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A Characterization of Convex and Semicoercive Functionals

Samir Adly
LACO, Université de Limoges,
123 Avenue A. Thomas, 87060 Limoges Cedex, France.
e-mail: adly@unilim.fr

Emil Ernst*
Institute of Mathematics of the Romanian Academy,
P. Box 1-764, RO 70700, Bucharest, Romania.
e-mail: ernst@imar.ro

Michel Théra
LACO, Université de Limoges,
123 Avenue A. Thomas, 87060 Limoges Cedex, France.
e-mail: michel.thera@unilim.fr

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In this paper we prove that every proper convex and lower semicontinuous functional Φ defined on a real reflexive Banach space X is semicoercive if and only if every small uniform perturbation of Φ attains its minimum value on X.

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1. Introduction

Throughout the paper X denotes a real reflexive Banach space, X^* its topological dual and $\langle \cdot, \cdot \rangle$ the associated duality pairing on $X \times X^*$. We write "$\rightarrow$" and "$\rightharpoonup$" to denote respectively the strong and the weak convergence on X. We denote by $B(x, r)$ (respectively by $\bar{B}(x, r)$) the open ball (respectively closed ball) with center x and radius $r > 0$. Following the standard notations used in convex analysis, $\Gamma_0(X)$ stands for the set of all convex lower semicontinuous proper (not identically equal to $+\infty$) extended-real-valued functionals $\Phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$.

By $\operatorname{argmin} \Phi$ we mean the (possibly empty) set of all $x \in X$ where Φ attains its minimum value, i.e.

$$\Phi(x) \leq \Phi(y), \ \forall\, y \in X.$$

Recall that a functional $\Phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be semicoercive if there exists a

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closed subspace V of X such that

$$\Phi(x) = \Phi(x + v),\ \forall x \in X,\ \forall v \in V,$$

and the quotient functional $\overline{\Phi} : X/V \to \mathbb{R} \cup \{+\infty\}$, defined by:

$$\overline{\Phi}(\overline{x}) = \Phi(x),\ \forall x \in \overline{x},\ \forall \overline{x} \in X/V,$$

is coercive, in the sense that $\left(\overline{\Phi}\right)^{-1}((-\infty, M])$ is bounded for every $M \in \mathbb{R}$.

Let us remark that every coercive functional is semicoercive. Let us also mention two of the most frequently encountered semicoercive functionals:

(i) The distance functional to a closed subspace F of an arbitrary Banach space X,

$$J_1 : X \to \mathbb{R},\ J_1(x) = \operatorname{dist}(x, F) = \inf_{y \in F} \|x - y\|.$$

(ii) If $\Omega \subseteq \mathbb{R}^n$ is a bounded subset with a smooth boundary, and $H^1(\Omega)$ is the corresponding Sobolev space,

$$J_2 : H^1(\Omega) \to \mathbb{R},\ J_2(u) = \int_\Omega |\nabla u(x)|^2\, dx.$$

By a uniform perturbation of Φ we mean a functional Ψ which satisfies for some $\varepsilon > 0$

$$\Phi(x) - \varepsilon \le \Psi(x) \le \Phi(x) + \varepsilon\ , \forall x \in X.$$

Obviously, every convex uniform perturbation of a semicoercive functional remains semicoercive. It is a well-known fact that every $\Gamma_0(X)$-functional, which is semicoercive attains its minimum on X. Consequently, in the class of $\Gamma_0(X)$-semicoercive functionals, the existence of a minimum on X is preserved under every uniform perturbation.

Our aim in this paper is to study if the converse of the preceding observation is true. More precisely, we prove (Theorem 4.2) that if $\Phi \in \Gamma_0(X)$ and if every small uniform perturbation of Φ within the class $\Gamma_0(X)$ reaches its minimum value on X, then Φ is necessarily semicoercive.

The paper is organized as follows. Section 2 contains some classical convex analysis results which are needed throughout the paper, together with the proof of the technical Lemma 2.1. Section 3 is devoted to the proof of Theorem 3.1, which characterizes unbounded, linearly bounded convex closed sets (that is unbounded sets with a recession cone reduced to the singleton $\{0\}$). This theorem plays a central role in the proof of the main result which is given in Section 4.

The final section contains the conclusions and some open problems.

2. Background material and preliminary results

Let us recall some basic definitions from convex analysis. For $\Phi \in \Gamma_0(X)$, we denote by $\operatorname{Dom}\Phi$, the *effective domain* of Φ which is defined by

$$\operatorname{Dom}\Phi = \{x \in X\ :\ \Phi(x) < +\infty\}$$

and by epi Φ the *epigraph* of Φ. Let K be a nonempty convex and closed subset of X. Following Rockafellar [9], the recession cone K_∞ of K is defined by:

$$K_\infty = \bigcap_{t>0} t\Big(K - x_0\Big), \quad x_0 \in K.$$

Note that K_∞ is independent of $x_0 \in K$ and that K_∞ is a closed convex cone of X and describes the global behavior of the convex set K.
For each $\Phi \in \Gamma_0(X)$, taking $C = \operatorname{epi}\Phi$ we define the *recession functional* of Φ as the function Φ_∞ such that $\operatorname{epi}\Phi_\infty = (\operatorname{epi}\Phi)_\infty$. Equivalently, it amounts to saying that

$$\Phi_\infty(x) := \lim_{t\to+\infty} \frac{\Phi(x_0+tx)-\Phi(x_0)}{t},$$

where x_0 is any point in $\operatorname{Dom}\Phi$.
In the sequel, $\operatorname{Ker}\Phi_\infty$ will denote the closed convex cone defined by

$$\operatorname{Ker}\Phi_\infty = \{x \in X \ : \ \Phi_\infty(x) = 0\}.$$

The following properties of Φ_∞ will be used in the sequel. The proofs of (1)–(3) are straightforward and were established in finite dimension in [9]; the reader is for instance refered to [5] for (4).

$$\Phi_\infty \in \Gamma_0(X) \tag{1}$$
$$\Phi(x+y) \le \Phi(x) + \Phi_\infty(y), \quad \forall x, y \in X \tag{2}$$
$$\Phi_\infty(\lambda x) = \lambda \Phi_\infty(x) \ \ \forall x \in X, \ \forall \lambda \ge 0 \tag{3}$$
$$\Phi_\infty(x) \le \liminf_{n\to\infty} \frac{\Phi(x_0+s_n x_n)}{s_n}\ , \tag{4}$$

where x_0 is any point in $\operatorname{Dom}\Phi$, $(x_n)_{n\in\mathbb{N}^*}$ is any sequence in X converging to x and $(s_n)_{n\in\mathbb{N}^*}$ is any real sequence converging to $+\infty$.

Throughout the paper, $x_0 \in X$ and $M \in \mathbb{R}$ are defined such that :

$$x_0 \in \operatorname{argmin}\Phi \text{ and } M := \Phi(x_0).$$

Denoting by $\operatorname{co}\{(y,N), \operatorname{epi}\Phi\}$ the convex hull of (y,N) and epi Φ, for $y \in X$ and $N < M$, we set

$$\mathcal{F}(x) = \{s \in \mathbb{R} : \ (x,s) \in \overline{\operatorname{co}}\{(y,N), \operatorname{epi}\Phi\}\},$$

the closure being taken with respect to the natural strong topology on $X \times \mathbb{R}$. Obviously, $\mathcal{F}(x)$ is a closed, convex, (possibly empty) subset of $\mathbb{R}$; since

$$\emptyset \ne \overline{\operatorname{co}}\{(y,N), \operatorname{epi}\Phi\} \subseteq X \times [N,+\infty) \subset X \times \mathbb{R},$$

it follows that

$$\mathcal{F}(x) \subset [N,\infty), \ \forall\, x \in X.$$

Therefore, the relation

$$\Phi_{y,N}(x) = \begin{cases} \min \mathcal{F}(x) & \text{if } \mathcal{F}(x) \ne \emptyset \\ +\infty & \text{if } \mathcal{F}(x) = \emptyset \end{cases}$$

defines a proper extended-real-valued functional, whose epigraph is $\overline{\text{co}}\{(y, N), \text{epi}\,\Phi\}$.

Some of the most important properties of $\Phi_{y,N}$ are stated in the following Lemma.

Lemma 2.1. *For every functional $\Phi_{y,N}$, the following holds:*

(i) $\Phi_{y,N}$ *belongs to* $\Gamma_0(X)$*;*

(ii) $\Phi_{y,N}$ *is a minorant of* Φ*:* $\Phi_{y,N}(x) \leq \Phi(x),\ \forall\, x \in X$*; If, in addition,* Φ *attains its minimum value at* y*, then*

$$\Phi(x) + N - M \leq \Phi_{y,N}(x),\ \forall\, x \in X;$$

(iii) *for every* $x \in X$ *such that* $\Phi_{y,N}(x) \leq M$ *we have*

$$\Phi_{y,N}(\lambda y + (1-\lambda)x) = \lambda \Phi_{y,N}(y) + (1-\lambda)\Phi_{y,N}(x),\ \forall\, \lambda \in (0,1), \tag{5}$$

(iv) $\text{argmin}\,\Phi_{y,N} = y + \text{Ker}\,\Phi_\infty$.

