Journal of Convex Analysis
Volume 8 (2001), No. 2, 409–415

When Can Points in Convex Sets be Separated by Affine Maps?

Reinhard Börger
Fachbereich Mathematik, FernUniversität - Gesamthochschule in Hagen, 58084 Hagen, Germany
reinhard.boerger@fernuni-hagen.de

Holger P. Petersson in Anerkennung und Dankbarkeit zum 60. Geburtstag gewidmet

Received December 8, 1999

For a class $\mathcal{A}$ of convex sets in (not necessarily finite-dimensional) real vector spaces, let $\mathrm{Sep}\,\mathcal{A}$ denote the class of all convex sets C such that the affine maps from C to elements of $\mathcal{A}$ separate points. If we restrict our attention to finite-dimensional convex sets, there are only four possibilities for $\mathrm{Sep}_{\mathrm{f}}\mathcal{A} := \mathrm{Sep}\mathcal{A} \cap \{\mathcal{C} : \mathcal{C}$ finite-dimensional convex set$\}$. Similarly, restriction to absolutely convex sets yields only three possibilities. In the general case, there are many possibilities for $\mathrm{Sep}\mathcal{A}$, at least as many as cardinals. In particular, there is no line-free convex set C such that for all linearly bounded convex sets D the affine maps from D to C separate points.

Keywords: (absolutely) convex set, (absolutely) affine map, linearly bounded convex set, line-free convex set

1991 Mathematics Subject Classification: 52A01, 52A05, 04A40, 18A99

1. Introduction

In this paper we shall study questions of the following type: Which convex sets D have the property that all affine maps from D to elements of a given class $\mathcal{A}$ of convex sets separate points of D? For which classes $\mathcal{A}$ of convex sets does there exist a $C \in \mathcal{A}$ such that for each $D \in \mathcal{A}$ the affine maps $D \to C$ separate points? It will turn out that a fairly easy complete answer can be obtained if we restrict the question either to finite-dimensional convex sets or to absolutely convex sets and zero-preserving affine maps. Categorically speaking, we are interested in regular-epireflective subcategories and cogenerators.

Throughout this paper, by a *convex set* we mean a (possibly empty) convex subset of a (not necessarily finite-dimensional) real vector space. For convex sets $D \subset V$, $C \subset W$ (in vector spaces V, W), a map $f : D \to C$ is called *affine* if $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$ holds for all $x, y \in D$, $\lambda \in [0,1]$. Obviously, f is affine if and only if there exist a $c \in C$ and a linear map $l : D \to C$ with $f(x) = c + l(x)$ for all $x \in C$. The "only if" part is obvious. The "if" part follows by direct construction if D generates V as a vector space; otherwise we extend a linear map by Zorn's lemma. Observe that the category of convex sets (and affine maps) is the same as the category of *pre-separated convex modules* in the sense of Pumplün and Röhrl [3], who describe the structure of a convex sets intrinsically (without reference to a vector space containing them).

ISSN 0944-6532 / \$ 2.50

If $\mathcal{A}$ is a class of convex sets, we denote by $\operatorname{Sep}\mathcal{A}$ the class of all convex sets D such that the affine maps from D to elements of $\mathcal{A}$ *separate points*, i.e. for all $x, y \in D$ with $x \neq y$ there exist a $C \in \mathcal{A}$ and an affine map $f : D \to C$ with $f(x) \neq f(y)$. Equivalently, $\operatorname{Sep}\mathcal{A}$ is the class of all convex sets which can by embedded affinely into a (cartesian) product of elements of $\mathcal{A}$. The product of convex sets can be formed in the canonical way as a convex subset of the product of the containing vector spaces. We call $\mathcal{A}$ *separation closed* if $\mathcal{A} = \operatorname{Sep}\mathcal{A}$ holds. For arbitrary $\mathcal{A}$, we easily see $\operatorname{Sep}\operatorname{Sep}\mathcal{A} = \operatorname{Sep}\mathcal{A}$ and $\operatorname{Sep}\mathcal{A}$ is the smallest separation closed class containing $\mathcal{A}$.

