Journal of Convex Analysis
Volume 8 (2001), No. 1, 171–195

On the Equilibrium Position of a Square on an Elastic Wire

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Received April 28, 2000

1. Introduction

We consider a solid square allowed to move freely on an elastic wire and we would like to determine its equilibrium position (see figure 1). We will denote by $2r$, $r > 0$ the length of the sides of the square and by G its weight.

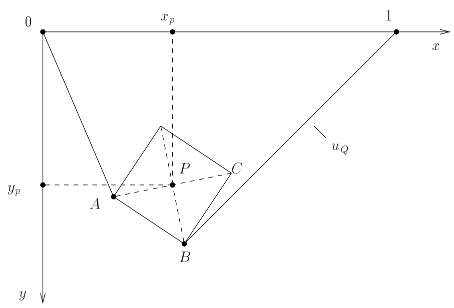

Figure 1

We denote also by $P = (x_p, y_p)$ the barycenter of this square and we will suppose that the wire, in its undeformed position, occupies the interval $\Omega = (0, 1)$. If u is an admissible deformation of the wire and if $P = (x_p, y_p)$ is the position of the center of the square then, the total energy corresponding to this configuration is given by

$$E = \frac{1}{2}\int_0^1 u_x^2(x)dx - Gy_p, \tag{1}$$

where u_x denotes the derivative in x of the function u. The first term in the expression above is a scaled elastic energy, the second a potential energy. If Q denotes the closed set of the points occupied by the square we will always suppose

$$Q \subset \Omega \times \mathbb{R}, \tag{2}$$

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and will impose all along

$$r < \frac{1}{4}. \tag{3}$$

Then, for the square fixed in a position such that (2) holds, an admissible deformation u is a function

$$u \in H_0^1(\Omega), \tag{4}$$

such that it holds:

$$Q \subset C_u = \{(x,y) \in \Omega \times \mathbb{R} / \;\; y \leq u(x)\}. \tag{5}$$

Note that we directed the y direction downward. Having chosen r such that (3) holds we would like to minimize (1) over the set of all admissible couples (Q,u) satisfying (5). (Recall that Q is the set of points occupied by the square and we allow this set to describe all the strip $\Omega \times \mathbb{R}$). So, the problem we would like to address is the following. Find

$$\mathrm{Inf}_{(Q,u)} \frac{1}{2} \int_\Omega u_x^2(x)dx - Gy_p^Q, \tag{6}$$

where Q is a square located in the strip $\Omega \times \mathbb{R}$, with barycenter y_p^Q and u is a function satisfying (4), (5). More precisely, we would like to show that this problem admits a minimizer (Q_0, u_0), unique, up to some symmetry.

We are minimizing over a set of isometric squares and admissible deformations so, the problem is a little bit unusual at first glance. However, we first show that it can be recast as a minimization problem in $\mathbb{R}^4$. Indeed, suppose that the position of the square Q is fixed in the strip $\Omega \times \mathbb{R}$. Denote by $I_Q = \Pi_x(Q)$ the subinterval of $(0,1)$ projection of Q parallel to the y axis (Π_x is the usual orthogonal projection on the x axis). For $x \in I_Q$ set

$$\psi_Q(x) = \;\mathrm{Sup}\{y / \; (x,y) \in Q\}. \tag{7}$$

Then, clearly, the function ψ_Q is a function describing the lower border of Q and is constant if the square has its sides parallel to the axis, a hat function otherwise. Then, the constraints (4), (5) will be satisfied for

$$u \in C_Q = \{u \in H_0^1(\Omega) / \;\; u(x) \geq \psi_Q(x) \text{ in } I_Q\}. \tag{8}$$

(Recall for the last time that our y-axis is directed downward).

Then, Q being fixed with center $P = (x_p, y_p)$, it first makes sense to look for $u = u_Q$ the solution to

$$\mathrm{Min}_{v \in C_Q} \frac{1}{2} \int_\Omega v_x^2(x)dx - Gy_p^Q \tag{9}$$

or equivalently since y_p^Q is fixed

$$\mathrm{Min}_{v \in C_Q} \frac{1}{2} \int_\Omega v_x^2(x)dx. \tag{10}$$

It is well known (see for instance [4, 5, 3]) that this problem has a unique solution which is the solution of the variational inequality

$$\begin{cases} u \in C_Q, \\ \int_\Omega u_x(v_x - u_x)dx \geq 0 \quad \forall v \in C_Q. \end{cases} \tag{11}$$

It is easy to show - see for instance the above references - that

$$-u'' \geq 0 \quad \text{in} \quad \Omega \tag{12}$$

- i.e. the function u is concave - and outside the coincidence set

$$\Lambda = \{x \in \Omega / \ u(x) = \psi_Q(x)\} \tag{13}$$

it holds

$$u'' = 0 \tag{14}$$

i.e. u is a straight line. A rapid inspection will show that $u = u_Q$ is of the following types.

- If $Q \subset \Omega \times (-\infty, 0]$, $u = u_Q = 0$, i.e. the square does not touch the wire.
- The coincidence set is reduced to a point (see figure 2).

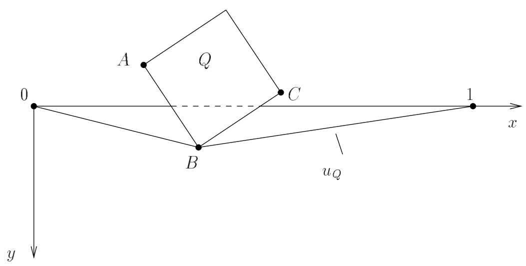

Figure 2

- The coincidence set is the projection of one side (see figure 1).
- The coincidence set is I_Q (see figure 3).