Proof of Lemma 2.1. (i) Since $\text{epi}\,\Phi_{y,N} = \overline{\text{co}}\{(y, N), \text{epi}\,\Phi\}$ is a non-empty, closed and convex set, $\Phi_{y,N} \in \Gamma_0(X)$.

(ii) As

$$\text{epi}\,\Phi \subset \overline{\text{co}}\{(y, N), \text{epi}\,\Phi\},$$

we have

$$\Phi_{y,N}(x) \leq \Phi(x)\ \forall\, x \in X.$$

If, in addition, $\Phi(y) = M$, we have $(y, N) \in \text{epi}(\Phi + N - M)$ and $\text{epi}\,\Phi \subset \text{epi}(\Phi + N - M)$, so

$$\overline{\text{co}}\{(y, N), \text{epi}\,\Phi\} \subset \text{epi}(\Phi + N - M),$$

that is

$$\Phi(x) + N - M \leq \Phi_{y,N}(x),\ \forall\, x \in X.$$

(iii) We shall distinguish the cases $\Phi_{y,N}(x) < M$, and $\Phi_{y,N}(x) = M$. Let us first suppose that $\Phi_{y,N}(x) < M$. Set $z = \lambda y + (1-\lambda)x$; for every $\delta > 0$ set

$$\alpha := \min\left\{\delta(1-\lambda), (M - \Phi_{y,N}(x))(1-\lambda)\right\}. \tag{6}$$

Since $(z, \Phi_{y,N}(z)) \in \overline{\text{co}}\{(y, N), \text{epi}\,\Phi\}$, there exist $(w, \theta) \in \text{epi}\,\Phi$ and $0 \leq \mu \leq 1$ (depending on δ), such that

$$\begin{aligned}
&\text{dist}_{X\times\mathbb{R}}\left(\mu(y, N) + (1-\mu)(w, \theta), (z, \Phi_{y,N}(z))\right)^2 \\
&\quad = \|(\mu y + (1-\mu)w) - z\|^2 + |(\mu N + (1-\mu)\theta) - \Phi_{y,N}(z)|^2 \\
&\quad \leq \alpha^2.
\end{aligned}$$

Therefore

$$\mu N + (1-\mu)\theta \leq \Phi_{y,N}(z) + \alpha.$$

Since $M \leq \theta$ and $\alpha \leq (M - \Phi_{y,N}(x))(1-\lambda)$, the previous relation yields

$$\mu N + (1-\mu)M \leq \Phi_{y,N}(z) + (M - \Phi_{y,N}(x))(1-\lambda). \tag{7}$$

Using the convexity of $\Phi_{y,N}$, we deduce that

$$\begin{aligned}\Phi_{y,N}(z) &+ (M-\Phi_{y,N}(x))(1-\lambda) \\ &\le \lambda\Phi_{y,N}(y) + (1-\lambda)\Phi_{y,N}(x) + (1-\lambda)(M-\Phi_{y,N}(x)) \\ &= \lambda N + (1-\lambda)M.\end{aligned} \tag{8}$$

By (7) and (8) we get

$$\mu N + (1-\mu)M \le \lambda N + (1-\lambda)M.$$

Consequently, $\lambda \le \mu$, and therefore $0 \le \lambda_1 = (\mu-\lambda)/(1-\lambda) \le 1$. This yields

$$\lambda_1(y,N) + (1-\lambda_1)(w,\theta) \in \operatorname{co}\{(y,N), \operatorname{epi}\Phi\}.$$

Thus, since the relation

$$\begin{aligned}&\operatorname{dist}\Big((x, \frac{1}{1-\lambda}\Phi_{y,N}(z) - \frac{\lambda}{1-\lambda}\Phi_{y,N}(y)),\ \lambda_1(y,N) + (1-\lambda_1)(w,\theta)\Big) \\ &= \frac{1}{1-\lambda}\operatorname{dist}\Big((z,\Phi_{y,N}(z)),\ \mu(y,N) + (1-\mu)(w,\theta)\Big) \\ &\le \alpha\frac{1}{1-\lambda} \le \delta,\end{aligned}$$

holds for every $\delta > 0$, we derive that

$$\Big(x, \frac{1}{1-\lambda}\Phi_{y,N}(z) - \frac{\lambda}{1-\lambda}\Phi_{y,N}(y)\Big) \in \overline{\operatorname{co}}\{(y,N), \operatorname{epi}\Phi\}.$$

Hence,

$$\Phi_{y,N}(x) \le \frac{1}{1-\lambda}\Phi_{y,N}(\lambda y + (1-\lambda)x) - \frac{\lambda}{1-\lambda}\Phi_{y,N}(y).$$

This relation combined with the convexity of $\Phi_{y,N}$ implies (5).

Let us now consider the case $\Phi_{y,N}(x) = M$. Set $x_n = \frac{1}{n}y + \frac{n-1}{n}x$; since $\Phi_{y,N}$ is convex,

$$\Phi_{y,N}(x_n) \le \frac{1}{n}\Phi_{y,N}(y) + \frac{n-1}{n}\Phi_{y,N}(x) = \frac{1}{n}N + \frac{n-1}{n}M < M.$$

Hence relation (5) holds for every x_n. Therefore, for every λ in $(0,1)$ and every n in $\mathbb{N}^*$ we have

$$\Phi_{y,N}(\lambda y + (1-\lambda)x_n) = \lambda\Phi_{y,N}(y) + (1-\lambda)\Phi_{y,N}(x_n). \tag{9}$$

On $[0,1]$, the mappings $\mu \mapsto \Phi_{y,N}(\mu y + (1-\mu)x)$ and $\mu \mapsto \Phi_{y,N}(\lambda y + (1-\lambda)(\mu y + (1-\mu)x))$ are convex and lower semicontinuous , thus continuous. Consequently, we obtain relation (5) by taking in (9) the limit as n goes to infinity.

(iv) Take x in X such that $\Phi_{y,N}(x) = N$. We may assume without loss of generality that $x \neq y$. Since (x,N) belongs to $\overline{\operatorname{co}}\{(y,N), \operatorname{epi}\Phi\}$, for every n in $\mathbb{N}^*$ there are (x_n,θ_n) in $\operatorname{epi}\Phi$ and μ_n in $[0,1)$ such that

$$\operatorname{dist}\Big(\mu_n(y,N) + (1-\mu_n)(x_n,\theta_n), (x,N)\Big) \le \frac{1}{n}. \tag{10}$$

By (10) we deduce, in particular, that $\mu_n N + (1-\mu_n)\theta_n \le N + \frac{1}{n}$, and since $\theta_n \ge M$, we obtain

$$1-\mu_n \le \frac{1}{n(M-N)}. \tag{11}$$

Relation (10) implies that $\lim_{n\to\infty} \Big|\mu_n N + (1-\mu_n)\theta_n - N\Big| = 0$; as from (11) it follows that $\lim_{n\to\infty} \mu_n = 1$, we deduce that $\lim_{n\to\infty}(1-\mu_n)\theta_n = 0$. Since $\Phi(x_n) \le \theta_n$, the previous relation yields

$$\limsup_{n\to\infty}(1-\mu_n)\Phi(x_n) \le 0. \tag{12}$$

On the other hand, relation (10) implies that

$$\lim_{n\to\infty} \Big\|\mu_n y + (1-\mu_n)x_n - x\Big\| = 0;$$

since $\lim_{n\to\infty} \mu_n = 1$, we obtain that

$$\lim_{n\to\infty} \Big\|(1-\mu_n)x_n - (x-y)\Big\| = 0,$$

which implies

$$\lim_{n\to\infty} \Big\|(1-\mu_n)(x_n - z_0) - (x-y)\Big\| = 0,$$

where z_0 is an arbitrary element of $\operatorname{Dom}\Phi$. We may therefore apply (4) for $x-y$ instead of x, $1/(1-\mu_n)$, and $(1-\mu_n)(x_n - z_0)$ instead of x_n and $s_n := 1/(1-\mu_n)$ (note that $\mu_n \neq 1$), and as

$$x_n = z_0 + \frac{1}{1-\mu_n}[(1-\mu_n)(x_n - z_0)],$$

we obtain

$$0 \le \Phi_\infty(x-y) \le \liminf_{n\to\infty}(1-\mu_n)\Phi(x_n). \tag{13}$$

By (12) and (13) it follows that $x-y$ belongs to $\operatorname{Ker}\Phi_\infty$, that is $x \in y + \operatorname{Ker}\Phi_\infty$.

