cdot \\ &\cdot \int_{B_t} (\wedge_{i-1} A_m \tilde{\odot}(Dw_m - Dw))(\sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_{j-1} D\phi_m \tilde{\odot} Dw) \, dz \to 0 \end{aligned}$$

by (H4), (43), lemma 2.1 and the weak convergence of $w_m$ (see also remark 6.1 to get rid of ambiguities). So we finally have that

$$D^i_m \to 0. \tag{89}$$

In order to estimate $E^i_m$ we distinguish two cases as done in section 6 to estimate the $B^i_m$ terms. If $i > \bar{k}$, where $\bar{k}$ is as in (40), then using growth condition (H4) we get

$$\mid E^i_m \mid \leq c(\mid A_m \mid^{-1} \mid \wedge_i A_m \mid)^{\frac{p-2}{2}} \int_{B_t} (\sum_{j=1}^{i} \lambda_m^{j-\frac{p}{2}} \mid \wedge_j A_m \mid^{\frac{p}{2}-1} \mid \wedge_j Dv_m \mid +1)\ dz$$
$$+c(\mid A_m \mid \lambda_m^{-1})^{\frac{2-p}{2}} \int_{B_t} (\sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid)^{p-1}\ dz \to 0,$$

by (35), (38) and (41).

If $i \leq \bar{k}$ then we put, as before

$$\begin{aligned} f^1_m &= \lambda_m^{2-p} \int_0^1 (1-\tau)[D^2F^i(\wedge_i A_m + \tau(\wedge_i(A_m + \lambda_m Dv_m) - \wedge_i A_m)) \\ &\qquad -D^2F^i(\wedge_i A_m)]\ d\tau, \\ f^2_m &= \lambda_m^{2-p} D^2F^i(\wedge_i A_m), \\ g_m &= [\sum_{j=2}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot (\wedge_j Dv_m - \wedge_j D\phi_m)][\sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j D\phi_m]. \end{aligned}$$

By (H5) and Young's inequality ($2\delta \leq p$) we obtain (recalling (42))

$$\int_{B_t} \mid f^1_m g_m \mid\ dz \leq c \mid\mid \wedge_i A_m \mid^{-1} \lambda_m \mid^{\delta} \int_{B_t} [(\sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid)^{2\delta}$$
$$+\sum_{j=1}^{i} \lambda_m^{2j-p} \mid \wedge_j A_m \mid^{p-2} \mid \wedge_j Dv_m \mid^2 +1]\ dz$$
$$+c(\mid A_m \mid^{-1} \lambda_m)^{\frac{p-2}{2}} \int_{B_t} (\sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid)^{p-1}\ dz \to 0,$$

by (35) and the fact that $i \leq \bar{k}$. Furthermore, with $\tilde{g}$ denoting a smooth, bounded $\wedge_i H^N_n$-valued function:

$$\mid \int_{B_t} f^2_m g_m\ dz \mid \leq c \mid\mid \wedge_i A_m \mid^{2-p} D^2F^i(\wedge_i A_m) \cdot$$
$$\cdot \int_{B_t} [(\sum_{j=2}^{i} \lambda_m^{j-\frac{p}{2}} \mid \wedge_j A_m \mid^{\frac{p}{2}-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m) \cdot$$
$$\cdot(\sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dw) + (\mid A_m \mid^{-1} \lambda_m)^{\frac{p-2}{2}} \tilde{g}]\ dz \mid \to 0$$

by the weak convergence in $(35)_6$, (43), (H4), the fact that $w$ is smooth and lemma 2.1 used once again to estimate $\mid \wedge_i A_m \mid \leq c \mid \wedge_j A_m \mid$, $c \equiv c(M)$. So we have proved that

$$E_m^i \to 0. \tag{90}$$

It remains to prove that also the first term in the right hand side of (80) tends to 0:

$$\begin{aligned}
&\lambda_m^{-p} \int_{B_t} DF_m^1(D\phi_m)(D\phi_m - Dv_m)\ dz \\
&= \int_{B_t} [\lambda_m^{-p}(\mid A_m \mid \lambda_m^{-1})^{2-p} \int_0^1 D^2 F_m^1(\tau D\phi_m)\ d\tau] Dw \otimes (Dw - Dw_m)\ dz \qquad (91)\\
&= \int_{B_t} [\mid A_m \mid^{2-p} \int_0^1 D^2 F^1(A_m + \tau\lambda_m D\phi_m)\ d\tau] Dw \otimes (Dw - Dw_m)\ dz \\
&\to 0.
\end{aligned}$$