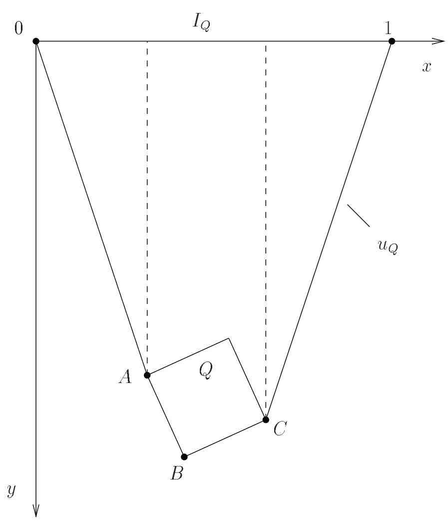

Figure 3

To see this, it is enough to notice that u is piecewise affine - i.e. $u'' = 0$ piecewise - and to integrate by parts in (11) on each interval where $u'' = 0$.

Having this in mind, (6) is equivalent to

$$\mathrm{Inf}_{Q\subset\Omega\times\mathbb{R}} \frac{1}{2}\int_\Omega u'^2_Q(x)dx - Gy_p^Q \tag{15}$$

i.e. one minimizes over Q only. (We denoted $(u_Q)_x$ by u'_Q for u_Q the solution to (11)). Now, the square Q is perfectly defined by the knowledge of two of its vertices - for instance A, B on the figures above - A being the further left vertex of Q, B the further down in such a way that if

$$A = (x_A, y_A)\ , \quad B = (x_B, y_B) \tag{16}$$

one can always assume

$$x_A < x_B\ , \quad y_A \le y_B. \tag{17}$$

Note that the two points A, B are not able to move freely but one has always the constraint

$$(x_A - x_B)^2 + (y_A - y_B)^2 = 4r^2. \tag{18}$$

Since A, B determine with no ambiguity Q one has

$$\frac{1}{2}\int_\Omega u'^2_Q(x)dx - Gy_p^Q = F(x_A, y_A, x_B, y_B) \tag{19}$$

i.e. the problem becomes the minimization of a function in $\mathbb{R}^4$ subjected to the constraint (18). This is what we would like to address now (we refer the reader to [1, 2, 6] for other issues).

The paper is divided as follows. In the next section we will show that the function defined by (19) is a C^1 function of its arguments. We will show also by a compactness argument that the problem (6) admits a minimizer, then, in the last section, we will determine effectively this minimizer.

Surprisingly, the answer depends on the intensity of G. For G small the minimum of the energy is achieved when the square has its sides parallel to the axis and is centered in the middle of the wire. G becoming larger the square tilts itself on the side and the minimum of the energy is achieved for two symmetric positions. A further increase in weight will cause the square to reach its minimum of energy when centered again but this time with its sides making a 45^0 angle with the coordinates axis. Finally when G reaches a higher level the square tilts itself again to achieve its minimum of energy for two symmetric positions.

2. Existence of a minimizer

We consider $A, B \in \Omega\times\mathbb{R}$ two points satisfying (17), (18). If $R_{\frac{\pi}{2}}$ denotes the rotation of angle $\frac{\pi}{2}$ the point $C = (x_C, y_C)$ such that

$$\begin{pmatrix} x_C \\ y_C \end{pmatrix} = \begin{pmatrix} x_B \\ y_B \end{pmatrix} + R_{\frac{\pi}{2}} \begin{pmatrix} x_A - x_B \\ y_A - y_B \end{pmatrix}$$

i.e. the point C such that

$$x_C = x_B - (y_A - y_B)\ , \quad y_C = y_B + (x_A - x_B) \tag{20}$$

is the third vertex of the square Q and the constraint

$$Q \subset \Omega \times \mathbb{R}$$

reads now

$$A, B, C \in \Omega \times \mathbb{R}. \tag{21}$$

For (x_A, y_A), (x_B, y_B) such that (21) holds we would like to get an expression for the function F defined in (19). As seen above different cases are possible.

- The square does not touch the wire.

In this case $B \in \Omega \times (-\infty, 0]$ and one has $u = u_Q = 0$ and

$$F(x_A, y_A, x_B, y_B) = -G y_p^Q. \tag{22}$$

Clearly

$$y_p^Q = \frac{y_A + y_C}{2} = \frac{y_A + y_B + x_A - x_B}{2}. \tag{23}$$

So, in this case

$$F(x_A, y_A, x_B, y_B) = -\frac{G}{2}(y_A + y_B + x_A - x_B). \tag{24}$$

In the case where $B \in \Omega \times (0, \infty]$ - i.e. when $y_B > 0$ - we have different situations corresponding - up to symmetry - to one of the figures above.

- The coincidence set reduces to a point.

We are in the case of the figure 2. This imposes the constraints

$$y_B > 0, \quad \frac{y_B}{x_B} < \frac{y_B - y_A}{x_B - x_A}, \quad \frac{y_B}{1 - x_B} < \frac{x_B - x_A}{y_B - y_A} \tag{25}$$

and one has clearly

$$F(x_A, y_A, x_B, y_B) = \frac{1}{2}\{\frac{y_B^2}{x_B} + \frac{y_B^2}{1 - x_B}\} - \frac{G}{2}(y_A + y_B + x_A - x_B). \tag{26}$$

- The coincidence set reduces to $\Pi_x([AB])$.