Since obviously $\Phi_{y,N}$ reaches its minimum value at every point of $y + \operatorname{Ker}\Phi_\infty$, the proof of Lemma 2.1 is established. □

3. Unbounded linearly bounded closed convex sets

Recall that a closed convex set K is *linearly bounded* if $K_\infty = \{0\}$.

Theorem 3.1 below states an important property of convex, closed, unbounded and linearly bounded sets. This result shows that, contrary to the finite dimensional setting (see [9], chapter 8), in a general linear space the recession cone does not characterize completely the behavior "at infinity" of a convex set.

Theorem 3.1. *Let K be a closed, convex, unbounded and linearly bounded subset of a reflexive Banach space X. Then, there exists $h \in X^*$ such that*

$$\inf_{w\in K} \langle h, w\rangle < \langle h, u\rangle \le 1,\ \forall u \in K. \tag{14}$$

For every closed convex subset K of X we define the *barrier cone* $\mathcal{B}(K)$ of K as the domain of the support functional σ_K of K defined by $\sigma_K(f) := \sup_{x\in K}\langle f, x\rangle$. In other words,

$$\mathcal{B}(K) = \{f \in X^* : \sigma_K(f) < +\infty\} = \operatorname{Dom}\sigma_K.$$

The following lemma is an elementary consequence of a known result.

Lemma 3.2. *Suppose K is a closed convex and linearly bounded set of X. Then $\mathcal{B}(K)$ is dense in X^*.*

Proof of Lemma 3.2. It is a well-known fact that the recession cone K_∞ of K is the polar of $\mathcal{B}(K)$ (see for instance [9] 14.2.1 in finite dimension and [3], Proposition 3.10 in infinite dimension). Therefore, by the bipolar theorem we obtain $X^* = \mathcal{B}(K)^{oo} = \overline{\mathcal{B}(K)}$ (since X is reflexive) and the proof is complete. □

Given $R > 0$ we define

$$\mathcal{B}_R(K) := \{f \in \mathcal{B}(K) : \exists x \in K \text{such that} \langle f, x\rangle > R\}.$$

Lemma 3.3. *Suppose K is a convex closed, unbounded and linearly bounded subset of X. If $\mathcal{B}(K)$ is a linear space, then for every $R > 0$, $\mathcal{B}_R(K)$ is dense in X^*.*

Proof of Lemma 3.3. Let us suppose that, for some $R > 0$, $\mathcal{B}_R(K)$ is not dense in X^*. Then, take f in X^* and $\varepsilon > 0$ such that

$$(f + B(0, \varepsilon)) \cap \overline{\mathcal{B}_R(K)} = \emptyset.$$

Using the previous Lemma, we observe that

$$f + B(0, \varepsilon) \subset X^* = \overline{\mathcal{B}(K)} = \overline{\mathcal{B}_R(K)} \cup \overline{\mathcal{B}(K) \setminus \mathcal{B}_R(K)},$$

and therefore, as $\mathcal{B}(K)\setminus\mathcal{B}_R(K) = \mathcal{B}(K)\cap \bigcap_{x\in K}\{f \in X^* : \langle f, x\rangle \le R\}$ is convex and closed,

$$f + B(0, \varepsilon) \subseteq \mathcal{B}(K) \setminus \mathcal{B}_R(K). \tag{15}$$

Accordingly, the linear space $\mathcal{B}(K)$ has a nonempty interior, from where it follows that $\mathcal{B}(K) = X^*$,which by the principle of uniform boundedness implies that K is bounded, a contradiction. □

We have proved that for every element $f \in X^*$ and for every $R > 0$, there exists a sequence $(f_n)_{n\in\mathbb{N}^*}$ in $\mathcal{B}_R(K)$ which converges to f and satisfies $\sigma_K(f_n) > R$.

This result has an immediate consequence.

Lemma 3.4. *Let f be in $\mathcal{B}(K)$. In the assumptions of* Lemma 3.3*, for every constant R, satisfying $R > \sigma_K(f)$, and every $\gamma > 0$, there is a sequence $(g_n)_{n\in\mathbb{N}^*}$ converging to f, such that*

$$\sigma_K(g_n) = R, \text{ and } \langle g_n, x\rangle < R, \forall x \in (K \cap \bar{B}(0, \gamma)), \ \forall n \in \mathbb{N}^*.$$

Proof of Lemma 3.4. According to Lemma 3.3, there is a sequence $(f_n)_{n\in\mathbb{N}^*}$ converging to f such that

$$R \leq \sigma_K(f_n) < +\infty.$$

As, for every n, the function $\lambda \to \sigma_K(\lambda f + (1-\lambda)f_n)$ is continuous on $[0,1]$, there is $\lambda_n \in [0,1]$ such that $\sigma_K(g_n) = R$, where

$$g_n = \lambda_n f + (1-\lambda_n)f_n.$$

Denoting by $\{\lambda_n\}_{n\in\mathbb{N}}$ a converging subsequence of $\{\lambda_n\}_{n\in\mathbb{N}}$, the sequence $\{g_n\}_{n\in\mathbb{N}}$ obviously converges to f.
Moreover, at least starting from a certain rank, this subsequence also satisfies the second condition of the lemma. Indeed, otherwise we would find a subsequence $(g_m)_{m\in\mathbb{N}^*}$ of $(g_n)_{n\in\mathbb{N}^*}$ such that $x_m \in K$, $\|x_m\| \leq \gamma$ and $\langle g_m, x_m\rangle = R$. Consequently,

$$|R - \langle f, x_m\rangle| = |\langle g_m - f, x_m\rangle| \leq \gamma\|f - g_m\|,$$

and, letting $m \to +\infty$, we would obtain

$$R > \sigma_K(f) \geq \lim_{m\to\infty} \langle f, x_m\rangle = R,$$

a contradiction and the result follows. □

We have now all the ingredients which are necessary to prove the main result of this section.

Proof of Theorem 3.1. Let us first remark that, if $\mathcal{B}(K)$ is not a linear space, then Theorem 3.1 is established. Indeed, in this case, there is $f \in \mathcal{B}(K)$ such that $-f \notin \mathcal{B}(K)$, and $h = f/\sigma_K(f)$ verifies (14), as $\inf_{w\in K} \langle h, w\rangle = -\infty$.

Let us now consider the case where $\mathcal{B}(K)$ is a linear space. Without loss of generality we may assume that $0 \in K$. In order to define the element h of X^*, we define by induction a sequence $(h_n, \gamma_n, \varepsilon_n, x_n)_{n\in\mathbb{N}^*} \subset X^* \times \mathbb{R} \times \mathbb{R} \times K$ as follows.

For $n = 0$, let us put $h_0 = 0$, $\gamma_0 = 1$, $\varepsilon_0 = 1$, $x_0 \in K$ arbitrary. For $n = 1$, take an element h_1 of X^* satisfying

$$\sup_{x\in K} \langle h_1, x\rangle = \frac{3}{4} \text{ and } \langle h_1, x\rangle < \frac{3}{4},\ \forall x \in K \cap \bar{B}(0,1),$$

(Lemma 3.4 applied for $f = 0$, $R = 3/4$ and $\gamma = 1$, ensures the existence of such an element). Consequently, there is some $x_1 \in K$ such that $\langle h_1, x_1\rangle = 2/3$; take $\gamma_1 = \max(2, \|x_1\|)$. Finally, set

$$\varepsilon_1 = \min\left(\frac{3}{4} - \sup_{x\in K\cap\bar{B}(0,1)} \langle h_1, x\rangle, \frac{1}{6\gamma_1}\right). \tag{16}$$

Let us now suppose that the sequence was defined for each i, $1 \leq i \leq n-1$ in such a way that the following relations hold for every $1 \leq i \leq n-1$:

$$\|h_i - h_{i-1}\| < \varepsilon_{i-1}, \quad (17)$$

$$\sup_{x \in K} \langle h_i, x \rangle = \frac{i+2}{i+3} \text{ and } \langle h_i, x \rangle < \frac{i+2}{i+3}, \forall x \in K \cap \bar{B}(0, \gamma_{i-1}),$$

$$\langle h_i, x_i \rangle = (i+1)/(i+2),$$

$$\gamma_i = \max(\gamma_{i-1} + 1, \|x_i\|),$$

$$\varepsilon_i = \min\left\{ \frac{i+2}{\gamma_{i-1}(i+3)} - \sup_{x \in K \cap \bar{B}(0,\gamma_{i-1})} \frac{\langle h_i, x \rangle}{\gamma_{i-1}}, \frac{1}{(i+1)(i+2)\gamma_i}, \varepsilon_{i-1} - \|h_i - h_{i-1}\| \right\}.$$