Our main result 4.2 is an improvement of Theorem 2.4 of [1].

2. The finite-dimensional situation

At first we shall study *finite-dimensional* (f.d.) convex sets, i.e. convex sets in finite-dimensional vector spaces. For a class $\mathcal{A}$ of finite-dimensional convex sets, let $\operatorname{Sep}_{\mathrm{f}}\mathcal{A}$ denote the class of all f.d. convex sets C such that the affine maps from C to elements of $\mathcal{A}$ separate points. We shall see that the finite-dimensional situation is much simpler than the general case.

Proposition 2.1. *There are only four classes, $\mathcal{A}$ with $\operatorname{Sep}_{\mathrm{f}} = \mathcal{A}$, namely*

(i) *the class of all convex sets with at most one point,*
(ii) *the class of all bounded f.d. convex sets,*
(iii) *the class of all line-free f.d. convex sets,*
(iv) *the class of all f.d. convex sets.*

Proof. "$\Leftarrow$" We show $\operatorname{Sep}_{\mathrm{f}}\mathcal{A} = \mathcal{A}$ in the cases (i)–(iv). The case (i) is trivial. Now consider a $C \subset \mathbb{R}^n$ such that the affine maps from C to bounded f.d. convex sets separate points. Since C is f.d. there exist finitely many maps from C to bounded f.d. convex sets jointly separating points. Since the coordinate functions on a f.d. set separate points and assume only values in an interval and since all finite-dimensional intervals can be affinely embedded into $[0,1] \subset \mathbb{R}$, there exists an injective affine map from C to $[0,1]^m$ for some $m \in \mathbb{N}$. Since $[0,1]^m \subset \mathbb{R}^m$ is bounded, C is affinely isomorphic to a bounded subset of $\mathbb{R}^n$; thus C is bounded, proving (ii).

Now assume that all affine maps from a convex set $D \subset \mathbb{R}^n$ to line-free convex sets separate points. Then we have to show that D is line-free, i.e. D does not contain a line, i.e. a convex subset affinely isomorphic to $\mathbb{R}$. Assume the contrary. Then there exists an injective affine map $l : \mathbb{R} \to D$; in particular we have $l(0) \neq l(1)$. Since the maps from D to line-free convex sets separate points, there exists an affine map $f : D \to C$ into a line-free convex set C with $f \circ l(0) \neq f \circ l(1)$. But each affine map with domain $\mathbb{R}$ is either injective or constant. Since $f \circ l : \mathbb{R} \to C$ is not constant, it must be injective, contradicting our hypothesis. This settles case (iii). Case (iv) is trivial.

"$\Rightarrow$" Now assume $\operatorname{Sep}_{\mathrm{f}}\mathcal{A} = \mathcal{A}$. Obviously, all convex sets with at most one point belong to $\mathcal{A}$. If there are no more sets in $\mathcal{A}$, we are in case (i) and hence finished.

So assume that $\mathcal{A}$ contains at least one set C with two distinct points a, b. Then the map $[0,1] \to C$, $\lambda \mapsto \lambda a + (1-\lambda)b$ is injective, and thus we have $[0,1] \in \mathcal{A}$. Since the coordinate functions on a bounded convex set $D \subset \mathbb{R}^n$ separate points and are bounded, the maps $D \to [0,1]$ separate points, and we obtain $D \in \mathcal{A}$. Thus $\mathcal{A}$ contains all bounded convex sets. If there are no more convex sets in $\mathcal{A}$, we are in case (ii) and hence finished.

