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Self Dual Operators on Convex Functionals Geometric Mean and Square Root of Convex Functionals

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Let $\mathrm{Conv}(X)$ be the set of the convex functionals defined on a linear space X, with values in $\mathbb{R} \cup \{+\infty\}$. In this paper we give an extension of the notion of duality for (convex) functionals to mappings which operate from $\mathrm{Conv}(X) \times \mathrm{Conv}(X)$ into $\mathrm{Conv}(X)$.
Afterwards, we present an algorithm which associates, under convenient assumptions, a self-dual operator to a given operator and its dual.
Finally, we give some examples which prove the generality and interest of our approach.

1. Introduction

In this paper, we present, mainly, a general algorithm to construct the square root of a convex functional. This square root is always a convex functional and generally distinct from the classical one. By the same algorithm we can associate a self dual operator to any convex operator and its dual. Some examples can be found in [1]. We have given only elementary applications of the theory of square root convex fuctionals (not to lengthen this paper). Many others could be developped for example in the theory of variational problems. To determine how to obtain - with convenient assumptions - the solution of the reverse problem of the square root, is not obvious and very interesting.

This paper is divided into four parts: Firstly, we give a basic algorithm which, beginning from arithmetic and harmonic means of two real positive numbers, converges to their geometric mean. In the second part, we present an extension of the previous algorithm to quadratic operators and one elementary physical example. Thirdly, we extend the results obtained in the second part to convex functionals. We construct the convex geometric mean of two convex functionals from which we deduce the construction of the convex square root of a convex functional. In the last part, we study some geometrical examples.

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2. The numerical square root algorithm

Definition 2.1. Let $\mathbb{R}_+^* =]0, +\infty[$. Given $a \in \mathbb{R}_+^*$, we set below $a^* = \frac{1}{a} \in \mathbb{R}_+^*$.

Proposition 2.2. *Let $a, b \in \mathbb{R}_+^*$; then:*

(i) $a^{**} = (a^*)^* = a$
(ii) *If $a \leq b$ then $a^* \geq b^*$*
(iii) *For all $\lambda \in]0,1[$, we have: $(\lambda a + (1-\lambda)b)^* \leq \lambda a^* + (1-\lambda)b^*$*

Proof. (i) and (ii) are obvious.

(iii) Because $x \to \frac{1}{x}$ is a convex mapping on $]0, +\infty[$. □

Definition 2.3. For any $a, b \in \mathbb{R}_+^*$, we set:

$$\alpha_0(a,b) = \frac{a+b}{2}\ , \quad \alpha_0^*(a,b) = (\alpha_0(a^*, b^*))^*$$
$$\forall n \in \mathbb{N}, \quad \alpha_{n+1}(a,b) = \frac{1}{2}(\alpha_n(a,b) + \alpha_n^*(a,b))$$
$$\text{where} \quad \alpha_n^*(a,b) = (\alpha_n(a^*, b^*))^*$$

Proposition 2.4. *Let $a, b \in \mathbb{R}_+^*$, then we have:*

(i) $\forall n \in \mathbb{N}$, $\alpha_n(a,b) \in \mathbb{R}_+^*$ *and* $\alpha_n^*(a,b) \in \mathbb{R}_+^*$
(ii) $\forall n \in \mathbb{N}$, $\alpha_n^*(a,b) \leq \alpha_n(a,b)$ *and* $\alpha_{n+1}(a,b) \leq \alpha_n(a,b)$.
(iii) $\forall n \in \mathbb{N}$, $\alpha_{n+1}(a,b) - \alpha_{n+1}^*(a,b) \leq \frac{1}{2}(\alpha_n(a,b) - \alpha_n^*(a,b))$
(iv) *The sequence $(\alpha_n(a,b), n \in \mathbb{N})$ converges to $\tau(a,b) \in \mathbb{R}_+^*$.*
(v) $\forall n \in \mathbb{N}$, $\alpha_n^*(a,b) = ab(\alpha_n(a,b))^*$
(vi) $\tau(a,b) = \sqrt{ab}$ *(geometric mean of a and b) and $\tau(a,1) = \sqrt{a}$ (positive square root of a)*

Proof. (i) We can verify that: $\alpha_0^*(a,b) = \frac{2ab}{a+b} \in \mathbb{R}_+^*$; we deduce the announced result by recurrence on $n \in \mathbb{N}$.

$$\begin{aligned}\text{(ii)} \quad \alpha_{n+1}^*(a,b) &= (\alpha_{n+1}(a^*, b^*))^* = (\frac{1}{2}\alpha_n(a^*, b^*) + \frac{1}{2}\alpha_n^*(a^*, b^*))^* \\ &\leq \frac{1}{2}(\alpha_n(a^*, b^*))^* + \frac{1}{2}(\alpha_n(a^{**}, b^{**}))^{**} && \text{(By Proposition 2.2(iii))} \\ &= \frac{1}{2}\alpha_n^*(a,b) + \frac{1}{2}\alpha_n(a,b) && \text{(By Proposition 2.2(ii)).}\end{aligned}$$

Thus, $\forall n \in \mathbb{N}$, $\alpha_n^*(a,b) \leq \alpha_n(a,b)$.

We deduce that:

$$\alpha_{n+1}(a,b) = \frac{1}{2}(\alpha_n(a,b) + \alpha_n^*(a,b)) \leq \ \alpha_n(a,b).$$

(iii) To simplify the writing below, we omit the a and b, so we write that:

$$\alpha_{n+1} = \frac{1}{2}(\alpha_n + \alpha_n^*) \quad \text{and} \quad \alpha_n^* \leq \ \alpha_{n+1}^* \leq \alpha_{n+1}$$

Since $\alpha_{n+1} \leq \alpha_n$ then $\alpha_n^* \leq \alpha_{n+1}^*$ (according to Proposition 2.2(ii)).

We can deduce that:

$$\alpha_{n+1} - \alpha_{n+1}^* \leq \frac{1}{2}(\alpha_n + \alpha_n^*) - \alpha_n^* \leq \frac{1}{2}(\alpha_n - \alpha_n^*).$$

(iv) From (iii) and (ii), we get that:

$$\forall n \in \mathbb{N}, \ \ 0 \leq \alpha_n - \alpha_n^* \leq \left(\frac{1}{2}\right)^n (\alpha_0 - \alpha_0^*)$$

Therefore $\forall a, b \in \mathbb{R}_+^*, \ \lim_n(\alpha_n(a,b) - \alpha_n^*(a,b)) = 0$.

Since the sequence $(\alpha_n(a,b); n \in \mathbb{N})$ is decreasing and lower bounded (resp. the sequence $(\alpha_n^*(a,b); n \in \mathbb{N})$ is increasing and upper bounded), then we have:

$$\lim_n \ \alpha_n(a,b) = \lim_n \ \alpha_n^*(a,b) = \tau(a,b)$$

and $\tau(a,b) \geq \alpha_0^*(a,b) = \frac{2ab}{a+b}$ thus $\tau(a,b) \in \mathbb{R}_+^*$.

(v) Given $a, b \in \mathbb{R}_+^*$ we let:

$$\forall n \in \mathbb{N}, \ \ \theta_n = \alpha_n \ (a,b) \quad \text{and} \quad \theta_n^\sharp = \alpha_n^*(a,b))$$

Then $\theta_0 = \frac{a+b}{2}$ and $\theta_0^\sharp = \frac{2ab}{a+b} = \frac{ab}{\theta_0}$.

Assume that $\forall p \in \mathbb{N}$, $p \leq n$, $\theta_n^\sharp = \frac{ab}{\theta_n}$, then: $\theta_{n+1} = \frac{1}{2}(\theta_n + \theta_n^\sharp) = \frac{1}{2}(\theta_n + \frac{ab}{\theta_n})$ and

$$\theta_{n+1}^\sharp = \alpha_{n+1}^*(a,b) = \left[\frac{1}{2}(\alpha_n(a^*,b^*) + (\alpha_n(a,b))^*)\right]^*$$

$$\theta_{n+1}^\sharp = \left[\frac{1}{2}(\frac{1}{\alpha_n^*(a,b)} + \frac{1}{\alpha_n(a,b)})\right]^* = \left[\frac{1}{2}(\frac{\theta_n}{ab} + \frac{1}{\theta_n})\right]^*$$

$$\theta_{n+1}^\sharp = \frac{1}{\frac{1}{2}(\frac{\theta_n}{ab} + \frac{1}{\theta_n})} = \frac{ab}{\frac{1}{2}(\alpha_n(a,b) + \frac{ab}{\alpha_n(a,b)})} = \frac{ab}{\theta_{n+1}}.$$

Then $\theta_{n+1}^\sharp = \frac{ab}{\theta_{n+1}}$ and thus $\alpha_n^*(a,b) = ab(\alpha_n(a,b))^*$.