Lemma 3.4, applied for $f = h_{n-1}$, $R = (n+2)/(n+3)$ and $\gamma = \gamma_{n-1}$, yields the existence of an element h_n of X^* such that

$$\|h_n - h_{n-1}\| < \varepsilon_{n-1},$$

and

$$\sup_{x \in K} \langle h_n, x \rangle = \frac{n+2}{n+3} \text{ and } \langle h_n, x \rangle < \frac{n+2}{n+3}, \forall x \in K \cap \bar{B}(0, \gamma_{n-1}).$$

Consequently, there exists x_n in K such that $\langle h_n, x_n \rangle = \dfrac{n+1}{n+2}$.
Set $\gamma_n = \max(\gamma_{n-1} + 1, \|x_n\|)$, and put ε_n for the following (strictly positive) expression:

$$\min\left\{ \frac{n+2}{\gamma_{n-1}(n+3)} - \sup_{x \in K \cap \bar{B}(0,\gamma_{n-1})} \frac{\langle h_n, x \rangle}{\gamma_{n-1}}, \frac{1}{(n+1)(n+2)\gamma_n}, \varepsilon_{n-1} - \|h_n - h_{n-1}\| \right\}.$$

The sequence $(h_n, \gamma_n, \varepsilon_n, x_n)_{n \in \mathbb{N}}$ defined inductively satisfies the relations (17).
The last relation in (17) implies that

$$h_i + \bar{B}(0, \varepsilon_i) \subset h_{i-1} + \bar{B}(0, \varepsilon_{i-1}), \forall i \geq 2;$$

since $\gamma_n \geq n+1$, we deduce from the previous relation that, for every $m > n$,

$$\|h_m - h_n\| < \varepsilon_n < \frac{n+2}{\gamma_{n-1}(n+3)} < \frac{1}{n}. \quad (18)$$

Relation (18) means that the sequence $(h_n)_{n \in \mathbb{N}^*}$ is a Cauchy sequence in X^* and therefore converges to some $h_\infty \in X^*$.
By relation (18), we derive that

$$\|h_n - h_\infty\| \leq \varepsilon_n \leq \frac{1}{\gamma_{n-1}} \left(\frac{n+2}{n+3} - \sup_{x \in K \cap \bar{B}(0,\gamma_{n-1})} \langle h_n, x \rangle \right).$$

It follows that, for every $x \in K \cap \bar{B}(0, \gamma_{n-1})$,

$$\begin{aligned} \langle h_\infty, x\rangle &= \langle h_n, x\rangle + \langle h_\infty - h_n, x\rangle \leq \\ &\leq \langle h_n, x\rangle + \|x\| \cdot \|h_\infty - h_n\| \\ &\leq \langle h_n, x\rangle + \gamma_{n-1}\frac{1}{\gamma_{n-1}}\left(\frac{n+2}{n+3} - \sup_{x\in K\cap\bar{B}(0,R_{n-1})}\langle h_n, x\rangle\right) \\ &= \frac{n+2}{n+3} + \left(\langle h_n, x\rangle - \sup_{x\in K\cap\bar{B}(0,\gamma_{n-1})}\langle h_n, x\rangle\right) \leq \frac{n+2}{n+3}. \end{aligned} \tag{19}$$

Let x be in K; there exists $n \in \mathbb{N}^*$ such that $\|x\| \leq \gamma_{n-1}$. Hence relation (19) implies

$$\langle h_\infty, x\rangle \leq \frac{n+2}{n+3} < 1,\ \forall x \in K, \tag{20}$$

and therefore

$$\sup_{x\in K}\langle h_\infty, x\rangle \leq 1. \tag{21}$$

Again from relation (18) we deduce that

$$\|h_n - h_\infty\| \leq \varepsilon_n \leq \frac{1}{(n+1)(n+2)\gamma_n},$$

that is

$$\begin{aligned} \langle h_\infty, x_n\rangle &= \langle h_n, x_n\rangle + \langle h_\infty - h_n, x_n\rangle \\ &\geq \frac{n+1}{n+2} - \|h_n - h_\infty\|\|x_n\| \\ &\geq \frac{n+1}{n+2} - \frac{1}{(n+1)(n+2)} = \frac{n}{n+1}. \end{aligned} \tag{22}$$

Consequently

$$\sigma_K(h_\infty) \geq \lim_{n\to\infty}\langle h_\infty, x_n\rangle = 1. \tag{23}$$

Combining (20), (21) and (23), it follows that, for any $x \in K$,

$$\langle h_\infty, x\rangle < 1 = \sigma_K(h_\infty).$$

The mapping h defined by

$$h = \begin{cases} h_\infty & \text{if } \inf_{w\in K}\langle h_\infty, w\rangle = -\infty \\ h_\infty \cdot \left[\min\{-1, \inf_{x\in K}\langle h_\infty, x\rangle\}\right]^{-1} & \text{if } \inf_{w\in K}\langle h_\infty, w\rangle > -\infty \end{cases} \tag{24}$$

satisfies the conclusion of Theorem 3.1. $\square$

4. The main result

We state the following result; its proof will be presented afterwards.

Theorem 4.1. *Let Φ be a $\Gamma_0(X)$-functional which achieves its minimum value on X. Assume that either*

(a) $\operatorname{Ker}\Phi_\infty$ *is not a linear subspace;*

or

(b) Φ *is non-semicoercive and* $\operatorname{Ker}\Phi_\infty$ *is a linear space.*

Then, for every $\varepsilon > 0$, there exists $\Phi^\varepsilon \in \Gamma_0(X)$, such that

$$\Phi(x) - \varepsilon \leq \Phi^\varepsilon(x) \leq \Phi(x), \quad \forall x \in X,$$

and $\operatorname{argmin}\Phi^\varepsilon = \emptyset$.

The result which follows is an immediate consequence of Theorem 4.1 and can be considered as the **main result** of the paper.

Theorem 4.2 (Main Result). *Let Φ be a $\Gamma_0(X)$-functional. Suppose that Φ and every small uniform perturbation of Φ (in the class $\Gamma_0(X)$) achieve its minimum value on X. Then, Φ is necessarily semicoercive.*

The proof given below is based on preceding results established in Sections 2 and 3.

Proof of Theorem 4.1.
Case (a): In this case, the functional Φ cannot be semicoercive. Let $x_0 \in \operatorname{argmin}\Phi$ and $M \in \mathbb{R}$ be such that $\Phi(x_0) = M$. Since Φ is bounded from below, $\Phi_\infty(x) \geq 0$ for every x in X. It follows from (2) and (3) that for every $v \in \operatorname{Ker}\Phi_\infty$, we have

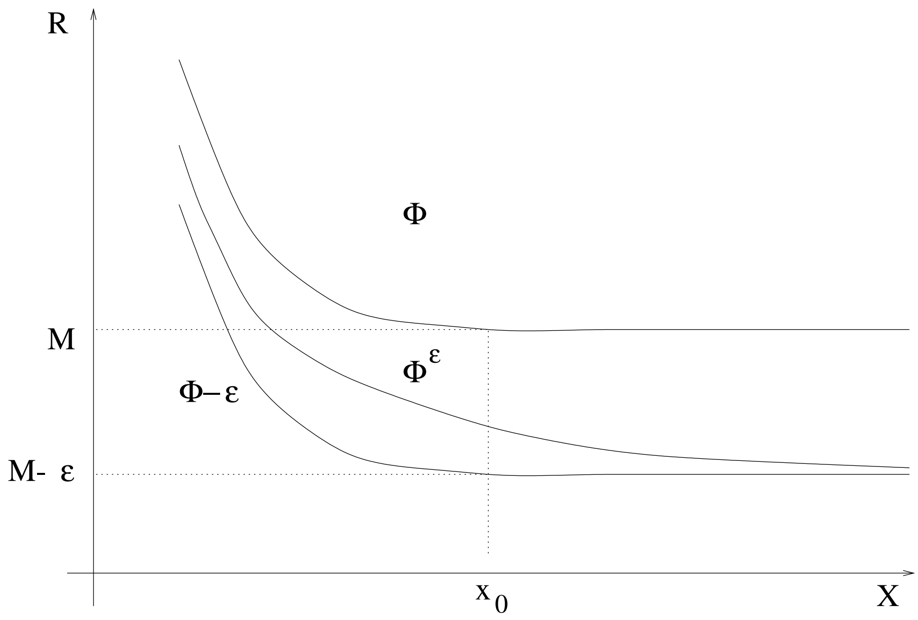
$$\Phi(x + tv) \leq \Phi(x), \ \forall\, t \geq 0 \text{ and } x \in X. \tag{25}$$

Since $\operatorname{Ker}\Phi_\infty$ is not a linear subspace of X, there exists v in X such that $\Phi_\infty(v) = 0$ and $\Phi_\infty(-v) > 0$. Let $\sigma : [0,1) \to \mathbb{R}$ be a convex and increasing function satisfying $\sigma(0) = 0$ and

$$\lim_{t \to 1^-} \sigma(t) = +\infty. \tag{26}$$

For example take $\sigma(t) = \dfrac{t}{1-t}$. We then define Φ^ε by

$$\Phi^\varepsilon(x) = \inf_{0 \leq t < 1} \left[\Phi(x - \sigma(t)v) - \varepsilon t\right]. \tag{27}$$



Let us prove that Φ^ε meets the requirements of Theorem 4.1.