Indeed, arguing as for the estimate of the term $I_m$ in section 6 and using (52), the quantity in square brackets in (91) is easily seen to converge strongly (in $L^\infty$) to a constant tensor while $w_m$ weakly converges (in $L^2$) to $w$. In this way also this term tends to zero and connecting together (83)–(91), all the terms in the right hand side of (80) tend to 0.

We now turn our attention to the left hand side of (80). The first integral is easily seen to control the following quantity:

$$\int_{B_t} |Dw_m - Dw|^2 + |Dv_m|^p\ dz - c(|A_m|\lambda_m^{-1})^{\frac{\delta(2-p)}{2}} \int_{B_t} (1 + |Dv_m|^2)\ dz$$

with $c \equiv c(||Dw||_{L^\infty}) \equiv c(M)$. Keeping into account $(35)_2$ and (38) it immediately follows from (83)–(91) that

$$\begin{aligned}
&Dw_m \to Dw \text{ strongly in } L^2(B_t), \qquad (92)\\
&Dv_m \to 0 \text{ strongly in } L^p(B_t),
\end{aligned}$$

for any $t < 1 - \epsilon$.

We now show that

$$\int_{B_t} \mid \wedge_i A_m \mid^{p-2} \lambda_m^{2i-p} \mid \wedge_i Dv_m \mid^2\ \ dz \to 0, \tag{93}$$

$$\int_{B_t} \lambda_m^{p(i-1)} \mid \wedge_i Dv_m \mid^p\ \ dz \to 0 \tag{94}$$

for any $2 \leq i \leq k$, $t < 1 - \epsilon$, thus completing (in view of the arbitrarieness of $\epsilon$ and of a standard diagonalization argument) the proof of $(54)_3$ and $(54)_4$. Using the triangle inequality in (80) and (35) together with the elementary estimate

$$\mid \wedge_j D\phi_m| \leq c(\mid A_m \mid \lambda_m^{-1})^{\frac{j(2-p)}{2}} \to 0$$

we easily get

$$\limsup_m \int_{B_t} (\mid \wedge_i A_m \mid \lambda_m^{-1})^{p-2} \mid \wedge_{i-1} A_m \tilde{\odot} (Dv_m - D\phi_m) \tag{95}$$

$$+\sum_{j=2}^{i-1}\lambda_m^{j-1}\wedge_{i-j}A_m\odot\wedge_j Dv_m+\lambda_m^{i-1}\wedge_i Dv_m\mid^2\ dz=0,$$

$$\limsup_m\int_{B_t}\mid\sum_{j=1}^{i-1}\lambda_m^{j-1}\wedge_{i-j}A_m\odot\wedge_j Dv_m+\lambda_m^{i-1}\wedge_i Dv_m\mid^p\ dz=0\,, \tag{96}$$

for any $2\leq i\leq k$. Now (93) can be inductively deduced from (95). Indeed we write (95) for $i=2$ and then use triangle inequality and (92) to get (93); then we proceed inductively, writing (95) for a general $i<k$ and using (93) for $j<i$ and the triangle inequality to get also this case. A similar inductive argument can be used also to derive (94) from (96). So the strong convergences as claimed in section 6 are proved.