(vi) One has $\lim_n \ \theta_n = \lim_n \ \alpha_n(a,b) = \tau(a,b)$.

Since $\theta_{n+1} = \frac{1}{2}(\theta_n + \theta_n^\sharp)$, we deduce that: $\tau(a,b) = \frac{ab}{\tau(a,b)}$, but $\tau(a,b) \in \mathbb{R}_+^*$, thus $\tau(a,b) = \sqrt{ab}$, and in particular $\tau(a,1) = \sqrt{a}$. □

Remark 2.5. We can prove the assertion (v) of Proposition 2.4 in a different way as follows:

By a recurrence, it is easy to verify that:

$$\forall a, b \in \mathbb{R}_+^*, \forall n \in \mathbb{N}, \ a^*.\alpha_n(a,b).b^* = \alpha_n(a^*,b^*)$$

then $\alpha_n^*(a,b) = (\alpha_n(a^*,b^*))^* = a(\alpha_n(a,b))^* b$, this concludes the proof.

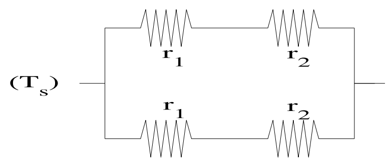
Remark 2.6 (Connected to Newton's algorithm)**.** Let us consider the Newton's algorithm to calculate the approximation of the square root of a positive number q: $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}\frac{q}{x_n}$ $(n \geq 0)$ with $x_0 > 0$ is given.

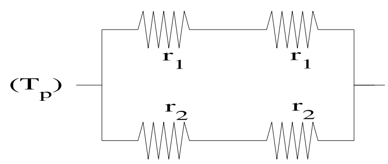
We know that for a fixed $x_0 > 0$, (x_n) converges to $\sqrt{q}$.

We note according to the above study that, if q can be written $q = ab$ with $a, b \in \mathbb{R}^*_+$, then $\alpha_n(a,b)$ converges, to $\sqrt{q} = \sqrt{ab}$ as rapidly as (x_n). Observe that in the following example: $q = 6 = 2\times 3(a = 2; b = 3); \alpha_0(2,3) = 2,5; \alpha^*_0(2,3) = 2,45$ and $(2,45)^2 = 6,0025$ thus $\sqrt{6} \simeq 2,45$. For (x_n), with $x_0 = 1, x_1 = 3,5, x_2 = 2,607...$

Now, we shall give an electrical interpretation of the previous algorithm.

Let r_1 and r_2 be two fixed electrical resistances, and we consider the following circuits:





The equivalent resistance of (T_s) is $\alpha_0(r_1, r_2) = \frac{r_1+r_2}{2}$.
The equivalent resistance of (T_p) is $\alpha^*_0(r_1, r_2) = \frac{1}{\frac{1}{2}(\frac{1}{r_1}+\frac{1}{r_2})} = \frac{2r_1r_2}{r_1+r_2}$.

Proposition 2.7. *Let r_1 and r_2 be two given electrical resistances, and the following algorithm:*

$$(R_n)\left\{\begin{array}{ll} R_0 = \begin{bmatrix} r_1 & r_2 \\ r_1 & r_2 \end{bmatrix} & R^*_0 = \begin{bmatrix} r_1 & r_1 \\ r_2 & r_2 \end{bmatrix} \\ & \\ R_{n+1} = \begin{bmatrix} R_n & R^*_n \\ R_n & R^*_n \end{bmatrix} & R^*_{n+1} = \begin{bmatrix} R_n & R_n \\ R^*_n & R^*_n \end{bmatrix} \end{array}\right.$$

$\tau(r_1, r_2) = \sqrt{r_1r_2}$ is the equivalent electrical resistance of the limit of R_n when n tends to infinity.

3. First extension: Square root of a symmetric positive definite matrix

3.1. Preliminaries

In the following, the space $\mathbb{R}^m$ (m integer ≥ 1) is endowed with the euclidian inner product $\langle .|. \rangle$ defined by:

$$\forall x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m, \ \forall y = (y_1, y_2, ..., y_m) \in \mathbb{R}^m, \quad \langle x|y\rangle = \sum_{j=1}^{m} x_jy_j$$

and its associated norm $\|x\| = (\langle x|x\rangle)^{\frac{1}{2}}$.

Let $m \in \mathbb{N}^*$. We recall that a matrix A of the type $(m \times m)$ is said symmetric positive definite (s.p.d) if: $A = A^t$ and $\forall u \in \mathbb{R}^m, u \neq 0, \ \langle Au|u\rangle > 0$.

Recall that, if A and B are two s.p.d matrices of type $(m \times m)$ then:

(i) Their sum $A + B$ is a s.p.d matrix of type $(m \times m)$
(ii) A (resp. B) is invertible and its inverse is s.p.d of type $(m \times m)$

Indeed, let $u \in \mathbb{R}^m, u \neq 0$, we have:

$$\langle (A+B)u|u\rangle = \langle Au + Bu|u\rangle = \langle Au|u\rangle + \langle Bu|u\rangle > 0$$

$\langle A^{-1}u|u\rangle = \langle A^{-1}u|A(A^{-1}u)\rangle > 0$ because $A^{-1}u \neq 0$.

Definition 3.1. Let $m \in \mathbb{N}^*$, and A be a matrix of type $(m \times m)$, we set:

$$\forall u \in \mathbb{R}^m, \ \ q_A(u) = \frac{1}{2}\ \langle Au|u\rangle$$

and $(q_A)^*(u) = \mathrm{Sup}\{\langle u|v\rangle - q_A(v), v \in \mathbb{R}^m\}$.

Below we write q_A^*, q_A^{**}... instead of $(q_A)^*, (q_A^*)^*$...

We say that $q_A \leq q_B$ if $\forall u \in \mathbb{R}^m$ $q_A(u) \leq q_B(u)$ (resp. $q_A < q_B$ if $\forall u \in \mathbb{R}^m, u \neq 0$ and $q_A(u) < q_B(u)$).

The following proposition is well-known.

Proposition 3.2. *Let A and B be two s.p.d. matrices of type $(m \times m)$.*

(i) $q_A^* = q_{A^{-1}}$ *and* $q_A^{**} = q_A$
(ii) $q_A \leq q_B \Longrightarrow q_A^* \geq q_B^*$
(iii) $\forall \lambda \in [0,1]$ $(\lambda q_A + (1-\lambda)q_B)^* \leq \lambda q_A^* + (1-\lambda)q_B^*$.

Proposition 3.3. *Let A and B be two s.p.d matrices of type $(m \times m)$. One has:*

(i) $\frac{1}{2}(q_A + q_B) = q_{\frac{A+B}{2}}$
(ii) $(\frac{1}{2}(q_A^* + q_B^*))^* = q_{(\frac{A^{-1}+B^{-1}}{2})^{-1}}$.