Lemma 4.3. *The functional Φ^ε defined in* (27) *is a $\Gamma_0(X)$-functional.*

Proof of Lemma 4.3. (i) The functional Φ^ε is an extended real-valued functional. In fact, since M is the minimum of Φ, we have

$$\Phi(x - \sigma(t)v) - \varepsilon t > M - \varepsilon, \ \ \forall x \in X, \ \ \forall t \in [0, 1).$$

By taking the infimum, in the previous relation, over $t \in [0, 1)$, we obtain

$$\Phi^\varepsilon(x) \geq M - \varepsilon > -\infty, \ \ \forall x \in X. \tag{28}$$

(ii) The functional Φ^ε is convex. Let x_1, x_2 be in X, t_1, $t_2 \in [0, 1)$ and $0 \leq \lambda \leq 1$. Using the convexity of σ and relation (25) for

$$\tilde{x} = \lambda(x_1 - \sigma(t_1)v) + (1 - \lambda)(x_2 - \sigma(t_2)v),$$

and

$$\tilde{t} = \lambda\sigma(t_1) + (1 - \lambda)\sigma(t_2) - \sigma(\lambda t_1 + (1 - \lambda)t_2)$$

(remark that $\tilde{t} \geq 0$), we obtain,

$$\begin{aligned} \Phi\Big(x_2 + \lambda(x_1 - x_2) - \sigma(t_2 + \lambda(t_1 - t_2))v\Big) &= \Phi(\tilde{x} + \tilde{t}v) \\ \leq \Phi(\tilde{x}) = \Phi\Big(\lambda(x_1 - \sigma(t_1)v) &+ (1 - \lambda)(x_2 - \sigma(t_2)v)\Big). \end{aligned} \tag{29}$$

Combining the convexity of Φ and (29), we derive

$$\begin{aligned} &\Phi\Big(x_2 + \lambda(x_1 - x_2) - \sigma(t_2 + \lambda(t_1 - t_2))v\Big) \leq \\ &\lambda\Phi\Big(x_1 - \sigma(t_1)v\Big) + (1 - \lambda)\Phi\Big(x_2 - \sigma(t_2)v\Big). \end{aligned} \tag{30}$$

Using (27) and (30), we derive for every $t_1, t_2 \in [0,1)$:

$$\Phi^\varepsilon(\lambda x_1 + (1-\lambda)x_2) \le \lambda\Phi(x_1 - \sigma(t_1)v) + (1-\lambda)\Phi(x_2 - \sigma(t_2)v) - \varepsilon(\lambda t_1 + (1-\lambda)t_2).$$

Taking the infimum over t_1 and t_2, in the previous relation yields

$$\Phi^\varepsilon(\lambda x_1 + (1-\lambda)x_2) \le \lambda\Phi^\varepsilon(x_1) + (1-\lambda)\Phi^\varepsilon(x_2).$$

Hence Φ^ε is convex.

(iii) The functional Φ^ε is lower semicontinuous. Let $(x_n)_{n\in\mathbb{N}^*}$, be a sequence in X such that $x_n \to x_0$ as $n \to +\infty$. Consider also a sequence $(t_n)_{n\in\mathbb{N}^*}$ in [0, 1) such that $\lim_{n\to\infty} t_n = t^*$.

If $t^* = 1$, relation (26) implies that $\lim_{n\to\infty} \sigma(t_n) = +\infty$. Hence,

$$v + \frac{(x_0 - x_n)}{\sigma(t_n)} \to v \text{ as } n \to +\infty.$$

We may therefore apply (4) for $-v$, $-\left(v + \dfrac{(x_0 - x_n)}{\sigma(t_n)}\right)$ and $\sigma(t_n)$, and obtain

$$0 < \Phi_\infty(-v) \le \liminf_{n\to\infty} \frac{\Phi(x_n - \sigma(t_n)v)}{\sigma(t_n)}. \tag{31}$$

As the sequence $(\sigma(t_n))_{n\in\mathbb{N}^*}$ tends to infinity, from (31) it follows that

$$\liminf_{n\to\infty} [\Phi(x_n - \sigma(t_n)v) - \varepsilon t_n] = +\infty.$$

Hence,

$$\liminf_{n\to\infty} [\Phi(x_n - \sigma(t_n)v) - \varepsilon t_n] \ge \Phi^\varepsilon(x_0). \tag{32}$$

If $t^* < 1$, as Φ is lower semicontinuous , we have

$$\liminf_{n\to\infty} [\Phi(x_n - \sigma(t_n)v) - \varepsilon t_n] \ge \Phi(x_0 - \sigma(t^*)v) - \varepsilon t^* \ge \Phi^\varepsilon(x_0). \tag{33}$$

Relations (32) and (33) imply that for every sequence $(t_n)_{n\in\mathbb{N}^*}, t_n \in [0,1)$

$$\liminf_{n\to\infty} [\Phi(x_n - \sigma(t_n)v) - \varepsilon t_n] \ge \Phi^\varepsilon(x_0).$$

According to the definition of Φ^ε, for every n in $\mathbb{N}^*$ there is t_n in $[0,1)$ such that

$$0 \le \Phi(x_n - \sigma(t_n)v) - \varepsilon t_n - \Phi^\varepsilon(x_n) \le \frac{1}{n}.$$

Consequently,

$$\liminf_{n\to\infty} \Phi^\varepsilon(x_n) = \liminf_{n\to\infty} [\Phi(x_n - \sigma(t_n)v) - \varepsilon t_n] \ge \Phi^\varepsilon(x_0),$$

that is, Φ^ε is lower semicontinuous. Hence, this completes the proof of Lemma 4.3. □

Lemma 4.4. *The functional* Φ^ε *satisfies:*

$$\Phi - \varepsilon \le \Phi^\varepsilon \le \Phi.$$

Proof of Lemma 4.4. Let x be in X. As $\sigma(0) = 0$, using (27) we observe that

$$\Phi^\varepsilon(x) \le \Phi(x - \sigma(0)v) - \varepsilon\, 0 = \Phi(x). \tag{34}$$

Since σ is positive, relation (25) implies

$$\Phi(x) - \varepsilon \le \Phi(x - \sigma(t)v) - \varepsilon \le \Phi(x - \sigma(t)v) - \varepsilon t, \ \forall\, t \in [0, 1).$$

By taking the infimum over t we derive

$$\Phi(x) - \varepsilon \le \Phi^\varepsilon(x). \tag{35}$$

The conclusion of the Lemma 4.4 follows immediately by summing up (34) and (35). □

Lemma 4.5. $\operatorname{argmin} \Phi^\varepsilon = \emptyset$.

Proof of Lemma 4.5. Let x_0 be in $\operatorname{argmin} \Phi$. For every $0 < \delta < 1$, we have

$$\Phi^\varepsilon(x_0 + \sigma(1-\delta)v) \le \Phi(x_0 + \sigma(1-\delta)v - \sigma(1-\delta)v) - \varepsilon(1-\delta) = M - \varepsilon + \delta\varepsilon.$$

Hence,

$$\inf_{x \in X} \Phi^\varepsilon(x) \le M - \varepsilon.$$

Relation (28) implies now that the infimum of Φ^ε is $M - \varepsilon$. Suppose that this infimum is reached, i.e. there is x in X such that

$$\Phi^\varepsilon(x) = \inf_{0 \le t < 1} [\Phi(x - \sigma(t)v) - \varepsilon t] = M - \varepsilon;$$

accordingly, there is a sequence $(t_n)_{n \in \mathbb{N}^*}$ in $[0, 1)$ such that

$$\lim_{n \to \infty} [\Phi(x - \sigma(t_n)v) - \varepsilon t_n] = M - \varepsilon.$$

Since $\Phi(x - \sigma(t_n)v) \ge M$, and $\varepsilon t_n \le \varepsilon$, the previous relation implies that

$$\lim_{n \to \infty} t_n = 1, \text{ and } \lim_{n \to \infty} \Phi(x - \sigma(t_n)v) = M.$$