That is to say if $[AB]$ denote the segment between A, B we suppose that we are in the case of figure 1. This imposes

$$y_B > 0, \quad \frac{y_B}{x_B} \geq \frac{y_B - y_A}{x_B - x_A}, \quad \frac{y_B}{1 - x_B} < \frac{x_B - x_A}{y_B - y_A} \tag{27}$$

and one has

$$F(x_A, y_A, x_B, y_B) = \frac{1}{2}\{\frac{y_A^2}{x_A} + \frac{(y_B - y_A)^2}{x_B - x_A} + \frac{y_B^2}{1 - x_B}\} - \frac{G}{2}(y_A + y_B + x_A - x_B). \tag{28}$$

- The coincidence set reduces to $\Pi_x([BC])$.

This is the symmetric situation to the one above. This imposes

$$y_B > 0, \quad \frac{y_B}{x_B} < \frac{y_B - y_A}{x_B - x_A}, \quad \frac{y_B}{1 - x_B} \geq \frac{x_B - x_A}{y_B - y_A} \tag{29}$$

and one has

$$F(x_A, y_A, x_B, y_B) = \frac{1}{2}\{\frac{y_B^2}{x_B} + \frac{(x_B - x_A)^2}{y_B - y_A} + \frac{y_C^2}{1 - x_C}\} - \frac{G}{2}(y_A + y_B + x_A - x_B), \tag{30}$$

where x_C, y_C are given by (20). Note that if $T = \dfrac{y_B - y_A}{x_B - x_A}$ is the slope of the segment $[AB]$, the slope of $[BC]$ in absolute value is $\dfrac{1}{T} = \dfrac{x_B - x_A}{y_B - y_A}$.

- The coincidence set is $\Pi_x(Q)$.

This the case is the case of figure 3. This imposes

$$y_B > 0, \quad \frac{y_B}{x_B} \geq \frac{y_B - y_A}{x_B - x_A}, \quad \frac{y_B}{1 - x_B} \geq \frac{x_B - x_A}{y_B - y_A} \tag{31}$$

and one has

$$F(x_A, y_A, x_B, y_B) = \\ \frac{1}{2}\{\frac{y_A^2}{x_A} + \frac{(y_B - y_A)^2}{x_B - x_A} + \frac{(x_B - x_A)^2}{y_B - y_A} + \frac{y_C^2}{1 - x_C}\} - \frac{G}{2}(y_A + y_B + x_A - x_B), \tag{32}$$

where x_C, y_C are given by (20). Using (24) for $y_B \leq 0$ and (30) for $y_B > 0$ it is clear that one can extend F continuously for $x_A = x_B$. So, for any r satisfying (3), we have defined a function F on the domain

$$D_r = \{(x_A, y_A, x_B, y_B)/\ x_A, x_B, x_C \in (0,1), x_A \leq x_B,\ y_A \leq y_B, \\ (x_A - x_B)^2 + (y_A - y_B)^2 = 4r^2\}. \tag{33}$$

Then we can show:

Theorem 2.1. *The function F defined above is continuous on*

$$D = \{(x_A, y_A, x_B, y_B)/\ x_A, x_B, x_C \in (0,1), x_A \leq x_B,\ y_A \leq y_B, \\ (x_A - x_B)^2 + (y_A - y_B)^2 < \frac{1}{4}\}. \tag{34}$$

Proof. A, B being fixed, the square Q is fixed and so are y_p^Q, u_Q so that

$$\begin{aligned} F(x_A, y_A, x_B, y_B) &= \frac{1}{2}\int_\Omega (u_Q')^2 dx - G y_p^Q \\ &= \frac{1}{2}\int_\Omega (u_Q')^2 dx - \frac{G}{2}(y_A + y_B + x_A - x_B). \end{aligned}$$

Now, clearly, when $A, B \to \overline{A}, \overline{B}$ the derivative $u_Q' \to u_{\overline{Q}}'$ uniformly on $(0,1)$.

($u_{\overline{Q}}$ is the function corresponding to $\overline{Q}$ the square defined by $\overline{A}, \overline{B}$). The continuity of F is then clear. $\square$

Furthermore we have

Theorem 2.2. *The function F is C^1 on*

$$D^0 = \{(x_A, y_A, x_B, y_B)/ \ x_A < x_B, \ y_A < y_B, \ (x_A - x_B)^2 + (y_A - y_B)^2 < \frac{1}{4}\}. \tag{35}$$

Proof. It is clear that F is C^1 inside of each subdomains of D^0 where (24) or (26), (28), (30), (32) holds. It is thus enough to show that the first partial derivatives match smoothly on the border of these domains. Let us first consider $\frac{\partial F}{\partial x_A}$. In the different subdomains one has clearly

$$\frac{\partial F}{\partial x_A} = -\frac{G}{2} \quad \text{in the case of (24),} \tag{36}$$

$$\frac{\partial F}{\partial x_A} = -\frac{G}{2} \quad \text{in the case of (26),} \tag{37}$$

$$\frac{\partial F}{\partial x_A} = \frac{1}{2}\{-(\frac{y_A}{x_A})^2 + (\frac{y_B - y_A}{x_B - x_A})^2\} - \frac{G}{2} \quad \text{in the case of (28),} \tag{38}$$

$$\frac{\partial F}{\partial x_A} = \frac{1}{2}\{2\frac{(x_A - x_B)}{y_B - y_A} + 2\frac{y_C}{1 - x_C}\} - \frac{G}{2} \quad \text{in the case of (30),} \tag{39}$$

$$\frac{\partial F}{\partial x_A} = \frac{1}{2}\{-(\frac{y_A}{x_A})^2 + (\frac{y_B - y_A}{x_B - x_A})^2 + 2\frac{(x_A - x_B)}{y_B - y_A} + 2\frac{y_C}{1 - x_C}\} - \frac{G}{2} \quad \text{in the case of (32),} \tag{40}$$

with $x_C = x_B - (y_A - y_B), \ y_C = y_B + (x_A - x_B)$.