Proof. (i) For all $x \in \mathbb{R}^m$, we have:

$$(\frac{1}{2}(q_A + q_B))(x) = \frac{1}{4}(\langle Ax|x\rangle + \langle Bx|x\rangle) = \frac{1}{2}\langle (\frac{A+B}{2})x|x\rangle = q_{\frac{A+B}{2}}(x).$$

(ii)

$$(\frac{1}{2}(q_A^* + q_B^*))^* = (\frac{1}{2}(q_{A^{-1}} + q_{B^{-1}}))^* = (q_{\frac{A^{-1}+B^{-1}}{2}})^* = q_{(\frac{A^{-1}+B^{-1}}{2})^{-1}}.$$

□

3.2. The matrix algorithm

Definition 3.4. Let A and B be two s.p.d matrices of type $(m \times m)$; we set in the following:

$$\begin{cases} \alpha_0(A,B) = \frac{1}{2}(q_A + q_B), \quad \alpha_0^*(A,B) = (\frac{1}{2}(q_A^* + q_B^*))^* \\ \forall n \in \mathbb{N},\ \alpha_{n+1}(A,B) = \frac{1}{2}(\alpha_n(A,B) + \alpha_n^*(A,B)) \\ \text{where} \quad \alpha_n^*(A,B)) = (\alpha_n(A^{-1},B^{-1}))^* \end{cases}$$

and

$$\begin{cases} \gamma_0(A,B) = \frac{1}{2}(A+B), \quad \gamma_0^*(A,B) = (\frac{1}{2}(A^{-1}+B^{-1}))^{-1} \\ \forall n \in \mathbb{N},\ \gamma_{n+1}(A,B) = \frac{1}{2}(\gamma_n(A,B) + \gamma_n^*(A,B)) \\ \text{where} \quad \gamma_n^*(A,B) = (\gamma_n(A^{-1},B^{-1}))^{-1} \end{cases}$$

We can easily verify that:

$$\forall n \in \mathbb{N},\ \alpha_n(A,B) = \alpha_n(B,A),\ \alpha_n^*(A,B) = \alpha_n^*(B,A)$$
$$\gamma_n(A,B) = \gamma_n(B,A),\ \gamma_n^*(A,B) = \gamma_n^*(B,A)$$

Proposition 3.5. *Let A and B be two s.p.d matrices of type $(m \times m)$; one has:*

(i) $\forall n \in \mathbb{N},\ \alpha_n(A,B) = q_{\gamma_n(A,B)}, \alpha_n^*(A,B) = q_{\gamma_n^*(A,B)}$,
(ii) $\forall n \in \mathbb{N},\ \alpha_n^*(A,B) \leq \alpha_n(A,B)$ *and* $\alpha_{n+1}(A,B) \leq \alpha_n(A,B)$
(iii) $\forall n \in \mathbb{N},\ \alpha_{n+1}(A,B) - \alpha_{n+1}^*(A,B) \leq \frac{1}{2}\left(\alpha_n(A,B) - \alpha_n^*(A,B)\right)$
(iv) *The sequence $(\alpha_n(A,B), n \in \mathbb{N})$ converges pointwise to a limit denoted by $\tau(A,B)$ (i. e: $\forall x \in \mathbb{R}^m,\ \lim_n(\alpha_n(A,B))(x) = (\tau(A,B))(x)$) such that: $\tau(A,B) = \tau(B,A)$ and $\tau(A,B) = \tau^*(A,B) = (\tau(A^{-1},B^{-1}))^*$*
(v) *There exists a s.p.d matrix of type $(m \times m)$ such that: $\tau(A,B) = q_{\sigma(A,B)}$ with $\sigma(A,B) = \sigma(B,A)$ and $\sigma(A,B) = \sigma^*(A,B) = (\sigma(A^{-1},B^{-1}))^{-1}$*
(vi) $\forall n \in \mathbb{N},\ \gamma_n^*(A,B) = A(\gamma_n(A,B))^{-1}\ B = B(\gamma_n(A,B))^{-1}A$
and $\sigma(A,B) = A(\sigma(A,B))^{-1}B = B(\sigma(A,B))^{-1}A$
(vii) $\sigma(A,B) = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}} = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$.

In particular $\sigma(A,I) = A^{\frac{1}{2}}$, and if A and B are commuting then $\sigma(A,B) = (AB)^{\frac{1}{2}} = A^{\frac{1}{2}}B^{\frac{1}{2}}$.

Further, if $B^{-1}A$ is s.p.d (resp. $A^{-1}B$ s.p.d) then we have: $\sigma(A,B) = B(B^{-1}A)^{\frac{1}{2}}$ (resp. $\sigma(A,B) = A(A^{-1}B)^{\frac{1}{2}}$).

Proof. (i) It's easy by recurrence on $n \in \mathbb{N}$.

(ii) (resp. (iii), (iv)) can be proved as (ii) (resp. (iii), (iv)) of Proposition 2.4.

(v) The existence of $\sigma(A,B)$ can be proved by using (i); the properties of $\sigma(A,B)$ can be deduce from (iv).

(vi) Remark that:

$$\begin{aligned}\gamma_0^*(A,B) &= (\tfrac{1}{2}(A^{-1}+B^{-1}))^{-1} &&= 2(A^{-1}(A+B)B^{-1})^{-1}\\ &= 2B(A+B)^{-1}A &&= B(\tfrac{A+B}{2})^{-1}A = B(\gamma_0(A,B))^{-1}A\end{aligned}$$

Since $\gamma_0^*(A,B) = \gamma_0^*(B,A)$ and $\gamma_0(A,B) = \gamma_0(B,A)$, then:

$$\gamma_0^*(A,B) = A(\gamma_0(A,B))^{-1}B$$

Assume that there exists $n \in \mathbb{N}^*$ such that: for all s.p.d matrices C and D of type $(m \times m)$ we have:

$$\gamma_{n-1}^*(C,D) = C(\gamma_{n-1}(C,D))^{-1}D = D(\gamma_{n-1}(C,D))^{-1}C$$

this property is true for $n = 1$; and

$$\begin{aligned}\gamma_n^*(A,B) &= (\gamma_n(A^{-1},B^{-1}))^{-1} = (\tfrac{1}{2}(\gamma_{n-1}(A^{-1},B^{-1}) + \gamma_{n-1}^*(A^{-1},B^{-1})))^{-1}\\ &= 2(\gamma_{n-1}(A^{-1},B^{-1}) + A^{-1}(\gamma_{n-1}(A^{-1},B^{-1}))^{-1}B^{-1})^{-1}\\ &= 2(\gamma_{n-1}(A^{-1},B^{-1}) + A^{-1}\gamma_{n-1}^*(A,B)B^{-1})^{-1}\\ &= 2(A^{-1}(A\gamma_{n-1}(A^{-1},B^{-1})B + \gamma_{n-1}^*(A,B))B^{-1})^{-1}\\ &= 2B(A\gamma_{n-1}(A^{-1},B^{-1})B + \gamma_{n-1}^*(A,B))^{-1}A\end{aligned}$$

But we have

$$A\gamma_{n-1}(A^{-1},B^{-1})B = A(\gamma_{n-1}^*(A,B))^{-1}B = \gamma_{n-1}^{**}(A,B) = \gamma_{n-1}(A,B)$$

then $\gamma_n^*(A,B) = B(\gamma_n(A,B))^{-1}A = A(\gamma_n(A,B))^{-1}B$ so when n tends to $+\infty$, we deduce that:

$$\sigma(A,B) = \sigma^*(A,B) = B(\sigma(A,B))^{-1}A = A(\sigma(A,B))^{-1}B$$

(vii) Now, we set $\sigma(A,B) = X$; according to (vi) we have: $X = BX^{-1}A = AX^{-1}B$ and thus $XB^{-1}X = A$ and $XA^{-1}X = B$.

Therefore:

$$(B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^2 = B^{-\frac{1}{2}}(XB^{-1}X)B^{-\frac{1}{2}} = B^{-\frac{1}{2}}AB^{-\frac{1}{2}},$$

and:

$$B^{-\frac{1}{2}}XB^{-\frac{1}{2}} = (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}$$

thus $X = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}$ and symmetrically, $X = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$.

If A and B are commuting thus $X = A^{\frac{1}{2}}B^{\frac{1}{2}}$ and in particular if $B = I$ then $X = A^{\frac{1}{2}}$.

Assume that $B^{-1}A$ is s.p.d; if we put $Y = B^{-1}X$ then $Y = Y^{-1}B^{-1}A$ implies that $Y^2 = B^{-1}A$ thus $Y = (B^{-1}A)^{\frac{1}{2}}$ and $X = B(B^{-1}A)^{\frac{1}{2}}$. $\square$

Definition 3.6. The matrix $\sigma(A,B)$, defined by Proposition 3.5, is called the geometric mean of A and B. In particular $\sigma(A,I) = A^{\frac{1}{2}}$ is the square root matrix of A.