Consequently,

$$0 < \Phi_\infty(-v) = \lim_{n \to \infty} \frac{\Phi(x - \sigma(t_n)v)}{\sigma(t_n)} = 0,$$

a contradiction. Hence the functional Φ^ε does not reach its infimum value. □

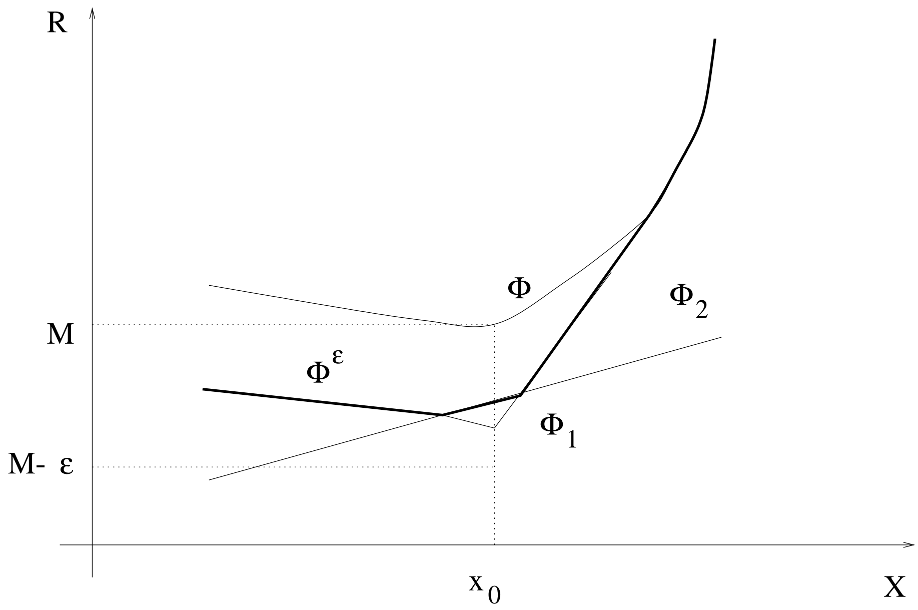
Lemmata 4.3–4.5 show that, if $\operatorname{Ker}\Phi_\infty$ is not a linear subspace of X, then the functional defined by (27) fulfills the conditions of Theorem 4.1. This thereby completes the proof of Case (a).

Case (b): The construction of functionals of type $\Phi_{y,N}$ allows us to deal with the case when $\operatorname{Ker}\Phi_\infty$ is a linear (and closed) subspace of X. Let us consider first a particular case, namely when $\operatorname{Ker}\Phi_\infty = \{0\}$. The desired functional Φ^ε will be in this case of the form

$$\Phi^\varepsilon(x) = \max\{\Phi_1(x), \Phi_2(x)\}, \tag{36}$$

where Φ_1 is defined by

$$\Phi_1(x) = \Phi_{x_0, M-\varepsilon}.$$



By Lemma 2.1, $\Phi_1 \in \Gamma_0(X)$ and

$$\Phi(x) - \varepsilon \le \Phi_1(x) \le \Phi(x),\ \forall\, x \in X. \tag{37}$$

In order to define Φ_2, let us consider the closed and convex set

$$P = \{x \in X : \Phi_1(x_0 + x) \le M\}. \tag{38}$$

Lemma 4.6. *The closed convex set P is unbounded, nevertheless it is linearly bounded.*

Proof of Lemma 4.6. Suppose P is bounded, i.e. there is a positive constant k such that $\|x\| \le k$ for all $x \in P$. Pick x in $\operatorname{Dom}\Phi$ and set

$$y = x_0 + \frac{\varepsilon}{\Phi(x) - M + \varepsilon}(x - x_0);$$

since

$$(y, M) = \frac{\varepsilon}{\Phi(x) - M + \varepsilon}(x, \Phi(x)) + \left(1 - \frac{\varepsilon}{\Phi(x) - M + \varepsilon}\right)(x_0, M - \varepsilon),$$

we have

$$(y, M) \in \overline{\text{co}}\{(x_0, M-\varepsilon), \operatorname{epi}\Phi\}.$$

Hence $\Phi_1(y) \le M$, that is $(y - x_0) \in P$.

Thus

$$\frac{\varepsilon}{\Phi(x) - M + \varepsilon}\|x - x_0\| = \|y - x_0\| \le k.$$

Consequently,

$$\frac{k}{\varepsilon}\Phi(x) \ge \|x\| - \|x_0\| + k\frac{M-\varepsilon}{\varepsilon},$$

which means that Φ is coercive, a contradiction. Therefore P is unbounded.

Fix now $u \in P_\infty$; then for every positive constant s we have $su \in P$. Relation (37) combined with (38) yields

$$\Phi(x_0 + su) \le \Phi_1(x_0 + su) + \varepsilon \le M + \varepsilon.$$

Hence due to (4) $\Phi_\infty(u) = 0$, that is $u \in \operatorname{Ker}\Phi_\infty$. Therefore $u = 0$ and, consequently, $P_\infty = \{0\}$. □

According to Theorem 3.1, take an element $f \in X^*$ such that

$$\inf_{w \in P} \langle f, w\rangle < \langle f, u\rangle \le 1, \ \forall\, u \in P. \tag{39}$$

Define

$$\Phi_2(x) := \frac{\varepsilon}{2}\langle f, x - x_0\rangle + M - \frac{3\varepsilon}{4}. \tag{40}$$

It easily follows from the definition of Φ_1 and Φ_2 that the functional Φ^ε defined in (36) belongs to $\Gamma_0(X)$, and satisfies $\Phi(x) - \varepsilon \le \Phi^\varepsilon(x)$. In order to prove that $\Phi^\varepsilon(x) \le \Phi(x)$, we use the following result.

Lemma 4.7. *Let x in X. If $\Phi_2(x) > M$ then $\Phi_1(x) > \Phi_2(x)$.*

Proof of Lemma 4.7. Take $\lambda_0 = \dfrac{M - \Phi_2(x_0)}{\Phi_2(x) - \Phi_2(x_0)}$. We have

$$\lambda_0\Phi_2(x) + (1-\lambda_0)\Phi_2(x_0) = \Phi_2(\lambda_0 x + (1-\lambda_0)x_0) = M. \tag{41}$$

Since,

$$\begin{aligned}\Phi_2(x) &= \frac{\varepsilon}{2}\langle f, x - x_0\rangle + M - \frac{3\varepsilon}{4}\\ &\le \frac{\varepsilon}{2} + M - \frac{3\varepsilon}{4} = M - \frac{\varepsilon}{4}\\ &< M, \ \forall\, x \in (x_0 + P),\end{aligned}$$

we deduce from (41) that

$$(\lambda_0 x + (1-\lambda_0)x_0) \notin (x_0 + P).$$

Therefore,

$$\Phi_1(\lambda_0 x + (1-\lambda_0)x_0) > M.$$

Since $\Phi_2(x_0) = M - (3/4)\varepsilon < M$ (see (40)), we obtain $0 < \lambda_0 \leq 1$. As Φ_1 is convex, we have

$$\lambda_0 \Phi_1(x) + (1-\lambda_0)\Phi_1(x_0) \geq \Phi_1(\lambda_0 x + (1-\lambda_0)x_0) > M. \tag{42}$$

Substracting (41) from (42) yields

$$\Phi_1(x) - \Phi_2(x) > \frac{1-\lambda_0}{4\lambda_0}\varepsilon \geq 0, \tag{43}$$

and establishes Lemma 4.7. □

By Lemma 4.7, $\Phi_2(x) \leq \Phi(x)$ whenever $\Phi_2(x) > M$; in other words,

$$\Phi_2(x) \leq \max\{M, \Phi(x)\} = \Phi(x).$$

This allows us to conclude that $\Phi^\varepsilon(x) \leq \Phi(x)$.

We conclude the proof of the particular case $\operatorname{Ker}\Phi_\infty = \{0\}$ by proving the following statement.

Lemma 4.8. *The functional Φ^ε does not attain its infimum value.*

Proof of Lemma 4.8. Define $B := \{x \in X : \Phi_2(x) \geq \Phi_1(x)\}$ and let us show that, if Φ^ε attains its minimum value over X, then Φ_2 also reaches its minimum value over B.