First when $y_B > 0, \ y_B \to 0$ the only possible definition for $\frac{\partial F}{\partial x_A}$ is (37), (38) or (39), and $\frac{\partial F}{\partial x_A}$ is continuous across the line $y_B = 0$ as it is easy to see.

When one passes from the formula (28) to (32) one has

$$\frac{y_C}{1 - x_C} - \frac{x_B - x_A}{y_B - y_A} \to 0 \ \ (\text{and } \frac{y_C}{1 - x_C} - \frac{y_B}{1 - x_B} \to 0) \tag{41}$$

and the formula (40) converges toward (38). When one passes from the formula (30) to (32) one has

$$\frac{y_B - y_A}{x_B - x_A} - \frac{y_A}{x_A} \to 0 \ \ (\text{and } \frac{y_B - y_A}{x_B - x_A} - \frac{y_B}{x_B} \to 0) \tag{42}$$

and the formula (40) converges toward (39). Finally when one passes from the formulae (28), (30) to (26) one has

$$\frac{y_B - y_A}{x_B - x_A} - \frac{y_A}{x_A} \to 0, \ (\text{and } \frac{y_B - y_A}{x_B - x_A} - \frac{y_B}{x_B} \to 0)$$

or

$$\frac{x_B - x_A}{y_B - y_A} - \frac{y_C}{1 - x_C} \to 0, \text{ (and } \frac{y_C}{1 - x_C} - \frac{y_B}{1 - x_B} \to 0) \tag{43}$$

and the formulae (38), (39) converge towards (37). We proceed similarly for the other derivatives. For instance

$$\frac{\partial F}{\partial x_B} = \frac{G}{2} \quad \text{in case (24),} \tag{44}$$

$$= \frac{1}{2}\{-(\frac{y_B}{x_B})^2 + (\frac{y_B}{1 - x_B})^2\} + \frac{G}{2} \quad \text{in case (26),} \tag{45}$$

$$= \frac{1}{2}\{-(\frac{y_B - y_A}{x_B - x_A})^2 + (\frac{y_B}{1 - x_B})^2\} + \frac{G}{2} \quad \text{in case (28),} \tag{46}$$

$$= \frac{1}{2}\{-(\frac{y_B}{x_B})^2 + 2\frac{x_B - x_A}{y_B - y_A} - 2\frac{y_C}{1 - y_C} + (\frac{y_C}{1 - x_C})^2\} + \frac{G}{2} \quad \text{in case (30),} \tag{47}$$

$$= \frac{1}{2}\{-(\frac{y_B - y_A}{x_B - x_A})^2 + 2\frac{x_B - x_A}{y_B - y_A} - 2\frac{y_C}{1 - y_C} + (\frac{y_C}{1 - x_C})^2\} + \frac{G}{2} \quad \text{in case (32).} \tag{48}$$

Using (41)–(43) one sees easily that the transition occurs also smoothly in this case.

One has now

$$\frac{\partial F}{\partial y_A} = -\frac{G}{2} \quad \text{in case (24),} \tag{49}$$

$$= -\frac{G}{2} \quad \text{in case (26),} \tag{50}$$

$$= \frac{1}{2}\{\frac{2y_A}{x_A} + 2\frac{y_A - y_B}{x_B - x_A}\} - \frac{G}{2} \quad \text{in case (28),} \tag{51}$$

$$= \frac{1}{2}\{(\frac{x_B - x_A}{y_B - y_A})^2 - (\frac{y_C}{1 - x_C})^2\} - \frac{G}{2} \quad \text{in case (30),} \tag{52}$$

$$= \frac{1}{2}\{\frac{2y_A}{x_A} + 2\frac{y_A - y_B}{x_B - x_A} + (\frac{x_B - x_A}{y_B - y_A})^2 - (\frac{y_C}{1 - x_C})^2\} - \frac{G}{2} \quad \text{in case (32).} \tag{53}$$

It follows then again from (41)–(43) that the transition is smooth. Finally

$$\frac{\partial F}{\partial y_B} = -\frac{G}{2} \quad \text{in case (24),} \tag{54}$$

$$= \frac{1}{2}\{\frac{2y_B}{x_B} + 2\frac{y_B}{1 - x_B}\} - \frac{G}{2} \quad \text{in case (26),} \tag{55}$$

$$= \frac{1}{2}\{2\frac{y_B - y_A}{x_B - x_A} + 2\frac{y_B}{1 - x_B}\} - \frac{G}{2} \quad \text{in case (28),} \tag{56}$$

$$= \frac{1}{2}\{\frac{2y_B}{x_B} - (\frac{x_B - x_A}{y_B - y_A})^2 + 2\frac{y_C}{1 - x_C} + (\frac{y_C}{1 - x_C})^2\} - \frac{G}{2} \quad \text{in case (30),} \tag{57}$$

$$= \frac{1}{2}\{2\frac{y_B - y_A}{x_B - x_A} - (\frac{x_B - x_A}{y_B - y_A})^2 + 2\frac{y_C}{1 - x_C} + (\frac{y_C}{1 - x_C})^2\} - \frac{G}{2} \quad \text{in case (32),} \tag{58}$$

and again the transition is smooth. This completes the proof of the theorem. □

We can now establish the main theorem of this section - namely:

Theorem 2.3. *The problem* (6) *admits a minimizer.*

Proof. Due to our above analysis it is enough to show that $F(x_A, y_A, x_B, y_B)$ admits a minimizer on D_r.

Step 1. One can suppose

$$-2r \le y_A \le y_B \le C \tag{59}$$

for some constant C.