Corollary 3.7. *Let A and B be two s.p.d matrices, then the equation: Find a s.p.d matrix X such that $XAX = B$ has one and only one solution given by $X = \sigma(A^{-1}, B)$.*

Remark 3.8. Without any difficulties, the above definitions and results can be generalized to the case where A and B are two symmetric positive invertible operators from a Hilbert H into H.

We can give a physical signification of the algorithm of $\alpha_n(A,B)$, and we have the:

Proposition 3.9. *Let R_1 and R_2 be two given matrices (s.p.d) of some electrical resistances, and we consider the following algorithm:*

$$(T_n)\begin{cases} T_0 = \begin{bmatrix} R_1 & R_2 \\ R_1 & R_2 \end{bmatrix} & T_0^* = \begin{bmatrix} R_1 & R_1 \\ R_2 & R_2 \end{bmatrix} \\ \\ T_{n+1} = \begin{bmatrix} T_n & T_n^* \\ T_n & T_n^* \end{bmatrix} & T_{n+1}^* = \begin{bmatrix} T_n & T_n \\ T_n^* & T_n^* \end{bmatrix} \end{cases}$$

$\sigma(R_1,R_2)$ is an equivalent electrical resistance of the limit of T_n when n tends to infinity.

4. Second extension: Square root of a convex functional

Let E be a Hilbert space and $\langle . \mid . \rangle$ its scalar product.

4.1. Preliminary

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, we extend the structure of $\mathbb{R}$ on $\bar{\mathbb{R}}$ by setting $\forall x \in \mathbb{R}$: $-\infty < x < +\infty$, $+\infty + x = +\infty, -\infty + x = -\infty, +\infty + (-\infty) = +\infty, 0.(+\infty) = +\infty$.

Definition 4.1. We say that $f : E \longrightarrow \bar{\mathbb{R}}$ is convex if:

$$\forall x_1, x_2 \in E,\ \ \forall \lambda \in]0,1[, \quad f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

If $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$, we define the effective domain of f by:

$$\operatorname{dom} f = \{x \in E; f(x) < +\infty\}.$$

Definition 4.2. Let : $E \longrightarrow \bar{\mathbb{R}}$; we call the Legendre - Fenchel transform (or polar or conjugate or dual) of f, the functional denoted by f^*, defined as follows:

$$\forall y \in E, \quad f^*(y) = \sup\{\langle y|x\rangle - f(x); x \in E\}.$$

We denote by $\Gamma_0(E)$ the cone of lower semi-continuous (l.s.c) convex functionals from E into $\mathbb{R} \cup \{+\infty\}$ not identically equal to $+\infty$.

We recall the following results:

Proposition 4.3. *Let f and g be two functions in $\Gamma_0(E)$, then:*

(i) $f^* \in \Gamma_0(E)$ *and* $(f^*)^* = f$

(ii) $f \le g \Longrightarrow f^* \ge g^*$

(iii) $\forall \lambda \in [0,1]\ \ (\lambda f + (1-\lambda)g)^* \le \lambda f^* + (1-\lambda)g^*$

Proof. (i) and (ii) are classical.

(iii) It is a simple exercise, analoguous to (iii) of Proposition 3.2. □

4.2. The fundamental algorithm

Suppose that f and g belong to $\Gamma_0(E)$, we set:

$$\begin{cases} \alpha_0(f,g) = \frac{1}{2}(f+g) \quad , \quad \alpha_0^*(f,g) = (\frac{1}{2}(f^*+g^*))^* \\ \forall n \in \mathbb{N},\ \ \alpha_{n+1}(f,g) \ = \ \frac{1}{2}(\alpha_n(f,g) + \alpha_n^*(f,g)) \\ \text{where} \quad \alpha_n^*(f,g) \ = \ (\alpha_n(f^*,g^*))^* \end{cases}$$

Proposition 4.4. *For all f and g in $\Gamma_0(E)$, we have:*

(i) $\forall n \in \mathbb{N},\ \alpha_n(f,g) \in \Gamma_0(E)$ *and* $\alpha_n^*(f,g) \in \Gamma_0(E)$

(ii) $\forall n \in \mathbb{N},\ \alpha_n^*(f,g) \le \alpha_n(f,g), \alpha_{n+1}(f,g) \le \alpha_n(f,g)$ *and* $\alpha_n^*(f,g) \le \alpha_{n+1}^*(f,g)$

(iii) $\forall n \in \mathbb{N},\ \alpha_{n+1}(f,g) - \alpha_{n+1}^*(f,g) \le \frac{1}{2}(\alpha_n(f,g) - \alpha_n^*(f,g))$

(iv) *If we assume* $\mathrm{dom}(\alpha_0(f,g)) = \mathrm{dom}(\alpha_0^*(f,g))$*, then the sequence* $(\alpha_n(f,g), n \in \mathbb{N})$ *converges pointwise to a limit function* $\tau(f,g) \in \Gamma_0(E)$*, furthermore:* $\mathrm{dom}\,\tau(f,g) = \mathrm{dom}\,f \cap \mathrm{dom}\,g$*, and* $\tau(f,g) = \tau(g,f), \tau(f,g) = \tau^*(f,g) = (\tau(f^*,g^*))^*$.

Proof. (i) If $f, g \in \Gamma_0(E)$, then

$$\alpha_0(f,g) = \frac{1}{2}(f+g) \in \Gamma_0(E) \quad \text{and} \quad \alpha_0^*(f,g) = (\frac{1}{2}(f^*+g^*))^* \in \Gamma_0(E).$$

By recurrence we deduce that: $\forall n \in \mathbb{N},\ \alpha_n(f,g) \in \Gamma_0(E)$ and $\alpha_n^*(f,g) \in \Gamma_0(E)$.

(ii) We have:

$$\alpha_0^*(f,g) = (\frac{1}{2}f^* + \frac{1}{2}g^*)^* \le \frac{1}{2}f^{**} + \frac{1}{2}g^{**} = \alpha_0(f,g)$$

$$\forall n \in \mathbb{N},\ \ \alpha_{n+1}^*(f,g) = (\alpha_{n+1}(f^*,g^*))^* = (\frac{1}{2}\alpha_n(f^*,g^*) + \frac{1}{2}(\alpha_n(f,g))^*)^*$$

$$\le \frac{1}{2}\alpha_n^*(f,g) + \frac{1}{2}\alpha_n(f,g) = \alpha_{n+1}(f,g).$$

We deduce that:

$$\forall n \in \mathbb{N},\ \ \alpha_{n+1}(f,g) = \frac{1}{2}\alpha_n(f,g) + \frac{1}{2}\alpha_n^*(f,g) \le \alpha_n(f,g)$$

and consequently: $\forall n \in \mathbb{N},\ \alpha_n^*(f,g) \le \alpha_{n+1}^*(f,g)$.

(iii) $\forall n \in \mathbb{N},\ \alpha_{n+1}(f,g) - \alpha_{n+1}^*(f,g) \le \frac{1}{2}\alpha_n(f,g) + \frac{1}{2}\alpha_n^*(f,g) - \alpha_n^*(f,g)$.

If $\alpha_n^*(f,g)$ is finite then we have:

$$\forall n \in \mathbb{N},\ \ \alpha_{n+1}(f,g) - \alpha_{n+1}^*(f,g) \le \frac{1}{2}(\alpha_n(f,g) - \alpha_n^*(f,g)). \tag{1}$$

If $\alpha_n^*(f,g)) = +\infty$ then $\alpha_n(f,g) = +\infty$ because $\alpha_n^*(f,g)) \leq \alpha_n(f,g)$, thus the second member of (1) is equal to $+\infty$ and (1) holds.

(iv) From (ii) we deduce that:

$$\forall n \in \mathbb{N},\ \ \alpha_0^*(f,g) \leq ..... \leq \alpha_n^*(f,g) \leq \alpha_n(f,g) \leq .... \leq \alpha_0(f,g). \tag{2}$$

Assume that $\mathrm{dom}(\alpha_0(f,g)) = \mathrm{dom}(\alpha_0^*(f,g))$ and let $x \in E$.