Now assume that $\mathcal{A}$ contains at least one unbounded set $C \subset \mathbb{R}^n$. Then from Proposition III, 2.2.3 (p. 109) of [3] we see that its closure $\overline{C}$ contains a ray, i.e. a subset which is affinely isomorphic to $\mathbb{R}^+ := \{\xi \in \mathbb{R} : \xi \geq 0\}$. Thus there are $a, b \in \mathbb{R}^n$ with $b \neq 0$ and $a + \xi b \in \overline{C}$ for all $\xi \in \mathbb{R}^+$. From Theorem III, 2.1.3 (p.103) of [2] or Theorem 1.1.12 (p.7) of [4] we see that $\overline{C}$ has an inner point c. Now from Theorem III, 2.1.6 (p.104) of [2] we see that $\frac{1}{2}(a+c) + \xi b = \frac{1}{2}c + \frac{1}{2}(a + 2\xi b) \in C$ for all $\xi \in \mathbb{R}^+$. Thus $\xi \mapsto \frac{1}{2}(a+c) + \xi b$ yields an affine embedding from $\mathbb{R}^+$ to C. Hence we have $\mathbb{R}^+ \in \mathcal{A}$, and we shall show that for each line-free convex set $D \subset \mathbb{R}^n$ the affine maps $D \to \mathbb{R}^+$ separate points.

By the same argument as above we see that $\overline{D}$ is also line-free. Thus without loss of generality we may assume that $D \subset \mathbb{R}^n$ is closed. Then D is an intersection of half-spaces (cf.[2], Theorem III, 4.1.1, p.121) or Theorem 1.3.4 (p.12) or Corollary 1.3.5 (p.13) of [4].

Now assume $x, y \in D,\ x \neq y$. Since D is line-free by hypothesis, D does not contain the line L connecting x and y. Thus there is a $z \in L$ with $z \notin D$. Since D is an intersection of half spaces, there exists a linear map $f : D \to \mathbb{R}$ and an $\alpha \in \mathbb{R}$ with $f(u) \geq \alpha$ for all $u \in C$ and $f(z) < \alpha$. In particular, we have $f(x) \geq \alpha > f(z)$ because $x \in D$, and since $x, z \in L$ we see that f is not constant on L. But then f must be injective on L; in particular we get $f(x) \neq f(y)$. Therefore the affine map $= D \to \mathbb{R}^+,\ u \mapsto f(a) - \alpha$ separates x and y. This shows that the affine maps from D to $\mathbb{R}^+$ separate points, proving $D \in \mathcal{A}$. Thus we have shown that $\mathcal{A}$ contains all line-free f.d. convex sets. If D contains no more sets, we are in case (iii) and therefore finished.

Finally, assume that $\mathcal{A}$ contains at least one convex set C which is not line-free. Then there exists an injective affine map from $\mathbb{R}$ to C, and we conclude $\mathbb{R} \in \mathcal{A}$. Since the coordinate maps $\mathbb{R}^n \to \mathbb{R}$ separate points, we obtain $\mathbb{R}^n \in \mathcal{A}$ for all $n \in \mathbb{N}$, and thus all convex subsets of $\mathbb{R}^n$ are also in $\mathcal{A}$. Thus we are in case (iv) and hence finished. □

For an arbitrary class $\mathcal{A}$ of f.d. convex sets, $\mathrm{Sep}_\mathrm{f}\,\mathcal{A}$ is the smallest class containing $\mathcal{A}$ and closed under the operator Sep_f. For a single f.d. convex set $C, \mathrm{Sep}_\mathrm{f}\{C\}$ is the class of all f.d. convex sets D such that the affine maps from D to C separate points. So from 3.1 we immediately obtain the following

Corollary 2.2. *For a f.d. convex set C,*
$\mathrm{Sep}_\mathrm{f}\{C\}$ *is:*

(i) *the class of all f.d. convex sets D with at most one point if C has at most one point,*
(ii) *the class of all bounded f.d. convex sets if C is bounded but contains at least two points,*
(iii) *the class of all line-free f.d. convex sets if C is unbounded but line-free,*
(iv) *the class of all f.d. convex sets if C contains a line.*