Indeed for $y_A < -2r$ one has $y_B < 0$ and

$$\begin{aligned} F(x_A, y_A, x_B, y_B) &= -\frac{G}{2}(y_A + y_B + x_A - x_B) \\ &= \frac{G}{2}(x_B - x_A - y_B - y_A) \\ &\ge rG = F(x_A, -2r, x_A, 0). \end{aligned} \tag{60}$$

Thus, one can suppose

$$-2r \le y_A \le y_B. \tag{61}$$

Next, due to the Poincaré inequality, one has for some constant C

$$\begin{aligned} F(x_A, y_A, x_B, y_B) &= \frac{1}{2}\int_\Omega (u_Q')^2 dx - \frac{G}{2}(y_A + y_B + x_A - x_B) \\ &\ge \frac{C}{2}\int_\Omega u_Q^2 dx - G y_B. \end{aligned}$$

Now for $y_B > 2r$ one has y_A, $y_C \in \Omega \times (0, +\infty)$,

$$u_Q \ge y_B - 2r \qquad \text{on} \quad I_Q$$

and the above inequality becomes

$$F(x_A, y_A, x_B, y_B) \ge C(y_B - 2r)^2 r - G y_B.$$

Since the right hand side of this inequality converges toward $+\infty$ when $y_B \to +\infty$ one can assume y_B bounded - i.e. (59).

Step 2. The infimum (6) cannot be achieved by a sequence of points $(x_A, y_A) \to (0, 0)$.

If $x_A = x_B$ then when $(x_A, y_A) \to 0, y_B > 0$, the formula (30) applies and $F \to +\infty$ which is impossible. If $x_A < x_B$ then one can clearly extend F by continuity at $(0, 0, x_B, y_B)$. Following the position of B one has two configurations possible described on the figures below:

Figure 4

In the case (I) moving the whole figure to the right will cause F to be defined by (26)

$$F(x_A, 0, x_B, y_B) = \frac{1}{2}\{\frac{y_B^2}{x_B} + \frac{y_B^2}{1 - x_B}\} - \frac{G}{2}(y_B + x_A - x_B).$$

One has $x_B = x_A + cst$, $y_A = 0$, y_B fixed. Thus

$$\frac{d}{dx_A}F(x_A, 0, x_B, y_B) = \frac{1}{2}\{-\frac{y_B^2}{x_B^2} + \frac{y_B^2}{(1 - x_B)^2}\} < 0 \quad \text{since} \quad x_B < 1 - x_B$$

which renders incompatible a minimum at 0. ($x_B < 1 - x_B$ since $2x_B \leq 4r < 1$).

Similarly, in case (II), for $x_A > 0$ one will have F defined by (30) i.e.

$$F(x_A, 0, x_B, y_B) = \frac{1}{2}\{\frac{y_B^2}{x_B} + \frac{(x_B - x_A)^2}{y_B} + \frac{y_C^2}{1 - x_C}\} - \frac{G}{2}(y_B + x_A - x_B).$$

It follows as above that one cannot have a minimum at 0 since

$$\frac{d}{dx_A}F(x_A, 0, x_B, y_B) = \frac{1}{2}\{-(\frac{y_B}{x_B})^2 + (\frac{y_C}{1 - x_C})^2\} < 0,$$

for x_A small enough. To see this, note that when $A = 0$, it is enough to show that

$$\begin{aligned} \frac{y_B}{x_B} > \frac{y_B - x_B}{1 - x_B - y_B} &\Leftrightarrow (1 - x_B - y_B)y_B > x_B(y_B - x_B) \\ &\Leftrightarrow y_B - x_B y_B - y_B^2 > x_B y_B - x_B^2 \\ &\Leftrightarrow y_B - 2x_B y_B - y_B^2 + x_B^2 > 0. \end{aligned}$$

This inequality holds true since

$$\begin{aligned} y_B - 2x_B y_B - y_B^2 + x_B^2 &\geq y_B - x_B^2 - y_B^2 - y_B^2 + x_B^2 \\ &= y_B(1 - 2y_B) \geq y_B(1 - 4r) > 0 \quad \text{by (3)}. \end{aligned} \tag{62}$$

Step 3. End of the proof.

We know that F is continuous on D_r. If we show that the minimum of F is achieved on a compact subset of D_r we will be done. Thanks to (59), and for symmetry reasons, if we show that one can assume

$$x_A \geq \delta \tag{63}$$

for some δ we will be able to conclude (since by symmetry one would have also $x_B \leq 1-\delta$).

If we are looking for a minimum in the region

$$y_B \leq 0$$

then one has

$$F(x_A, y_A, x_B, y_B) = -\frac{G}{2}(y_A + y_B + x_A - x_B).$$

and moving the square horizontally will not change the energy so that one can assume in this case that (63) holds.

Let us consider now $y_B > 0$. If $y_A \leq 0$ then for $x_A > 0$, F will be defined by the formula (26) or (30) if $y_A > 0$ it will be defined by (28) or (32) for x_A small enough. Let us consider the different cases:

- F is defined by (26)

This is the case of the figure 2. Rotating the square until it touches the wire will decrease its energy (u is inchanged but y_p^Q increases). Thus this case can be included in the case of (28) or (30) and we do not have to consider it.

- F is defined by (28)

Then, moving the square horizontally one has in this case

$$\frac{d}{dx_A}F(x_A, y_A, x_B, y_B) = \frac{1}{2}\{-(\frac{y_A}{x_A})^2 + (\frac{y_B}{1-x_B})^2\}.$$

This derivative is nonnegative for

$$\frac{|y_A|}{x_A} \leq \frac{y_B}{1-x_B} \Leftrightarrow x_A \geq \frac{1-x_B}{y_B}|y_A| \geq \frac{1-2r}{C}|y_A| \tag{64}$$

where C is the constant in (59). Thus, one can impose (64). If there is no δ such that

$$x_A \geq \delta \quad \text{or} \quad |y_A| \geq \delta \tag{65}$$

then the infimum is “achieved” for $A = 0$ this is impossible. Thus, (65) holds and also (63) by (64) for perhaps some other δ.