**(2) Second case.**

This time we choose a sequence of smooth test functions in (80), $\{\phi_s\}_{s\in\mathbb{N}}$, in such a way that $\phi_s\to v$ strongly in $W^{1,p}(B_1;\mathbb{R}^n)$. In this way, (note that we have not proved yet that $v$ is smooth, since, in contrast to the first case, we emploied the claimed strong convergences to prove that $v(z)+l\tilde{A}z$ is a $p$-harmonic mapping) we a priori have no control on the sequence $\{\|\ D\phi_s\ \|_\infty\}_{s\in\mathbb{N}}$, which may turn out to be unbounded, thus making the constant $\tilde{c}$ in (80) blow up; so we have to carefully analyze the various terms in (80). We keep the same notation for $\epsilon$ fixed for the first case. Note that, up to subsequences we may always suppose as in the first case (by Sobolev embedding Theorem), that, for a.e. $t\in(1/2,1-\epsilon)$

$$\lim_m\mid v_m-\phi_s\mid=\mid v-\phi_s\mid\ \text{ in } L^\infty(\partial B_t)\,,$$

$$\lim_s(\sup_{\partial B_t}\mid v-\phi_s\mid)=0\,,$$

$$\lim_s\int_{\partial B_t}\mid Dv-D\phi_s\mid^p+\mid Dv-D\phi_s\mid^2\ d\mathcal{H}^{n-1}=0\,.$$

As far as the $X_i$ terms are concerned (look at (73)), we note that, having fixed $\phi_s$, every product containing $\tilde{c}$, contains also a power of $\lambda_m$. So we fix $\phi_s$ and let $m\to\infty$, thus getting, by (81) (which, of course, can be supposed to be in force also in this case)

$$\begin{aligned}\limsup_m\mid X_m^i\mid\leq c\delta[1+\int_{\partial B_t}\mid D\phi_s\mid^2\ d\mathcal{H}^{n-1}+\int_{\partial B_t}\mid D\phi_s\mid^p\ d\mathcal{H}^{n-1}\\ +\delta^{-2}\sup_{\partial B_t}\mid v-\phi_s\mid^2+\delta^{-p}\sup_{\partial B_t}\mid v-\phi_s\mid^p],\end{aligned}\tag{97}$$

with $c$ independent of $\|\ D\phi_s\ \|_\infty$.

For $Y_m^i$ we have

$$\begin{aligned}\mid Y_m^i\mid\leq c\int_{B_t-B_{t-\delta}}[(\mid A_m\mid\lambda_m^{-1})^{p-2}\mid D\phi_s\mid^2+\mid D\phi_s\mid^p]\ dz\\ +c\sum_{j=1}^{i}\int_{B_t-B_{t-\delta}}(\mid A_m\mid\lambda_m^{-1})^{p-2}\lambda_m^{2j-2}\mid\wedge_j D\phi_s\mid^2+\lambda_m^{p(j-1)}\mid\wedge_j D\phi_s\mid^p\ dz,\end{aligned}$$

by (74) and Lemma 2.1.

Letting $m \to \infty$, by (59) we have

$$\limsup_m \mid Y_m^i \mid \leq c \int_{B_t - B_{t-\delta}} \mid D\phi_s \mid^2 \ dz + c \int_{B_t - B_{t-\delta}} \mid D\phi_s \mid^p \ dz. \tag{98}$$

Now we proceed estimating $D_m^i, E_m^i$ and $C_m^i$.

Using growth conditions (H4), formula (14), and keeping into account (79),

$$\begin{aligned} \mid D_m^i \mid + \mid E_m^i \mid &\leq \tilde{c}[\sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_{i-j} A_m \mid] \int_{B_t} [(\mid A_m \mid \lambda_m^{-1})^{p-2} \\ &\quad + \mid \sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid\mid^{p-2}][1 + \sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid] \ dz \\ &\leq \tilde{c}[\sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_{i-j} A_m \mid] \int_{B_t} (1 + \sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_j Dv_m \mid)^{p-1} \ dz, \end{aligned}$$

where $\tilde{c}$ essentially depends on $\mid\mid D\phi_s \mid\mid_\infty$ and $l$. Now observing that in the second case $\mid \wedge_l A_m \mid \to 0$ for any $l \leq k$ (see (59)), and that here $i \geq 2$ we have that

$$\limsup_m \mid D_m^i \mid + \mid E_m^i \mid = 0. \tag{99}$$

A similar argument works also for $C_m^i$ and yields

$$\limsup_m \mid C_m^i \mid = 0. \tag{100}$$

Note that both (97) and (98) are valid for any given $\phi_s$. Also in this case we have that