3. Absolutely Convex Sets

The aim of this section is to study the situation for *absolutely convex sets*, i.e. convex sets C with $0 \in C$ and $-x \in C$ for all $x \in C$. The results of this section are also valid for absolutely convex sets over the field $\mathbb{C}$ of complex numbers, i.e. convex sets C with $0 \in C$ and $\alpha x \in C$ for all $x \in C,\ \alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. For real (or complex) vector spaces V, W

and absolutely convex sets $D \subset V$, $C \subset W$ we call a map $f : D \to C$ *absolutely affine* if it is a restriction of a linear map from V to W. A map between real absolutely convex sets is absolutely affine if and only if it is affine and preserves the zero; in the complex situation, $f : D \to C$ is absolutely affine if and only if f is absolutely affine as a map between real absolutely convex sets and if moreover $f(ix) = if(x)$ holds for all $x \in D$. For a class $\mathcal{A}$ of absolutely convex sets, we denote by $\mathrm{Sep_a}\,\mathcal{A}$ the class of all absolutely convex sets D such that the absolutely affine maps from D to elements of $\mathcal{A}$ separate points. But observe that point separation by affine maps is sufficient for $D \in \mathrm{Sep_a}\,\mathcal{A}$. Indeed, if $f : D \to C$ is affine with $f(a) \neq f(b)$ for some $a, b \in D$, then the map $D \to C$, $x \mapsto \frac{1}{2}f(x) - \frac{1}{2}f(0)$ is absolutely affine over $\mathbb{R}$ and separates a and b. In the complex case, separation by absolutely convex spaces suffices because 3.1 below holds over both $\mathbb{R}$ and $\mathbb{C}$.

We need an infinite-dimensional analogue of boundedness. An arbitrary subset C of a real vector space is called linearly bounded if its intersection with each line is bounded (as a subset of the line). Since all convex subsets of $\mathbb{R}$ are intervals, a convex sets C is *linearly bounded* if and only if C contains no *ray*, i.e. no subset, which is affinely isomorphic to $\mathbb{R}^+$. This is equivalent to saying that each affine map from $\mathbb{R}^+$ to C is constant or – equivalently – that for all vectors a, b with $a+\xi b \in C$ for all $\xi \in \mathbb{R}^+$ it follows that $b = 0$. An absolutely convex set is already linearly bounded if it does not contain a vector subspace $\neq 0$. Indeed, if $a + \xi b \in C$ for all $\xi \in \mathbb{R}^+$, then $(\xi - \eta)b = \frac{1}{2}(a + 2\xi b) - \frac{1}{2}(a + 2\eta b) \in C$ for all $\xi, \eta \in \mathbb{R}^+$, hence $\mathbb{R}b \subset C$ and even $\mathbb{C}b \subset C$ in the complex case.

Proposition 3.1. *There exist only three classes $\mathcal{A}$ of absolutely convex sets with* $\mathrm{Sep_a}\,\mathcal{A} = \mathcal{A}$*, namely:*

(i) *the class of all one-point absolutely convex sets,*
(ii) *the class of all linearly bounded absolutely convex sets,*
(iii) *the class of all absolutely convex sets.*

Proof. "$\Rightarrow$" In case (i), $\mathrm{Sep_a}\,\mathcal{A} = \mathcal{A}$ is obvious. In case (ii), $\mathrm{Sep_a}\,\mathcal{A} = \mathcal{A}$ is clear because for every $C \in \mathrm{Sep}\,\mathcal{A}$ the absolutely affine maps from C to linearly bounded maps from C to linearly bounded affine maps separate points; hence C cannot contain an absolutely affine copy of $\mathbb{R}$ (or $\mathbb{C}$) and is therefore linearly bounded. In case (iii) $\mathrm{Sep_a}\,\mathcal{A} = \mathcal{A}$ is trivial.

"$\Leftarrow$" Assume $\mathrm{Sep_a}\,\mathcal{A} = \mathcal{A}$. If we are not in case (i), there are $C \in \mathcal{A}$ and $a \in C$ with $a \neq 0$. Then for $B := \{\xi :\mid \xi \mid\leq 1\}$ (in $\mathbb{R}$ or $\mathbb{C}$), the map $B \to C$, $\xi \mapsto \xi a$ is absolutely affine and injective; thus we have $B \in \mathcal{A}$. We show that $\mathcal{A}$ contains all linearly bounded absolutely convex sets. So let $D \subset V$ be absolutely convex in a (real or complex) vector space V. Without loss of generality we can assume that D generates V as a vector space; otherwise replace V by the linear span of D. Now on V the Minkowski functional $\|\cdot\|_D$ is a norm, where $\|x\|_D := \inf\{\xi \in \mathbb{R}^+ : x \in \xi D\}$. By the Hahn-Banach theorem, the linear maps of norm ≤ 1 from V to $\mathbb{R}$ (or $\mathbb{C}$) separate points, and they map D into B. Thus the absolutely affine maps from D to B separate points, and since $B \in \mathcal{A}$ we conclude $D \in \mathrm{Sep_a}\,\mathcal{A} = \mathcal{A}$. Therefore $\mathcal{A}$ contains all linearly bounded spaces.