- F is defined by (30)

In this case one has

$$\frac{d}{dx_A}F(x_A, y_A, x_B, y_B) = \frac{1}{2}\{-(\frac{y_B}{x_B})^2 + (\frac{y_C}{1-x_C})^2\} < 0$$

for x_A small enough (see (62) and note that $\dfrac{y_C}{1-x_C} = \dfrac{y_B - x_B + x_A}{1 - x_B - y_B + y_A} \geq \dfrac{y_B - x_B + x_A}{1 - x_B - y_B}$ since we are in a case where $y_A \leq 0$). Thus in this case one can assume that (63) holds also.

- F is defined by (32).

Then it holds

$$\frac{d}{dx_A}F(x_A, y_A, x_B, y_B) = \frac{1}{2}\{-(\frac{y_A}{x_A})^2 + (\frac{y_C}{1-x_C})^2\}.$$

This quantity is nonnegative for

$$\begin{aligned} x_A \geq \frac{1-x_C}{y_C} y_A &\geq \frac{1-x_B+y_A-y_B}{C} y_A \\ &> \frac{1-4r}{C} y_A \end{aligned}$$

since $y_A > 0$. Then one concludes as for F defined by (28). This shows that one can always assume (63) and completes the proof □

3. Computation of the minimizers

As seen above we have to minimize F on D_r - or in other words F on D with the constraint (18). First one notices that for $y_B \leq 0$ it holds

$$\begin{aligned} F(x_A, y_A, x_B, y_B) &= -\frac{G}{2}(y_A + y_B + x_A - x_B) \\ &\geq \frac{G}{2}(y_A + x_A - x_B) = F(x_A, y_A, x_B, 0) \end{aligned} \tag{66}$$

and one can always assume $y_B \geq 0$. Then - recall that F is C^1 - one can assume F given by one of the formulae (26), (28), (30), (32). In fact, due to the symmetry of the problem, a minimizer found through (30) should lead to a minimizer through (28) and conversely. So, we can restrict ourselves to the cases of (26), (28), (32). Moreover, as we already have seen in the existence part, one can avoid the case (26) by moving the square of the figure 2 until it touches the wire. So, we restrict ourselves to the cases (28) and (32).

Case 1: A minimizer of F is inside D on the set of points where F is given by (28).

Due to the usual theory of Lagrange multiplyers (see (18)), at this point, one must have for some λ:

$$\begin{cases} \frac{\partial F}{\partial x_A} = \frac{1}{2}\{-(\frac{y_A}{x_A})^2 + (\frac{y_B-y_A}{x_B-x_A})^2\} - \frac{G}{2} = 2\lambda(x_A - x_B), \\ \frac{\partial F}{\partial x_B} = \frac{1}{2}\{-(\frac{y_B-y_A}{x_B-x_A})^2 + (\frac{y_B}{1-x_B})^2\} + \frac{G}{2} = 2\lambda(x_B - x_A), \\ \frac{\partial F}{\partial y_A} = \frac{1}{2}\{2\frac{y_A}{x_A} + 2\frac{y_A-y_B}{x_B-x_A}\} - \frac{G}{2} = 2\lambda(y_A - y_B), \\ \frac{\partial F}{\partial y_B} = \frac{1}{2}\{2\frac{y_B-y_A}{x_B-x_A} + 2\frac{y_B}{1-x_B}\} - \frac{G}{2} = 2\lambda(y_B - y_A). \end{cases} \tag{67}$$

Adding the two first equations and the two last ones, one obtains easily

$$\frac{y_A}{x_A} = \frac{y_B}{1-x_B} = \frac{G}{2}. \tag{68}$$

Keeping also the first and third equations it comes

$$-\frac{G^2}{8} + \frac{1}{2}(\frac{y_B - y_A}{x_B - x_A})^2 - \frac{G}{2} = 2\lambda(x_A - x_B), \tag{69}$$

$$\frac{y_A - y_B}{x_B - x_A} = 2\lambda(y_A - y_B), \tag{70}$$

$$(x_A - x_B)^2 + (y_A - y_B)^2 = 4r^2. \tag{71}$$

Since one is supposed to be inside D one has $y_A < y_B$, $x_A < x_B$ and from (70) λ is different of 0 and given by

$$\lambda = \frac{1}{2(x_B - x_A)}.$$

Replacing in (69) it comes

$$\begin{aligned}(\frac{y_B - y_A}{x_B - x_A})^2 &= \frac{G^2}{4} + G - 2 = \frac{1}{4}\{G^2 + 4G - 8\} \\ &= \frac{1}{4}\{(G+2)^2 - 12\}.\end{aligned} \tag{72}$$

Thus, this case is only possible when (recall that $y_A < y_B$)

$$(G+2)^2 - 12 > 0 \Leftrightarrow G > 2(\sqrt{3} - 1). \tag{73}$$

Combining (71), (72) one obtains first

$$y_B - y_A = \frac{\sqrt{G^2 + 4G - 8}}{2}(x_B - x_A)$$

then

$$x_B - x_A = \frac{4r}{\sqrt{G^2 + 4G - 4}}, \quad y_B - y_A = 2r\frac{\sqrt{G^2 + 4G - 8}}{\sqrt{G^2 + 4G - 4}}. \tag{74}$$