Lemma 4.9. *For every x in X, there is $b(x)$ in B such that $\Phi^\varepsilon(x) \geq \Phi_2(b(x))$.*

Proof of Lemma 4.9. If x is such that $\Phi^\varepsilon(x) \geq M$, then $b(x) := x_0$ satisfies Lemma 4.9; and if x belongs to B, we set $b(x) := x$. It remains to define $b(x)$ when $M > \Phi^\varepsilon(x)$ and $\Phi_1(x) > \Phi_2(x)$. In this case, set

$$\lambda_0 = \frac{\Phi_1(x) - \Phi_2(x)}{\frac{\varepsilon}{4} + \Phi_1(x) - \Phi_2(x)};$$

Lemma 2.1 (iii) implies that

$$\begin{aligned}
&\Phi_1(\lambda_0 x_0 + (1-\lambda_0)x)\\
&\quad = \lambda_0\Phi_1(x_0) + (1-\lambda_0)\Phi_1(x)\\
&\quad = \frac{\Phi_1(x) - \Phi_2(x)}{\frac{\varepsilon}{4} + \Phi_1(x) - \Phi_2(x)}(M - \varepsilon) + \frac{\frac{\varepsilon}{4}}{\frac{\varepsilon}{4} + \Phi_1(x) - \Phi_2(x)}\Phi_1(x)\\
&\quad = \frac{\Phi_1(x) - \Phi_2(x)}{\frac{\varepsilon}{4} + \Phi_1(x) - \Phi_2(x)}(M - \frac{3\varepsilon}{4}) + \frac{\frac{\varepsilon}{4}}{\frac{\varepsilon}{4} + \Phi_1(x) - \Phi_2(x)}\Phi_2(x)\\
&\quad = \lambda_0\Phi_2(x_0) + (1-\lambda_0)\Phi_2(x)\\
&\quad = \Phi_2(\lambda_0 x_0 + (1-\lambda_0)x),
\end{aligned}$$

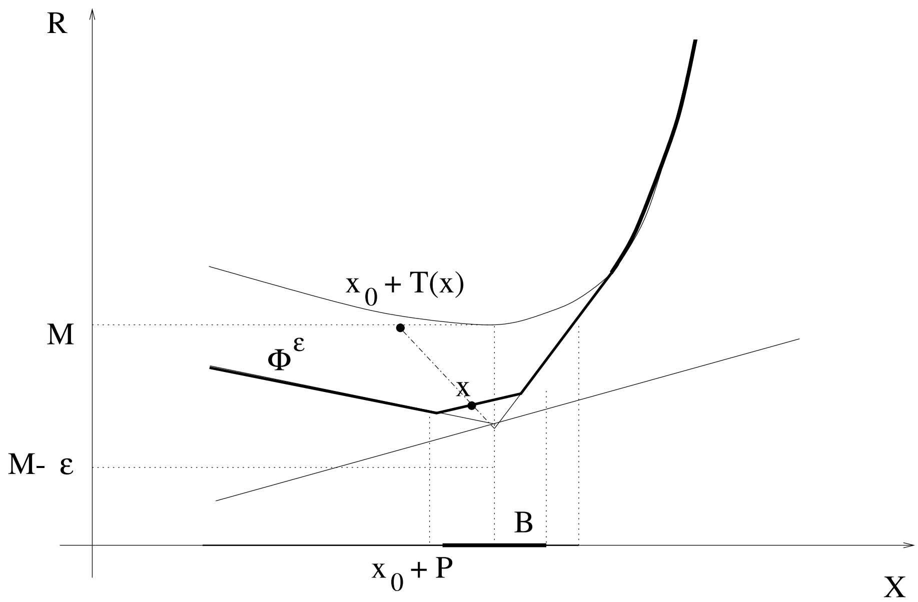
and the conclusion of Lemma 4.9 follows by taking $b(x) = \lambda_0 x_0 + (1-\lambda_0)x$. □

Lemma 4.8 will be established if we show that Φ_2 does not reach its infimum value on B. To this end, we prove that we can rewrite Φ_2 on B as

$$\Phi_2(x) = M - \varepsilon + \frac{\varepsilon}{2} \cdot \frac{1}{2 - \langle f, T(x) \rangle}, \tag{44}$$

where T is defined by $T(x) = \dfrac{\varepsilon(x - x_0)}{\Phi_2(x) - M + \varepsilon} = \dfrac{4(x - x_0)}{1 + 2\langle f, x - x_0 \rangle}$.

Lemma 4.10. *The operator $T : B \to P$ is one-to-one and onto.*



Proof of Lemma 4.10. Let us first prove that T is well-defined. As $T(x_0) = 0 \in P$, it suffices to show that $T(x) \in P$ if x lies in B and $x \neq x_0$.

By Lemma 2.1 (iv), $\operatorname{argmin} \Phi_1 = \{x_0\}$ and thus $\Phi_1(x) > \Phi_1(x_0) = M - \varepsilon$. Let us set $s_0 = \dfrac{\varepsilon}{\Phi_1(x) - M + \varepsilon}$. As $\Phi_1(x) \leq \Phi_2(x)$, according to Lemma 4.7 we deduce that $\Phi_2(x) \leq M$. Therefore,

$$s_0 \geq \frac{\varepsilon}{\Phi_2(x) - M + \varepsilon} = \frac{4}{1 + 2\langle f, x - x_0 \rangle}.$$

Suppose $\Phi_1(x_0 + s_0(x - x_0)) > M$. As the function $[0, s_0] \ni s \mapsto \Phi_1(x_0 + s(x - x_0))$ is convex and lower semicontinuous , hence continuous, there is s_1 in $[0, s_0)$ such that $\Phi_1(x_0 + s_1(x - x_0)) = M$. Applying Lemma 2.1 (iii) with $\Phi_{x_0, M-\varepsilon}$ for Φ, $x_0 + s_1(x - x_0)$ for x and $s_1/(1 + s_1)$ for λ, and using the convexity of Φ_1 we obtain

$$M = \Phi_1(x_0 + s_1(x - x_0)) \leq \Phi_1(x_0) + s_1(\Phi_1(x) - \Phi_1(x_0)),$$

that is

$$s_1 \geq \frac{\varepsilon}{\Phi_1(x) - M + \varepsilon} = s_0,$$

a contradiction. Consequently,

$$\Phi_1(x_0 + s_0(x - x_0)) \leq M. \tag{45}$$

For simplicity, let us define

$$\alpha_1 = \frac{s_0 - \dfrac{4}{1 + 2\langle f, x - x_0 \rangle}}{s_0} \text{ and } \alpha_2 = 1 - \frac{s_0 - \dfrac{4}{1 + 2\langle f, x - x_0 \rangle}}{s_0}.$$

The following relation

$$x_0 + T(x) = \alpha_1 x_0 + \alpha_2(x_0 + s_0(x - x_0))$$

together with the convexity of Φ_1 and (45) imply that

$$\Phi_1(x_0 + T(x)) \le \alpha_1(M - \varepsilon) + \alpha_2 s_0 M \le M.$$

Hence, $T(x)$ belongs to P.

Now fix w in P and set $y = x_0 + w$,

$$\lambda_0 = \frac{3 - 2\langle f, y - x_0 \rangle}{4 - 2\langle f, y - x_0 \rangle} = \frac{3 - 2\langle f, w \rangle}{4 - 2\langle f, w \rangle},$$

(remark that relation (14) implies $0 < \lambda_0 < 1$), and $x = \lambda_0 x_0 + (1 - \lambda_0)y$. After straightforward calculations we deduce that

$$\Phi_2(x) = M - \varepsilon + \frac{\varepsilon}{2} \cdot \frac{1}{2 - \langle f, y - x_0 \rangle} = M - \varepsilon\lambda_0. \tag{46}$$

Since $w \in P$, we have $\Phi_1(y) \le M$. Lemma 2.1 (iii) implies that

$$\Phi_1(x) = \lambda_0 \Phi_1(x_0) + (1 - \lambda_0)\Phi_1(y) \le M - \varepsilon\lambda_0 = \Phi_2(x).$$

Thus x belongs to B, and since

$$T(x) = \frac{4(x - x_0)}{1 + 2\langle f, x - x_0 \rangle} = \frac{4(1 - \lambda_0)}{1 + 2\langle f, x - x_0 \rangle} w = w,$$

it follows that T is onto. Since the operator T is obviously one-to-one, this completes the proof of Lemma 4.10. □

Let us return to the proof of Lemma 4.8. Relation (44) follows now from Lemma 4.10 and relation (46).