If $x \in \mathrm{dom}(\alpha_0(f,g))$ then $\forall n \in \mathbb{N},\ \ x \in \mathrm{dom}(\alpha_n(f,g))$ and the sequence $(\alpha_n(f,g)(x), n \in \mathbb{N})$ converges in $\mathbb{R}$.

If $x \notin \mathrm{dom}(\alpha_0(f,g)) = \mathrm{dom}(\alpha_0^*(f,g))$ then $\forall n \in \mathbb{N},\quad x \notin \mathrm{dom}(\alpha_n(f,g))$ and thus $\alpha_n(f,g)(x) = +\infty$, consequently $\alpha_n(f,g)(x)$ tends to $+\infty$.

We set:

$$\forall x \in E,\ \ \tau(f,g)(x) = \begin{cases} \lim \alpha_n(f,g)(x) & \text{if } x \in \mathrm{dom}(\alpha_0(f,g)) \\ +\infty & \text{otherwise.} \end{cases}$$

and we obtain that $(\alpha_n(f,g))_n$ converges pointwise to $\tau(f,g)$ and

$$\mathrm{dom}(\tau(f,g)) = \mathrm{dom}(\alpha_0(f,g)) = \mathrm{dom}\, f \cap \mathrm{dom}\, g.$$

Finally, by a recurrence on $n \in \mathbb{N}$, we prove that: $\tau(f,g) = \tau(g,f)$ and by using (1) and (2), we deduce that: $\tau(f,g) = \tau^*(f,g) = (\tau(f^*,g^*))^*$. □

Remark 4.5. If f and g belong to $\Gamma_0(E)$ with finite values then the condition $\mathrm{dom}(\alpha_0(f,g)) = \mathrm{dom}(\alpha_0^*(f,g))$ holds.

Proposition 4.6. *For f, g and h in $\Gamma_0(E)$, one has*

(i) $\tau(f,g) = \tau(g,f) = \tau^*(f,g)$ *and* $(\frac{1}{2}(f^* + g^*))^* \leq \tau(f,g) \leq \frac{1}{2}(f+g)$
(ii) τ *is not associative that is to say:* $\tau(\tau(f,g),h) \neq \tau(f,\tau(g,h))$
(iii) $\forall f \in \Gamma_0(E),\ \ \tau(f,f) = f$ *and if* $\mathrm{dom}\, f = \mathrm{dom}\, f^* = E,\ \tau(f,f^*) = \frac{1}{2}\|.\|^2$ *where* $\|.\|$ *is the euclidian norm of* E.
(iv) $\forall f \in \Gamma_0(E)$ *such that* $\mathrm{dom}\, f = \mathrm{dom}\, f^* = E$, *one has* $\tau(f,0) = -\frac{1}{2}f^*(0)$ *and* $\mathrm{Inf}\{f(x), x \in E\} = 2.\tau(f,0)$
(v) τ *is increasing: let* $f_1, g_1 \in \Gamma_0(E)$ *with* $f \leq f_1$ *and* $g \leq g_1$, *then* $\tau(f,g) \leq \tau(f_1,g_1)$.

Proof. (i) It is proved in the above proposition.

(ii) Let $a > 0, b > 0, c > 0$ and put for every $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2}ax^2, g(x) = \frac{1}{2}bx^2, h(x) = \frac{1}{2}cx^2,$$

then we have

$$\tau(\tau(f,g),h)(x) = \frac{1}{2}((ab)^{\frac{1}{2}}c)^{\frac{1}{2}}x^2$$
$$\tau(f,\tau(g,h))(x) = \frac{1}{2}(a(bc)^{\frac{1}{2}})^{\frac{1}{2}}x^2.$$

Thus, in general, τ is not associative.

(iii) We prove, without difficulty, by a recurrence that:

$$\forall n \in \mathbb{N}, \ \ \alpha_n(f,f) = f, \text{ then } \tau(f,f) = f.$$

Further, we know that: $\forall f \in \Gamma_0(E)$ with $\operatorname{dom} f = \operatorname{dom} f^* = E$, $\tau(f,f^*) = (\tau(f,f^*))^*$. But, the only function $f_\sigma \in \Gamma_0(E)$ such that $f_\sigma^* = f_\sigma$ is $f_\sigma = \frac{1}{2}\|.\|^2$ thus $\tau(f,f^*) = \frac{1}{2}\|.\|^2$. Observe that $\tau(f,f^*)$ is independent of f.

(iv) Suppose that $g = 0$. Then for any $y \in E$,

$$g^*(y) = \delta(y,0) = \begin{cases} 0 & \text{if } y = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

we can verify that for every $n \in \mathbb{N}$,

$$\begin{aligned} \alpha_n(f,0) &= \frac{1}{2^{n+1}} f - \frac{2^n - 1}{2^{n+1}} f^*(0) \\ \alpha_n^*(f,0) &= \frac{1-2^n}{2^{n+1}} f^*(0) + \frac{1}{2^{n+1}} f^*. \end{aligned}$$

If $\operatorname{dom} f = \operatorname{dom} f^* = E$ we have

$$\tau(f,0) = \lim \alpha_n(f,0) = \lim \alpha_n^*(f,0) = -\frac{1}{2} f^*(0)$$

and thus $\operatorname{Inf}\{f(x), x \in \mathbb{R}^m\} = 2.\tau(f,0)$.

(v) If $f \leq f_1$ and $g \leq g_1$, we prove easily by a recurrence that for all $n \in \mathbb{N}$, the inequality $\alpha_n(f,g) \leq \alpha_n(f_1,g_1)$ implies $\tau(f,g) \leq \tau(f_1,g_1)$. □

Definition 4.7. Let f, g be in $\Gamma_0(E)$; $\tau(f,g)$ is called the convex geometric mean functional of f and g. In particular, if $g = f_\sigma = \frac{1}{2}\|.\|^2$, then $\tau(f,f_\sigma)$, denoted by $f^{[\frac{1}{2}]}$, is called the convex square root functional of f.

Remark 4.8. The definition above extends the classical ones given in the section 3.

Proposition 4.9. *Let $\phi : \Gamma_0(E) \longrightarrow \Gamma_0(E)$, we set: $\forall f \in \Gamma_0(E)$, $\phi^*(f) = (\phi(f^*))^*$ (this implies that $\phi^{**} = \phi$).*

We assume that:

$$(p_1) \quad \begin{cases} \forall f, g \in \Gamma_0(E), \phi(\alpha_0(f,g)) & = \alpha_0(\phi(f), \phi(g)) \\ \phi^*(\alpha_0(f,g)) & = \alpha_0(\phi^*(f), \phi^*(g)). \end{cases}$$

(p_2) *For any sequence $(h_n, n \in \mathbb{N})$ of elements of $\Gamma_0(E)$ which converges pointwise in E, to a function $h \in \Gamma_0(E)$, the sequence $(\phi(h_n))_n$ converges pointwise to $\phi(h)$.*

Let $f_0, g_0 \in \Gamma_0(E)$ such that $\operatorname{dom} f_0 = \operatorname{dom} g_0$ then $\tau(\phi(f_0), \phi(g_0)) = \phi(\tau(f_0, g_0))$ and $\tau(\phi^*(f_0), \phi^*(g_0)) = \phi^*(\tau(f_0, g_0))$.

Proof. Let $\mu_0 = \phi(f_0)$ and $\nu_0 = \phi(g_0)$ and for each $n \in \mathbb{N}$

$$\mu_{n+1} = \frac{1}{2}(\mu_n + \nu_n) = \alpha_0(\mu_n, \nu_n), \quad \nu_{n+1} = (\frac{1}{2}(\mu_n^* + \nu_n^*))^* = (\alpha_0(\mu_n^*, \nu_n^*))^*.$$

Suppose that there exists $n \in \mathbb{N}$ such that: $\mu_n = \phi(f_n), \nu_n = \phi(g_n)$, then

$$\mu_{n+1} = \alpha_0(\phi(f_n), \phi(g_n)) = \phi(\alpha_0(f_n, g_n)) = \phi(f_{n+1})$$

where $f_{n+1} = \alpha_0(f_n, g_n)$ and

$$\begin{aligned}\nu_{n+1} &= (\alpha_0((\phi(f_n))^*, (\phi(g_n))^*)^* = (\alpha_0(\phi^*(f_n^*), \phi^*(g_n^*)))^* = (\phi^*(\alpha_0(f_n^*, g_n^*)))^* = \\ &= (\phi((\alpha_0(f_n^*, g_n^*))^*))^{**} = \phi((\alpha_0(f_n^*, g_n^*))^*) = \phi(g_{n+1})\end{aligned}$$

where $g_{n+1} = (\alpha_0(f_n^*, g_n^*))^*$.