$$\begin{aligned} &\int_{B_t} \lambda_m^{-p} DF_m^1(D\phi_s)(D\phi_s - Dv_m) \ dz \\ &= \int_{B_t} \lambda_m^{1-p}[DF^1(A_m + \lambda_m D\phi_s) - DF^1(A_m)](D\phi_s - Dv_m) \ dz \end{aligned}$$

converges to

$$\begin{aligned} &L \int_{B_t} [\mid l\bar{A} + D\phi_s \mid^{p-2} (l\bar{A} + D\phi_s) + l^{p-1}\bar{A}](D\phi_s - Dv) \ dz, \\ &\leq c \left( \int_{B_1} 1 + |D\phi_s|^p \ dx \right)^{\frac{p-1}{p}} ||D\phi_s - Dv||_{L^p(B_t)} \end{aligned} \tag{101}$$

where we have used (H6) and (58) (also look at section 7).

In the left hand side of (80) we have

$$\int_{B_t} [(\mid \wedge_i A_m \mid \lambda_m^{-1})^{p-2} + \mid \sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m \mid^{p-2}] \cdot$$

$$\cdot[\sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j D\phi_s]^2 \; dz \tag{102}$$

$$\leq \tilde{c}(\sum_{j=1}^{i} \lambda_m^{j-1} \mid \wedge_{i-j} A_m \mid)^2 \to 0,$$

by $(35)_7$ and (59). As in the first case and roughly estimating $\mid Dv_m - Dv \mid \leq \mid Dv_m - D\phi_s \mid + \mid Dv - D\phi_s \mid$ we find that the first integral in (80) controls the following quantity:

$$\int_{B_t} |Dv_m - Dv|^p \; dz - c \int_{B_t} |D\phi_s - Dv|^p \; dz \, .$$

So collecting (97)–(102) and letting first $m \to \infty$ and then $s \to \infty$, we finally obtain by the triangle inequality

$$\limsup_m \int_{B_t} \mid Dv_m - Dv \mid^p \; dz$$

$$+ \limsup_m \sum_{i=2}^{k} \int_{B_t} [(\mid \wedge_i A_m \mid \lambda_m^{-1})^{p-2} + \mid \sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m \mid^{p-2}] \cdot$$

$$\cdot[\sum_{j=1}^{i} \lambda_m^{j-1} \wedge_{i-j} A_m \odot \wedge_j Dv_m]^2 \; dz$$

$$\leq c\delta[1 + \int_{\partial B_t} \mid Dv \mid^2 \; d\mathcal{H}^{n-1} + \int_{\partial B_t} \mid Dv \mid^p \; d\mathcal{H}^{n-1}]$$

$$+ c \int_{B_t - B_{t-\delta}} \mid Dv \mid^2 \; dz + c \int_{B_t - B_{t-\delta}} \mid Dv \mid^p \; dz.$$

Letting $\delta \to 0$ and arguing by induction as done for the first case, we can finally prove the strong convergences as stated in (61). This completes the proof of theorem 5.2.

## 10. Proof of the Main Theorem

In this section we finally prove the Main Theorem stated in section 5. The proof rests on a standard iteration argument involving $U(x, r)$, essentially based on Theorem 5.2.

**Lemma 10.1.** *Let $0 < \alpha < 1$ and $M > 0$, then there exists $0 < \tau < \frac{1}{2}$ and $\epsilon > 0$, both depending on $\alpha$ and $M$, such that if:*

$$B(x, r) \subset \Omega, \qquad \mid (Du)_{x,r} \mid \leq M,$$

$$U(x, r) \leq \epsilon,$$

*then*

$$U(x, \tau^l r) \leq (\tau^l)^{\mu\alpha} U(x, r), \tag{103}$$

*for each $l \in \mathbb{N}$ and $\mu$ has been introduced in* Theorem 4.1.

**Proof.** Just follow Lemma 6.1 from [21], iterating Theorem 5.2 and adapting the proof to the different structure of $U(x, r)$ and the different statement of Theorem 5.2 which involves the exponent $\mu$ rather than the exponent 2. □

Before ending, we still need a result from [21].