So, if we are neither in case (i) nor in case (ii), $\mathcal{A}$ must contain at least one space C which is not linearly bounded. Thus C contains a non-trivial vector subspace and hence an affine copy of $\mathbb{R}$ (or $\mathbb{C}$), proving $\mathbb{R} \in \mathcal{A}$ ($\mathbb{C} \in \mathcal{A}$ resp.) Since the maps from a vector space to the base field separate points, $\mathcal{A}$ contains all vector spaces. Since every absolutely convex set can be embedded in a vector space, $\mathcal{A}$ contains all absolutely convex sets. Therefore we are in case (iii). □

By the same argument as in section 2 we obtain the following

Corollary 3.2. *For an absolutely convex set* $C, \mathrm{Sep}_{\mathrm{a}}\{C\}$ *is*

(i) *the class of all sets isomorphic to* 0 *if* $C = 0$;
(ii) *the class of all linearly bounded absolutely convex sets, if* $C \neq 0$ *is linearly bounded;*
(iii) *the class of all absolutely convex sets if* C *is not linearly bounded.*

4. The General Case

From 3.2 we easily obtain a characterization of those convex sets for which the bounded affine functionals separate points. For each convex set C in a vector space, the pointwise difference $C - C := \{x - y : x, y \in C\}$ is symmetric about the orign, and if $C \neq \emptyset$ we have $0 \in C$, and $C - C$ is therefore absolutely convex (over $\mathbb{R}$). The following proposition is a special case of proposition 2.3 from [1], but here we shall give a proof that avoids the abstract machinery used there.

Proposition 4.1. *A convex set* C *belongs to* $\mathrm{Sep}\{[-1,1]\}$ *(i.e. the bounded linear real-valued affine maps separate points) if and only if* $C - C$ *is linearly bounded.*

Proof. "⇒" Assume that $C - C$ contains a ray. Then there are sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in C such that $a_1 \neq b_1$ and $a_n - b_n = n(a_1 - b_1)$ for all $n \in \mathbb{N}$. If $C \in \mathrm{Sep}\{[0,1]\}$ then there even exists a linear map f from the vector space containing C to $\mathbb{R}$ such that $f(a_1) \neq f(b_1)$ and $f(C) \subset [-1,1]$. But then for all $n \in \mathbb{N}$ we obtain

$$nf(a_1 - b_1) = f(a_n - b_n) = f(a_n) - f(b_n) \in [-2,2]$$

because $f(a_n), f(b_n) \in f(C) \subset [-1,1]$. But this yields $f(a_1) - f(b_1) = f(a_1 - b_1) = 0$, i.e. $f(a_1) = f(b_1)$, contradicting our hypothesis.

"⇐" For $C = \emptyset$ the statement is trivial. Otherwise, fix some $c \in C$. Then the map $C \to C - C,\ x \mapsto x - c$ is affine and injective. From 3.2 we get $C - C \in \mathrm{Sep}\ \{[0,1]$ and hence $C \in \mathrm{Sep}\{[0,1]\}$. □

A nicer condition than linear boundedness of $C - C$ would be linear boundedness of C itself, but we shall see soon that this condition is strictly weaker than linear boundedness, of $C - C$. Since the class of linearly bounded convex sets is clearly separation-closed, one might hope to find at least a single (linearly bounded) convex set C such that $\mathrm{Sep}\{C\}$ is the class of all linearly bounded convex sets. We shall see below that this is not the case. There is not even a set $\mathcal{C}$ of convex sets such that $\mathrm{Sep}\,\mathcal{C}$ is the class of all linearly bounded sets. Indeed, without loss of generality we could assume $\emptyset \notin \mathcal{C}$, and for then the (cartesian) product C of all elements of $\mathcal{C}$ we should get $\mathrm{Sep}\,\mathcal{C} = \mathrm{Sep}\{C\}$.