Thus for any $p \in \mathbb{N},\ \mu_p = \phi(f_p), \nu_p = \phi(g_p)$, but

$$\lim_n \mu_n = \lim_n \nu_n = \tau(\phi(f_0)), \phi(g_0)) = \lim_n \phi(f_n) = \lim_n \phi(g_n) = \phi(\tau(f_0, g_0)).$$

Furthermore:

$$\begin{aligned}\tau(\phi^*(f_0), \phi^*(g_0)) &= \tau((\phi(f_0^*))^*, (\phi(g_0^*))^*) \\ &= (\tau(\phi(f_0^*), \phi(g_0^*)))^* = (\phi(\tau(f_0^*, g_0^*)))^* \\ &= (\phi((\tau(f_0, g_0))^*))^* = \phi^*(\tau(f_0, g_0)).\end{aligned}$$

□

Proposition 4.10. *Let $f, g \in \Gamma_0(E)$ such that* $\mathrm{dom}(\alpha_0(f, g)) = \mathrm{dom}(\alpha_0^*(f, g))$. *Then one has:*

(i) $\forall \alpha \in \mathbb{R},\ \ \tau(f + \alpha, g) = \tau(f, g) + \frac{\alpha}{2}$

(ii) $\forall \lambda \in \mathbb{R}_+^*\ \ \tau(\lambda f, \lambda g) = \lambda \tau(f, g)$, *and if we set:* $\forall h \in \Gamma_0(\mathbb{R}^m),\ \ h.\lambda = \lambda h(\frac{\cdot}{\lambda})$, *then we have:* $\tau(f.\lambda, g.\lambda) = \tau(f, g).\lambda$

(iii) $\forall y \in \mathbb{R}^m$,

$$\begin{gathered}\tau(f(.+y), g(.+y)) = \tau(f, g)(.+y) \\ \tau(f^* - \langle y|.\rangle, g^* - \langle y|.\rangle) = \tau(f^*, g^*) - \langle y|.\rangle\end{gathered}$$

(iv) *If S is an invertible linear mapping from E onto E, then we have $\tau(f \circ S, g \circ S) = (\tau(f, g)) \circ S$.*

Proof. This proposition is a corollary of the above Proposition 4.9, by setting:

(i) $\phi(f) = f + \frac{\alpha}{2}$

(ii) $\phi(f) = \lambda f$ (resp. $\phi(f) = f.\lambda$)

(iii) $\phi(f) = f(.+y)$ (resp. $\phi(f^*) = f^* - \langle y|.\rangle$)

(iv) $\phi(f) = f \circ S$.

□

Proposition 4.11. *Let $f, g \in \Gamma_0(E)$ and let for each $n \in \mathbb{N}$, $\alpha_n(f, g) = h_n$ and $\alpha_n^*(f, g) = k_n$. Then one has for all $n, p \in \mathbb{N}$*

(i) $\alpha_n(f^*, g^*) = k_n^*$ *and* $\alpha_n^*(f^*, g^*) = h_n^*$

(ii) $h_{n+p+1} = \alpha_p(h_n, k_n)$ *and* $k_{n+p+1} = \alpha_p^*(h_n, k_n)$
$h_{n+p+1}^* = \alpha_p^*(h_n^*, k_n^*)$ *and* $k_{n+p+1}^* = \alpha_p(h_n^*, k_n^*)$

(iii) $\tau(f,g) = \tau(h_n, k_n)$.

Proof. (i) $\alpha_n(f^*, g^*) = (\alpha_n(f^*, g^*))^{**} = (\alpha_n^*(f,g))^* = k_n^*$ and
$\alpha_n^*(f^*, g^*) = (\alpha_n(f^{**}, g^{**}))^* = (\alpha_n(f,g))^* = h_n^*$.

(ii) First, observe that for each $n \in \mathbb{N}$, we have

$$h_{n+1} = \alpha_{n+1}(f,g) = \alpha_0(\alpha_n(f,g), \alpha_n^*(f,g)) = \alpha_0(h_n, k_n)$$

and

$$\begin{aligned} k_{n+1} &= \alpha_{n+1}^*(f,g) = (\alpha_{n+1}(f^*, g^*))^* = (\alpha_0(\alpha_n(f^*, g^*), \alpha_n^*(f^*, g^*)))^* \\ &= (\alpha_0(k_n^*, h_n^*))^* = (\alpha_0(h_n^*, k_n^*))^* = \alpha_0^*(h_n, k_n). \end{aligned}$$

In the same way, we prove that

$$h_{n+1}^* = \alpha_0^*(h_n^*, k_n^*) \quad \text{and} \quad k_{n+1}^* = \alpha_0(h_n^*, k_n^*).$$

Now, suppose that there exists $p \in \mathbb{N}^*$ such that

$$\begin{cases} h_{n+p} = \alpha_{p-1}(h_n, k_n), & k_{n+p} = \alpha_{p-1}^*(h_n, k_n) \\ h_{n+p}^* = \alpha_{p-1}^*(h_n^*, k_n^*), & k_{n+p}^* = \alpha_{p-1}(h_n^*, k_n^*). \end{cases}$$

We deduce that

$$\begin{aligned} h_{n+p+1} &= \alpha_0(h_{n+p}, k_{n+p}) = \alpha_0(\alpha_{p-1}(h_n, k_n), \alpha_{p-1}^*(h_n, k_n)) = \alpha_p(h_n, k_n) \\ k_{n+p+1} &= (\alpha_{n+p+1}(f^*, g^*))^* = (\alpha_0(\alpha_{n+p}(f^*, g^*), \alpha_{n+p}^*(f^*, g^*)))^* \\ &= (\alpha_0(k_{n+p}^*, h_{n+p}^*))^* = (\alpha_0(\alpha_{p-1}(h_n^*, k_n^*), \alpha_{p-1}^*(h_n^*, k_n^*)))^* \\ &= (\alpha_p(h_n^*, k_n^*))^* = \alpha_p^*(h_n, k_n) \end{aligned}$$

Furthermore

$$\begin{aligned} k_{n+p+1}^* &= \alpha_{n+p+1}(f^*, g^*) = \alpha_0(\alpha_{n+p}(f^*, g^*), \alpha_{n+p}^*(f^*, g^*)) \\ &= \alpha_0(k_{n+p}^*, h_{n+p}^*) = \alpha_0(\alpha_{p-1}(h_n^*, k_n^*), \alpha_{p-1}^*(h_n^*, k_n^*)) = \alpha_p(h_n^*, k_n^*) \\ h_{n+p+1}^* &= \alpha_{n+p+1}^*(f^*, g^*) = (\alpha_{n+p+1}(f,g))^* \\ &= (\alpha_0(\alpha_{n+p}(f,g), \alpha_{n+p}^*(f,g)))^* = (\alpha_0(h_{n+p}, k_{n+p}))^* \\ &= (\alpha_0(\alpha_{p-1}(h_n, k_n), \alpha_{p-1}^*(h_n, k_n)))^* \\ &= (\alpha_p(h_n, k_n))^* = \alpha_p^*(h_n^*, k_n^*). \end{aligned}$$

Then, we conclude by recurrence.