**Lemma 10.2.** *Let $u \in \wedge_k W^{1,p}(\Omega)$, then*

$$\lim_{r\to 0} \fint_{B_{x,r}} | \wedge_i (Du - (Du)_{x,r}) |^p \ dx = 0$$

*for almost every $x \in \Omega$, $1 \le i \le k$.*

The proof of the Main Theorem is now a consequence of a standard iteration procedure based on Lemma 10.1 (see for example [12] or [24]).

The singular set turns out to be contained in the complement of:

$$\begin{aligned} \Omega_0 \ &= \ \{x \in \Omega \mid \ \lim_{r\to 0} (Du)_{x,r} = Du(x) \text{ and} \\ &\lim_{r\to 0} \fint_{B_{x,r}} | \wedge_i (Du - (Du)_{x,r}) |^p \ dx = 0 \ \text{ for } i = 1, ..., k\}, \end{aligned}$$

so that by Lemma 10.2 we have

$$| \Omega - \Omega_0 | = 0.$$

We finally remark that from (103) and Campanato's integral characterization of Hölder continuity it easily follows that the Hölder exponent of $Du$ is at least $\frac{\mu}{p}$, where $\mu$ is the exponent provided by Theorem 4.1.

## 11. Final Remarks

As we have mentioned in the introduction, one of the main reasons for introducing (and studying) polyconvex integral functionals is that they play a central role in the theory of (hyper) elastic materials, that is those whose equilibrium configurations are found by minimizing a stored energy functional:

$$\int_\Omega W(Du) + g(x, u) \ dx \tag{104}$$

where $\Omega \subset \mathbb{R}^3$ and $W : \mathbb{R}^9 \to \mathbb{R}^+$ is the energy density; this function is usually assumend to be polyconvex (see [7]). In this case a basic requirement in Nonlinear Elasticity is the blow-up condition:

$$\begin{array}{lll} W(z) \to +\infty & \text{if} & \det z \to 0 \\ W(z) = +\infty & \text{if} & \det z \le 0 \end{array} \tag{105}$$

that is designed in order to prevent the interpenetration of the matter.

It is clear that neither our result nor the ones developed in previous papers (see [12, 21, 22, 16, 17]) cover this important case; this remains a major problem of the issue.

Needless to say, the aim of this paper is more modest; we intend to show how it is possible to treat, in the context of polyconvex energy densities with polynomial growth, degenerate integrals of the type:

$$\int_\Omega |Du|^p + W(Du) \ dx \tag{106}$$

$W$ being a degenerate polyconvex function in the sense described in section 5. With this respect we hope that some of the methods presented here will be useful also when new techniques will be developed in order to attack the problem of regularity under the realistic condition (105).

In anyway we would like to mention that degenerate polyconvex energies of the type (106), that is with a convex leading part that is degenerate elliptic in a $p$-laplacian fashion (see hypothesis (H6), section 5), have been considered in several papers in the last years, in connection with problems from nonlinear elasticity (see for instance [8, 36] and related references).

Finally we conclude the section by listing possible extensions to our results that can be easily worked out with a few modifications to the techniques presented here.

Of course it is possible to consider $k < \min\{n, N\}$ (see (25)). It is also possible to consider anisotropic growth conditions of the functions $F_i$, (assigned, in view of the growth condition (H4), throught the second derivatives $D^2 F_i$) leading to an inequality of the type:

$$0 \leq D^2 F^i(z) \leq c \mid z \mid^{p_i - 2}, \qquad z \in \wedge_i \mathrm{H}_n^N, \qquad 2 \leq i \tag{107}$$

for a suitable choice of exponents $p_i$. In this case the model would be:

$$\int_\Omega |Du|^p + \sum_{i=2}^{k} | \wedge_i Du|^{p_i} \, dx \, .$$

The choice in (107) eventually leads (at least to determine existence theorems as in section 3) to considering anisotropic spaces of the type $\wedge_i W^{1,p}$ defined by the condition $\wedge_i Du \in L^{p_i}$.

It is also possible to consider non-splitting functionals of the type:

$$\int_\Omega F^1(Du) + \tilde{F}(\wedge_2 Du, \wedge_3 Du, \ldots, \wedge_k Du) \, dx$$

with $F^1$ being as in section 5 and suitable growth conditions imposed on $D^2 \tilde{F}$. The proof for this case involves a linearization procedure more delicate than the one presented in section 5.

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