This explains why we have to use cardinality arguments, because for every cardinality κ there is a set $\mathcal{C}$ of convex sets such that each convex set of cardinality $< \kappa$ is affinely isomorphic to an element of $\mathcal{C}$.

Similarly, the set of all *line-free convex sets* (i.e. convex sets containing no affine copy of $\mathbb{R}$) is the largest separation-closed class which is different from the class of all convex sets.

Indeed, it is clearly separation closed, but for each C that contains a line, $\mathrm{Sep}\{C\} = \mathrm{Sep}\,\mathbb{R}$ is the class of all convex sets, because linear functionals on a vector space separate points. Now 2.1 might suggest that it is equal to $\mathrm{Sep}\{\mathbb{R}^+\}$. We shall see as well that it is not even of the form $\mathrm{Sep}\{C\}$ for a single convex set C (or even $\mathrm{Sep}\,\mathcal{C}$ for a set $\mathcal{C}$ of convex sets). Moreover, there is no class between linearly bounded and line-free convex sets, which is of the form $\mathrm{Sep}\{C\}$, i.e. there is no C such that $\mathrm{Sep}\{C\}$ contains all linearly bounded convex sets but not all convex sets. The following results generalizes Theorem 2.4. of [1].

Theorem 4.2. *There exists no line-free convex set C such that for each linearly bounded convex set D the affine maps from D to C separate points.*

Proof. Assume that C is such a convex set, let I be an uncountable set of cardinality $\#I > \#C$, and let V be a vector space with a basis consisting of $\#I$ many vectors $e_i,\ i \in I$ and one more vector d. Then each vector in V has a unique representation $\sum_{i\in I} \alpha_i e_i + \beta d$ as a formally infinite real linear combinations of basis vectors, i.e. $\alpha_i \in \mathbb{R}$ for all $i \in I,\ \#\{i \in I : \alpha_i \neq 0\} < \aleph_0,\ \beta \in \mathbb{R}$.

Now let $D \subset V$ be the set of vectors for which the above representation satisfies the following conditions:

(i) $\alpha_i \geq 0$ for all $i \in I$,

(ii) $\sum\limits_{i\in I} \alpha_i \leq 1$,

(iii) $|\beta| \leq \#\{i \in I : \alpha_i > 0\} + 1$.

Then D is a convex set, because for $\sum\limits_{i\in I} \alpha_i = e_i + \beta d, \sum\limits_{i\in I} \gamma e_i + \delta d \in D,\ \lambda = \in]0,1[$ we have

$$\begin{aligned}|\lambda\beta + (1-\lambda)\delta| &\leq \max(|\beta|,|\delta|) \leq \max\{\#\{i \in I : \alpha_i > 0\}, \#\{i \in I : \gamma_i > 0\} + 1 \\ &\leq \#(\{i \in I : \alpha_i > 0\} \cup \{i \in I : \gamma_i > 0\}) + 1 = \#\{i \in I : \lambda\alpha_i + (1-\lambda)\gamma_i > 0\} + 1.\end{aligned}$$

We claim that D is linearly bounded. Indeed, assume

$$\sum_{i\in I}(\alpha_i + \lambda\gamma_i)e_i + (\beta + \lambda\delta)d = \sum_{i\in I}\alpha_i e + \beta d + \lambda(\sum_{i\in I}\gamma_i e_i + \delta d) \in D$$

for all $\lambda \in \mathbb{R}^+$. For each $i \in I$, conditions (i) and (ii) give

$$0 \leq \alpha_i + \lambda\gamma_i \leq 1 \text{ for all } \lambda \in \mathbb{R}^+, \text{ hence } \gamma_i = 0.$$

Now from condition (iii) we obtain

$$|\beta + \lambda\delta| \leq \#\{i \in I : \alpha_i + \lambda\gamma_i > 0\} + 1 = \#\{i \in I : \alpha_i > 0\} + 1$$

for all $\lambda \in \mathbb{R}^+$, hence $\delta = 0$ and therefore $\sum\limits_{i\in I}\gamma_i e_i + \delta d = 0$, proving that D is linearly bounded.