(iii) It is immediate from (ii). □

5. A Geometrical interpretation

The above fundamental algorithm, associated with the functions f and g, can be written otherwise:

Put

$$\alpha_p(f,g) = h_p \text{ and } \alpha_p^*(f,g) = \alpha_p(f^*, g^*))^* = k_p.$$

Then

$$h_{p+1} = \alpha_{p+1}(f,g) = \frac{1}{2}\,(\alpha_p(f,g) + \alpha_p^*(f,g)),$$

thus

$$h_{p+1} = \frac{1}{2}(h_p + k_p) = \alpha_0(h_p, k_p).$$

Also

$$\begin{aligned} k_{p+1} &= \alpha_{p+1}^*(f,g) = (\alpha_{p+1}(f^*,g^*))^* \\ &= (\frac{1}{2}(\alpha_p(f^*,g^*) + \alpha_p^*(f^*,g^*)))^* \\ &= (\frac{1}{2}((\alpha_p^*(f,g))^* + (\alpha_p(f,g))^*))^*, \end{aligned}$$

then

$$k_{p+1} = (\frac{1}{2}(h_p^* + k_p^*))^* = \alpha_0^*(h_p, k_p).$$

Note that by (A) the following algorithm:

$$\begin{cases} h_0, k_0 \in (\mathbb{R}^m) & \text{are given} \\ h_{p+1} & = \frac{1}{2}(h_p + k_p) = \alpha_0(h_p, k_p) \ \ (p \geq 0) \\ k_{p+1} & = (\frac{1}{2}(h_p^* + k_p^*))^* = \alpha_0^*(h_p, k_p) \ \ (p \geq 0). \end{cases} \tag{A}$$

The fundamental algorithm, studied in the above section, is a particular case of (A) with

$$h_0 = \alpha_0(f,g) \ \text{ and } \ k_0 = \alpha_0^*(f,g).$$

Now, let $M \in \mathbb{R}^2, M \neq 0$. We denote by M^* the inverse of M in the inversion with center 0 and ratio 1, and thus we have: $\overrightarrow{OM^*} = \frac{1}{|\overrightarrow{OM}|^2}\overrightarrow{OM}$.

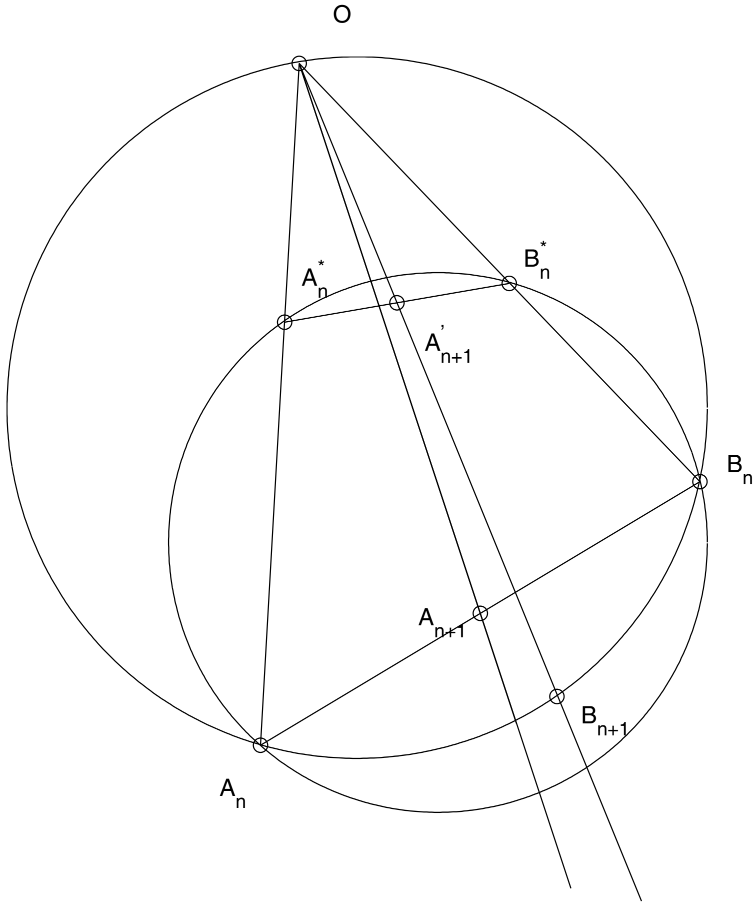
Let $A_0, B_0 \in \mathbb{R}^2, A_0 \neq O, B_0 \neq O$. We suppose that $\overrightarrow{OA}_0.\overrightarrow{OB}_0 > 0$, i.e, the angle of the vectors $\overrightarrow{OA}_0$ and $\overrightarrow{OB}_0$ is sharp. We consider the following algorithm:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad \overrightarrow{OA}_{n+1} &= \frac{1}{2}(\overrightarrow{OA}_n + \overrightarrow{OB}_n) \\ \overrightarrow{OA'}_{n+1} &= \frac{1}{2}(\overrightarrow{OA^*}_n + \overrightarrow{OB^*}_n) \end{aligned}$$

where

$$\overrightarrow{OB}_n = \frac{\overrightarrow{OA'_n}}{|\overrightarrow{OA'}_n|^2}.$$

We can represent that by the following graph:



We can observe that the points A_n, B_n, B^*_n, A^*_n are cocyclical, the point B_{n+1} is at the intersection of the straight line which supports the vector $\overrightarrow{OA'}_{n+1}$ and of the circle limited by the points OA_nB_n. That circle is the inverse of the straight line which is the support of the vector $\overrightarrow{A^*_nB^*_n}$.

Remark 5.1. When $\overrightarrow{OA_0} = \mu.\overrightarrow{OB_0}$ with $\mu \in \mathbb{R}$, we find again the case of the first section.

We put for all $n \in \mathbb{N}$:

$$|\overrightarrow{OA_n}| = a_n, |\overrightarrow{OA'}_n| = a'_n, |\overrightarrow{OB_n}| = b_n, \omega_n = \arccos \frac{\overrightarrow{OA_n}.\overrightarrow{OB_n}}{a_nb_n}.$$

Then $|\overrightarrow{OA}_{n+1}|^2 = |\frac{\overrightarrow{OA_n}+\overrightarrow{OB_n}}{2}|^2 = \frac{1}{4}(a_n^2 + b_n^2 + 2a_nb_n \cos\omega_n)$, and thus $a_{n+1} = \frac{1}{2}(a_n^2 + b_n^2 + 2a_nb_n \cos\omega_n)^{\frac{1}{2}}$. Also, we can write

$$\begin{aligned}|\overrightarrow{OA'}_{n+1}|^2 &= |\frac{1}{2}(\frac{\overrightarrow{OA_n}}{a_n^2} + \frac{\overrightarrow{OB_n}}{b_n^2})|^2 = \frac{1}{4}(\frac{1}{a_n^2} + \frac{1}{b_n^2} + \frac{2}{a_nb_n}\cos\omega_n) \\ &= \frac{1}{a_n^2b_n^2}\ |\overrightarrow{OA}_{n+1}|^2\end{aligned}$$

and then $\quad a'_{n+1} = \frac{a_{n+1}}{a_n b_n}$.

Since $|\overrightarrow{OB}_{n+1}| = \frac{1}{|\overrightarrow{OA'}_{n+1}|}$, then we deduce that for each $n \in \mathbb{N}$

$$b_{n+1} = \frac{a_n b_n}{a_{n+1}} \iff a_{n+1}\, b_{n+1} = a_n b_n.$$

Now we set

$$a_1 = \alpha_1\ (a_0, b_0)$$

then

$$b_1 = (\alpha_1(\frac{1}{a_0}, \frac{1}{b_0}))^{-1} \doteq \alpha_1^*(a_0, b_0).$$

Proposition 5.2. *For each $n \in \mathbb{N}$, $a_n = \alpha_n(a_0, b_0)$ and $b_n = \alpha_n^*(a_0, b_0)$.*

Proof. By recurrence, assume that there exists $p \in \mathbb{N}$ such that: $a_p = \alpha_p(a_0, b_0)$ and $b_p = (\alpha_p(\frac{1}{a_0}, \frac{1}{b_0}))^{-1} = \alpha_p^*(a_0, b_0)$.