Relations (14) and (44) imply that, if $\inf_{w \in P} \langle f, w \rangle > -\infty$, then

$$\begin{aligned} \Phi_2(x) &= M - \varepsilon + \frac{\varepsilon}{2} \cdot \frac{1}{2 - \langle f, T(x) \rangle} \qquad (47) \\ &> M - \varepsilon + \frac{\varepsilon}{2} \cdot \frac{1}{2 - \inf_{w \in P} \langle f, w \rangle} \\ &= \inf_{v \in B} \Phi_2(v), \ \forall x \in B, \end{aligned}$$

and that, if $\inf_{w\in P}\langle f, w\rangle = -\infty$, then

$$\Phi_2(x) = M - \varepsilon + \frac{\varepsilon}{2}\cdot\frac{1}{2-\langle f, T(x)\rangle} > M - \varepsilon = \inf_{v\in B}\Phi_2(v),\ \forall x\in B.$$

In both cases, we have proved that Φ_2 does not reach its infimum value on B, which allows us to complete the proof of Theorem 4.1. □

Let us return to the general case, where $\operatorname{Ker}\Phi_\infty$ is an arbitrary closed subspace of X. As a consequence of (25), we have

$$\Phi(x+v) = \Phi(x),\ \text{for all}\, x\in X,\ \text{and all}\ v\in\operatorname{Ker}\Phi_\infty. \tag{48}$$

We may therefore factorize X by $\operatorname{Ker}\Phi_\infty$; the quotient functional $\overline{\Phi}$ is a non- semicoercive $\Gamma_0(X/\operatorname{Ker}\Phi_\infty)$-functional which attains its minimum value, and satisfies $\operatorname{Ker}\overline{\Phi}_\infty = \{0\}$. We may thus define $\overline{\Phi}^\varepsilon$ as before, and set

$$\Phi^\varepsilon(x) = \overline{\Phi}^\varepsilon(\overline{x}),\ \text{for all}\, x\in\overline{x},\ \text{and all}\, x\in X.$$

This functional obviously satisfies the requirements of Theorem 4.1. □

Following the lines of Theorem 4.1 we can also derive the following result:

Theorem 4.11. *Suppose that Φ is a $\Gamma_0(X)$-functional which achieves its minimum value on X. Moreover, assume that either*

(a) $\operatorname{Ker}\Phi_\infty$ *is not a subspace;*

or

(b) Φ *is non-semicoercive and* $\operatorname{Ker}\Phi_\infty$ *is a linear space.*

Then, for every $\varepsilon > 0$ and $R > 0$, there exists $\Phi^{\varepsilon,R}\in\Gamma_0(X)$ such that

- $\operatorname{argmin}\Phi^{\varepsilon,R}\neq\emptyset$;
- *for each $x\in X$, $\Phi(x)-\varepsilon\le\Phi^{\varepsilon,R}(x)\le\Phi(x)$;*
- *if $u\in\operatorname{argmin}\Phi^{\varepsilon,R}$, then $\|u\|\ge R$.*

Proof of Theorem 4.11.
Case (a): If $\operatorname{Ker}\Phi_\infty$ is not a linear subspace of X, then there is v in $\operatorname{Ker}\Phi_\infty$ such that $-v\notin\operatorname{Ker}\Phi_\infty$. Let $\delta := \operatorname{dist}(-v,\operatorname{Ker}\Phi_\infty)$ and set $y := x_0 + \frac{R+\|x_0\|}{\delta}v$ and define $\Phi^{\varepsilon,R}(x) := \Phi_{y,M-\varepsilon}(x)$; by virtue of Lemma 2.1, $\Phi^{\varepsilon,R}$ lies in $\Gamma_0(X)$ and

$$\Phi(x)-\varepsilon\le\Phi^{\varepsilon,R}(x)\le\Phi(x),\ \forall x\in X.$$

Moreover, if $z\in\operatorname{argmin}\Phi^{\varepsilon,R}$, then $z\in(y+\operatorname{Ker}\Phi_\infty)$. Thus there is $w\in\operatorname{Ker}\Phi_\infty$ such that

$$z = x_0 + \frac{R+\|x_0\|}{\delta}v + w$$

Since $\delta = \operatorname{dist}(-v,\operatorname{Ker}\Phi_\infty)$, we have

$$\left\|\frac{R+\|x_0\|}{\delta}v + w\right\| \ge \frac{R+\|x_0\|}{\delta}\left\|-v - \frac{\delta}{R+\|x_0\|}w\right\| \ge R+\|x_0\|;$$

accordingly,

$$\|z\| \geq \left\| \frac{R + \|x_0\|}{\delta} v + w \right\| - \|x_0\| \geq R.$$

Hence $\Phi^{\varepsilon,R}$ fulfills the conditions of Theorem 4.11.

Case (b): If $\operatorname{Ker}\Phi_\infty$ is a linear subspace of X, consider $\overline{\Phi}^\varepsilon$ as constructed in the proof of Theorem 4.1, and let $(\overline{x}_n)_{n\in\mathbb{N}^*}$ be a minimizing sequence for $\overline{\Phi}^\varepsilon$. Since the functional $\overline{\Phi}^\varepsilon$ does not attain its infimum value and the space X is reflexive, the sequence $(\overline{x}_n)_{n\in\mathbb{N}^*}$ is unbounded.

According to relation (47), it follows that

$$M - \varepsilon \leq \inf_{x\in X} \overline{\Phi}^\varepsilon \leq M - \frac{3\varepsilon}{4}.$$

Thus take n_0 such that $\|\overline{x}_{n_0}\| \geq R$ and $\overline{\Phi}^\varepsilon(\overline{x}_{n_0}) < M$.

Consider now $\overline{\Phi}_{\overline{x}_{n_0},\overline{\Phi}^\varepsilon(\overline{x}_{n_0})}$. Lemma 2.1 implies that $\overline{\Phi}_{\overline{x}_{n_0},\overline{\Phi}^\varepsilon(\overline{x}_{n_0})}$ is a $\Gamma_0(X/\operatorname{Ker}\Phi_\infty)$-functional which reaches its minimum value only on $\overline{x}_{n_0}$ and satisfies

$$\overline{\Phi}_{\overline{x}_{n_0},\overline{\Phi}^\varepsilon(\overline{x}_{n_0})}(\overline{x}) \leq \overline{\Phi}(\overline{x}). \tag{49}$$

Since $(\overline{x}_{n_0}, \overline{\Phi}^\varepsilon(\overline{x}_{n_0})) \in \operatorname{epi}\overline{\Phi}^\varepsilon$, and $\overline{\Phi}^\varepsilon \leq \overline{\Phi}$, we obtain

$$\operatorname{epi}\overline{\Phi}_{\overline{x}_{n_0},\overline{\Phi}^\varepsilon(\overline{x}_{n_0})} = \overline{\operatorname{co}}\{(\overline{x}_{n_0}, \overline{\Phi}^\varepsilon(\overline{x}_{n_0})), \operatorname{epi}\overline{\Phi}\} \subset \operatorname{epi}\overline{\Phi}^\varepsilon.$$

Therefore

$$\overline{\Phi} - \varepsilon \leq \overline{\Phi}^\varepsilon \leq \overline{\Phi}_{\overline{x}_{n_0},\overline{\Phi}^\varepsilon(\overline{x}_{n_0})}. \tag{50}$$

The desired functional is now defined by setting

$$\Phi^{\varepsilon,R}(x) = \overline{\Phi}_{\overline{x}_{n_0},\overline{\Phi}^\varepsilon(\overline{x}_{n_0})}(\overline{x}),\ \forall\, x \in \overline{x},\ \forall\, \overline{x} \in X/\operatorname{Ker}\Phi_\infty.$$

By (49) and (50), it follows that $\Phi^{\varepsilon,R} \in \Gamma_0(X)$ and satisfies:

$$\Phi(x) - \varepsilon \leq \Phi^{\varepsilon,R}(x) \leq \Phi(x),\ \forall\, x \in X.$$

The functional $\Phi^{\varepsilon,R}$ reaches its minimum value only on the set $\overline{x}_{n_0}$. Since $\|\overline{x}_{n_0}\| \geq R$, we have $\|x\| \geq R$ for every x in $\overline{x}_{n_0}$, which means that $\Phi^{\varepsilon,R}$ fulfills all the conditions of Theorem 4.11. □

5. Concluding remarks

In this paper we have studied the stability under uniform perturbations of the existence of a solution for the simplest variational problem, namely the minimization of a proper, convex and lower semicontinuous functional. In summary, we established that the problem of finding a minimum point of a convex functional is stable under uniform perturbations only within the class of semicoercive $\Gamma_0(X)$-functionals.

The same question may be raised in some other variational contexts, such as the theory of variational inequalities.

This case has already been considered and several results establishing sufficient existence conditions for noncoercive problems have been obtained recently, using the so-called recession analysis (see for instance the work by Adly *et al* [1] and Attouch *et al* [2]). However, even in the case of a positive operator, the question of the existence of solutions of the perturbed initial variational inequality remains partially open.

Another interesting direction of research is the nonconvex case. We remark that this case does not subsume the convex one, since, even if the class of functionals is broader, so is the uniform neighborhood composed of functionals which must attain their minima. Accordingly, no simple relation can be established between the two problems.

Sufficient stability conditions have been obtained in the non-convex setting (see [4] and [5]), implying, inter alia, that semicoercive functionals are no longer the only functionals with a stable minimum. The problem of characterizing all lower semicontinuous functionals having a stable minimum is thus still open.

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