Together with (68) which reads

$$y_A = \frac{G}{2}x_A, \quad y_B = \frac{G}{2}(1 - x_B)$$

one deduces

$$y_A + y_B = \frac{G}{2} + \frac{G}{2}(x_A - x_B) = \frac{G}{2} - \frac{2Gr}{\sqrt{G^2 + 4G - 4}} \tag{75}$$

hence

$$\begin{aligned} y_A &= \frac{G}{4} - \frac{r}{\sqrt{G^2 + 4G - 4}}(G + \sqrt{G^2 + 4G - 8}), \\ y_B &= \frac{G}{4} - \frac{r}{\sqrt{G^2 + 4G - 4}}(G - \sqrt{G^2 + 4G - 8}), \end{aligned} \tag{76}$$

$$\begin{aligned} x_A &= \frac{1}{2} - \frac{r}{\sqrt{G^2 + 4G - 4}}(2 + \frac{2\sqrt{G^2 + 4G - 8}}{G}), \\ x_B &= \frac{1}{2} + \frac{r}{\sqrt{G^2 + 4G - 4}}(2 - \frac{2\sqrt{G^2 + 4G - 8}}{G}). \end{aligned} \tag{77}$$

If E denotes the value of F at this point one has - see (28), (68), (72), (74), (75) -

$$
\begin{aligned}
E &= \frac{1}{2}\{\frac{G}{2}(y_A+y_B)+\frac{(y_B-y_A)^2}{x_B-x_A}\}-\frac{G}{2}(y_A+y_B)-\frac{G}{2}(x_A-x_B)\\
&= -\frac{G}{4}(y_A+y_B)+\frac{G}{2}(x_B-x_A)+\frac{1}{2}\frac{(y_B-y_A)^2}{x_B-x_A}\\
&= -\frac{G}{4}\{\frac{G}{2}-\frac{2rG}{\sqrt{G^2+4G-4}}\}+\frac{2rG}{\sqrt{G^2+4G-4}}+\frac{1}{2}\frac{r(G^2+4G-8)}{\sqrt{G^2+4G-4}}\\
&= -\frac{G^2}{8}+\frac{r}{\sqrt{G^2+4G-4}}\{\frac{G^2}{2}+2G+\frac{G^2}{2}+2G-4\}\\
&= -\frac{G^2}{8}+r\sqrt{G^2+4G-4}. \qquad (78)
\end{aligned}
$$

Remark 3.1. In fact in order for $[AB]$ to be the only segment touching the wire one needs an extra condition namely

$$
\begin{aligned}
\frac{x_B-x_A}{y_B-y_A} = \frac{2}{\sqrt{G^2+4G-8}} &> \frac{G}{2}\\
&\Leftrightarrow 4 > G\sqrt{G^2+4G-8}\\
&\Leftrightarrow 0 > G^4+4G^3-8G^2-16\\
&\Leftrightarrow 0 > (G-2)(G^3+6G^2+4G+8)\\
&\Leftrightarrow G<2.
\end{aligned}
$$

We consider now the second case -i.e.

Case 2: A minimizer of F is inside D on the set of points where F is given by (32).

The Lagrange multiplyers system reads at this point

$$
\begin{cases}
\frac{\partial F}{\partial x_A} = \frac{1}{2}\{-(\frac{y_A}{x_A})^2+(\frac{y_B-y_A}{x_B-x_A})^2+2\frac{x_A-x_B}{y_B-y_A}+2\frac{y_C}{1-x_C}\}-\frac{G}{2} = 2\lambda(x_A-x_B),\\
\frac{\partial F}{\partial x_B} = \frac{1}{2}\{-(\frac{y_B-y_A}{x_B-x_A})^2+2\frac{x_B-x_A}{y_B-y_A}-2\frac{y_C}{1-x_C}+(\frac{y_C}{1-x_C})^2\}+\frac{G}{2} = 2\lambda(x_B-x_A),\\
\frac{\partial F}{\partial y_A} = \frac{1}{2}\{2\frac{y_A}{x_A}+2\frac{y_A-y_B}{x_B-x_A}+(\frac{x_B-x_A}{y_B-y_A})^2-(\frac{y_C}{1-x_C})^2\}-\frac{G}{2} = 2\lambda(y_A-y_B),\\
\frac{\partial F}{\partial y_B} = \frac{1}{2}\{2\frac{y_B-y_A}{x_B-x_A}-(\frac{x_B-x_A}{y_B-y_A})^2+2\frac{y_C}{1-x_C}+(\frac{y_C}{1-x_C})^2\}-\frac{G}{2} = 2\lambda(y_B-y_A).
\end{cases} \qquad (79)
$$

Adding the two first equations and the two last ones we deduce

$$\frac{y_A}{x_A} = \frac{y_C}{1-x_C} = \frac{G}{2}. \qquad (80)$$

Keeping also the first and the third equation of (79) it comes

$$\frac{1}{2}(\frac{y_B-y_A}{x_B-x_A})^2-\frac{x_B-x_A}{y_B-y_A}-\frac{G^2}{8} = 2\lambda(x_A-x_B), \qquad (81)$$

$$-\frac{y_B-y_A}{x_B-x_A}+\frac{1}{2}(\frac{x_B-x_A}{y_B-y_A})^2-\frac{G^2}{8} = 2\lambda(y_A-y_B). \qquad (82)$$

Let us set $T = \dfrac{y_B - y_A}{x_B - x_A}$. Note that T, $\dfrac{1}{T} \neq 0$ since one is inside D.