Obviously, we have $0, d \in D$, and we shall show $f(0) = f(d)$ for every affine map $f : D \to C$. Since $\#I > \#C > \aleph_0$, there must be infinitely (even uncountably) many $i \in I$ for which $f(e_i)$ is the same element of C; otherwise we should get $\#I \leq \aleph_0 \cdot \#C = \#C$. Thus there are a $u \in C$ and a sequence $(i_n)_{n\in\mathbb{N}}$ in I such that $i_n \neq i_m$ for $n \neq m$ and $f(e_{i_n}) = u$

for all $n \in \mathbb{N}$. Since f is affine, there is a linear map l from V to a vector space containing C with $f(x) = f(0) + l(x)$ for all $x \in D$, in particular $l(e_{i_n}) = u - f(0)$ for all $n \in \mathbb{N}$. For each $\lambda \in \mathbb{R}$ there is an $m \in \mathbb{N}$ with $|\lambda| \leq m + 1$ and hence $\frac{1}{m} \sum_{n=1}^{m} e_{i_n} + \lambda d \in D$. Since

$$\begin{aligned} f(\frac{1}{m}\sum_{n=1}^{m} e_{i_n} + \lambda d) &= f(0) + \frac{1}{m}\sum_{n=1}^{m} l(e_{i_n}) + \lambda l(d) \\ &= f(0) + \frac{1}{m}\sum_{n=1}^{m}(u - f(0)) + \lambda l(d) \\ &= f(0) + u - f(0) + \lambda l(d) = u + \lambda l(d), \end{aligned}$$

we obtain

$$u + \lambda l(d) \in f(D) \subset C \text{ for all } \lambda \in \mathbb{R}.$$

But C is line-free by hypothesis, thus we must have $l(d) = 0$ and therefore $f(d) = f(0)$, i.e. no linear map from D to C separates 0 and d. □

If we have $\#I \geq 2^{\aleph_0}$, then in the proof of 4.2 we obtain $\#D = \#I$. So for an arbitrary cardinal $\kappa \geq 2^{\aleph_0}$ and for $\mathcal{C}_\kappa$ the class of all convex sets or cardinality $< \kappa$, we see that $D \notin \operatorname{Sep} \mathcal{A}_\kappa$, but $D \in \mathcal{A}_{\kappa'} \subset \operatorname{Sep} \mathcal{A}_{\kappa'}$ for each cardinal $\kappa' > \kappa$. Thus the separation-closed class $\operatorname{Sep} \mathcal{A}_\kappa$ and $\operatorname{Sep} \mathcal{A}_{\kappa'}$ are different whenever $\kappa \neq \kappa'$, $\kappa, \kappa' \geq 2^{\aleph_0}$. So in general there are many more separation closed classes than in the finite-dimensional or absolutely convex situation.

Acknowledgements. I am indebted to D. Pallaschke for some bibliographical hints, to H. P. Petersson for a suggestion that led to a simplification of the proof of 4.2, and to D. Pumplün for some helpful directions.

References

[1] R. Börger, R. Kemper: A cogenerator for pre-separated superconvex spaces, Applied Categorical Structures 4 (1996) 361–370.

[2] J. B. Hiriart-Urruty, C. Lemaréchal: Convex Analysis and Minimazation Algorithms I, Grundlehren der mathematischen Wissenschaften 305, Springer-Verlag, Berlin et al., 1993.

[3] D. Pumplün, H. Röhrl: Banach spaces and totally convex spaces I, Comm. Algebra 12 (1984) 953–1019.

[4] R. Schneider: Convex Bodies: The Brunn-Minkowski Theory, Cambridge University Press, Cambridge (UK), 1993.