Then

$$a_{p+1} = \frac{1}{2}(a_p + b_p) = \frac{1}{2}(\alpha_p(a_0, b_0) + \alpha_p^*(a_0, b_0)) \doteq \alpha_{p+1}(a_0, b_0)$$

and

$$b_{p+1} = \frac{a_p b_p}{a_{p+1}} = \frac{2\alpha_p(a_0, b_0).\alpha_p^*(a_0, b_0)}{\alpha_p(a_0, b_0) + \alpha_p^*(a_0, b_0)}$$

but

$$\begin{aligned}(\alpha_{p+1}(\frac{1}{a_0}, \frac{1}{b_0}))^{-1} &= 2(\alpha_p(\frac{1}{a_0}, \frac{1}{b_0}) + \alpha_p^*(\frac{1}{a_0}, \frac{1}{b_0}))^{-1} \\ &= 2(\frac{1}{\alpha_p^*(a_0, b_0)} + \frac{1}{\alpha_p(a_0, b_0)})^{-1}\end{aligned}$$

Then $b_{p+1} = (\alpha_{p+1}(\frac{1}{a_0}, \frac{1}{b_0}))^{-1} \doteq \alpha_{p+1}^*(a_0, b_0)$. □

According to the study of the first section, we deduce the following proposition

Proposition 5.3. $\lim_n a_n = \lim_n b_n = \tau(a_0, b_0) = \sqrt{a_0 b_0}$.

When n tends to $+\infty$, the behaviour of the straight line which supports the vectors $\overrightarrow{OA}_n$ and $\overrightarrow{OB}_n$ is given as follows

Proposition 5.4.

(i) *For all $n \in \mathbb{N}, \cos\omega_{n+1} = \frac{2a_0b_0 + (a_n^2 + b_n^2)\cos\omega_n}{a_n^2 + b_n^2 + 2a_0b_0\cos\omega_n}$*

(ii) *If one put*

$$\cos\xi_n = \frac{\overrightarrow{OA}_{n-1}.\overrightarrow{OA}_n}{a_{n-1}.a_n}$$
$$\cos\eta_n = \frac{|\overrightarrow{OB}_{n-1}.\overrightarrow{OA'_n}|}{b_{n-1}.a'_n},$$

then $\cos\xi_n = \cos\eta_n$.

Proof. (i) We know that for all $n \in \mathbb{N}$, $\overrightarrow{OA}_n.\overrightarrow{OA'}_n = a_n.a'_n.\cos\omega_n$, then

$$\begin{aligned}
a_{n+1}.a'_{n+1}.\cos\omega_{n+1} &= \overrightarrow{OA}_{n+1}.\overrightarrow{OA'}_{n+1} \\
&= \frac{1}{4}\left(\overrightarrow{OA}_n + \overrightarrow{OB}_n\right).\left(\frac{\overrightarrow{OA}_n}{a_n^2} + \frac{\overrightarrow{OB}_n}{b_n^2}\right) \\
&= \frac{1}{4}\left(\frac{\overrightarrow{OA}_n^2}{a_n^2} + \frac{\overrightarrow{OB}_n^2}{b_n^2} + \left(\frac{1}{a_n^2} + \frac{1}{b_n^2}\right)\overrightarrow{OA}_n.\overrightarrow{OB}_n\right) \\
&= \frac{1}{4a_n^2 b_n^2}\left(2a_n^2 b_n^2 + (a_n^2 + b_n^2)a_n b_n \ \cos\omega_n\right).
\end{aligned}$$

If we recall that: $a'_{n+1}.a_{n+1} = \frac{a_{n+1}^2}{a_n b_n}$ and $a_n b_n = a_0 b_0$, then we deduce the result of (i).

(ii) For each $n \in \mathbb{N}$,

$$\cos\xi_n = \frac{1}{2a_n a_{n-1}}\left(\overrightarrow{OA}_{n-1}\right).\left(\overrightarrow{OA}_{n-1} + \overrightarrow{OB}_{n-1}\right)$$

and hence

$$\cos\xi_n = \frac{1}{2a_n a_{n-1}}\ \left(a_{n-1}^2 + a_{n-1}b_{n-1}\cos\omega_{n-1}\right).$$

Also

$$\begin{aligned}
\cos\eta_n &= \frac{1}{2a'_n b_{n-1}}\left(\overrightarrow{OB}_{n-1}\right).\left(\frac{\overrightarrow{OA}_{n-1}}{a_{n-1}^2} + \frac{\overrightarrow{OB}_{n-1}}{b_{n-1}^2}\right) \\
&= \frac{1}{2a'_n b_{n-1}}\left(1 + \frac{a_{n-1}b_{n-1}}{a_{n-1}^2}\cos\omega_{n-1}\right).
\end{aligned}$$

Then

$$\cos\eta_n = \frac{1}{2a'_n b_n a_{n-1}^2}\ \left(a_{n-1}^2 + a_{n-1}b_{n-1}\cos\omega_{n-1}\right).$$

But $a'_n b_{n-1} a_{n-1}^2 = a'_n(b_{n-1}a_{n-1})\ a_{n-1} = a'_n b_n a_n a_{n-1} = a_n.a_{n-1}$ and finally $\cos\xi_n = \cos\eta_n$. □

Proposition 5.5. *When n tends to $+\infty$, the vectors $\overrightarrow{OA}_n$ and $\overrightarrow{OB}_n$ converge to a vector $\overrightarrow{OA}_\infty$ which is supported by the bisector of the angle limited by the vectors $\overrightarrow{OA}_0$ and $\overrightarrow{OB}_0$. Moreover $|\overrightarrow{OA}_\infty| = \tau(a_0, b_0) = \sqrt{a_0 b_0}$.*

Proof. By the above proposition, we deduce that $\lim_n \cos\omega_n = 1$ and consequently $\lim_n \ \omega_n = 0$. Further, the straight lines which support the vectors $\overrightarrow{OA}_n$ and $\overrightarrow{OB}_n$ have the same bisector as the vectors $\overrightarrow{OA}_0$ and $\overrightarrow{OB}_0$. We deduce that the straight line which supports $\overrightarrow{OA}_n$ (resp.$\overrightarrow{OB}_n$) tends to the bisector of the angle limited by the vectors $\overrightarrow{OA}_0$ and $\overrightarrow{OB}_0$ when n tends to $+\infty$. Since $\lim_n \ a_n = \lim_n \ b_n = \sqrt{a_0 b_0}$, we have: $\lim_n \ \overrightarrow{OA}_n = \lim_n \ \overrightarrow{OB}_n = \overrightarrow{OA}_\infty$ with $|\overrightarrow{OA}_\infty| = \sqrt{a_0 b_0}$. This concludes the proof. □

The complex version of the previous algorithm can be given as follows: Let $z_0, t_0 \in \mathbb{C}^*$, $z_0 = a_0 \, e^{iu_0}$ and $t_0 = b_0 e^{iv_0}$ we put that for each $n \in \mathbb{N}$

$$z_{n+1} = \frac{1}{2}\,(z_n + t_n)$$

and

$$t_{n+1} = \frac{1}{\frac{1}{2}(\frac{1}{z_n} + \frac{1}{t_n})} = \frac{2 z_n t_n}{z_n + t_n}.$$

In fact, it is sufficent to note that if the point $M \in \mathbb{R}^2, M \neq 0$, has the affix $z = ae^{iu}$ then the inverse of M in the inversion of center 0 and the ratio 1 has the affix

$$z^* = \frac{1}{\bar{z}} = \frac{1}{a} e^{iu}.$$

According to the above study we deduce that:

Proposition 5.6. $\lim_n \ z_n = \lim_n \ t_n = z_\infty = \sqrt{a_0 b_0} \ e^{i\frac{u_0+v_0}{2}}$.

Observing that for each $n \in \mathbb{N} \quad t_{n+1} = \frac{z_n t_n}{z_{n+1}}$ then $z_{n+1} = \frac{1}{2}(z_n + \frac{z_0 t_0}{z_n})$, we deduce:

Corollary 5.7. *Let $\omega_0 \in \mathbb{C}$, $\omega_0 = r_0 \, e^{i\vartheta_0}$. The sequence of complex numbers $(z_n, n \in \mathbb{N})$ such that $z_0 \neq 0$ and $\forall n \in \mathbb{N} \ \ z_{n+1} = \frac{1}{2}(z_n + \frac{\omega_0}{z_n})$ converges to $z_\infty = \sqrt{r_0} \ e^{i\frac{\vartheta_0}{2}}$.*

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