We claim first that λ is necessarily different of 0. Indeed if not one has

$$\frac{1}{2}T^2 - \frac{1}{T} = -T + \frac{1}{2T^2} = \frac{G^2}{8}. \tag{83}$$

From the two first equations one deduces

$$\frac{1}{2}(T^2 - \frac{1}{T^2}) = -(T - \frac{1}{T}) \Leftrightarrow \frac{1}{2}(T - \frac{1}{T})(T + \frac{1}{T}) = -(T - \frac{1}{T})$$

Hence necessarily $T = 1$. But then (83) is impossible. Thus $\lambda \neq 0$. Dividing (82) by (81) one obtains

$$\begin{aligned}
T &= (-T + \frac{1}{2T^2} - \frac{G^2}{8})/(\frac{1}{2}T^2 - \frac{1}{T} - \frac{G^2}{8}) \\
&\Leftrightarrow T(\frac{1}{2}T^2 - \frac{1}{T} - \frac{G^2}{8}) = -T + \frac{1}{2T^2} - \frac{G^2}{8} \\
&\Leftrightarrow \frac{1}{2}T^3 - \frac{1}{2T^2} + (T-1) - \frac{G^2}{8}(T-1) = 0 \\
&\Leftrightarrow \frac{1}{2}\frac{T^5 - 1}{T^2} + (T-1) - \frac{G^2}{8}(T-1) = 0 \\
&\Leftrightarrow \frac{(T-1)}{2}\{\frac{1 + T + T^2 + T^3 + T^4}{T^2} + 2 - \frac{G^2}{4}\} = 0.
\end{aligned}$$

Thus we have

$$T = \frac{y_B - y_A}{x_B - x_A} = 1 \tag{84}$$

or

$$\frac{1 + T + T^2 + T^3 + T^4}{T^2} + 2 - \frac{G^2}{4} = 0. \tag{85}$$

First note that (84) is only possible - due to (80) and to be in the case of the formula (32) - if

$$G \geq 2. \tag{86}$$

The equation (85) reads also

$$\frac{1}{T^2} + T^2 + 1 + T + \frac{1}{T} + 2 - \frac{G^2}{4} = 0.$$

Setting $u = T + \dfrac{1}{T}$ and noting that $u^2 = T^2 + 2 + \dfrac{1}{T^2}$ we obtain

$$\begin{aligned}
&u^2 + u + 1 - \frac{G^2}{4} = 0 \\
\Leftrightarrow\ &(u + \frac{1}{2})^2 = \frac{G^2 - 3}{4}.
\end{aligned} \tag{87}$$

Thus, in order to have a solution to (87), one has to impose

$$G \geq \sqrt{3}. \tag{88}$$

One obtains then (recall that u has to be positive !)

$$\begin{aligned} u + \frac{1}{2} &= \frac{\sqrt{G^2 - 3}}{2} \\ \Leftrightarrow T^2 - T(\frac{\sqrt{G^2-3}-1}{2}) + 1 &= 0. \\ \Leftrightarrow (T - \frac{\sqrt{G^2-3}-1}{4})^2 &= (\frac{\sqrt{G^2-3}-1}{4})^2 - 1. \end{aligned} \tag{89}$$

In order for this equation to be solvable one must have

$$\sqrt{G^2-3} - 1 \geq 4 \Leftrightarrow G \geq 2\sqrt{7}. \tag{90}$$

One gets then

$$T = \frac{\sqrt{G^2-3}-1}{4} \pm \sqrt{(\frac{\sqrt{G^2-3}-1}{4})^2 - 1}. \tag{91}$$

It is clear that the solution to (87) are T and $\frac{1}{T}$ thus only one of these roots is to be considered - for instance we will consider the one with the sign + - indeed the other one corresponds to the symmetric position of the square.

Let us now compute x_A, x_B, y_A, y_B and F corresponding to these different values of T. First

- Case $T = 1$ $(G \geq 2)$.

Combining (80) that reads

$$y_A = \frac{G}{2}x_A, \quad y_B - (x_B - x_A) = \frac{G}{2}(1 - x_B + y_A - y_B)$$

and (84), (18) one obtains

$$y_B - y_A = x_B - x_A = \sqrt{2}r.$$

Then it follows easily that

$$x_A = \frac{1}{2} - \sqrt{2}r, \quad x_B = \frac{1}{2}, \quad y_A = \frac{G}{2}(\frac{1}{2} - \sqrt{2}r), \quad y_B = \frac{G}{4} + \sqrt{2}r(1 - \frac{G}{2}). \tag{92}$$

In this case the value E of F at this point is given by

$$E = -\frac{G^2}{8} + \sqrt{2}r(\frac{G^2}{4} + 1). \tag{93}$$

- Case $T = S + \sqrt{S^2 - 1}$, $\quad S = \dfrac{\sqrt{G^2-3}-1}{4}$, $\quad (G \geq 2\sqrt{7})$.

In this case one has from (80) and the definition of T

$$y_A = \frac{G}{2} x_A, \quad y_C = y_B - (x_B - x_A) = \frac{G}{2}(1 - x_B + y_A - y_B) \tag{94}$$

$$y_B - y_A = T(x_B - x_A) \tag{95}$$
$$(1 + T^2)(x_B - x_A)^2 = 4r^2 \qquad \text{(see (18))}.$$

One derives

$$x_B - x_A = \frac{2r}{\sqrt{1+T^2}}, \quad y_B - y_A = \frac{2rT}{\sqrt{1+T^2}}. \tag{96}$$

Summing the equations of (94) we get:

$$y_A + y_B = \frac{G}{2} - \frac{Gr}{\sqrt{1+T^2}} - \frac{GrT}{\sqrt{1+T^2}} + \frac{2r}{\sqrt{1+T^2}}. \tag{97}$$

It follows

$$y_A = \frac{G}{4} + \frac{r}{\sqrt{1+T^2}}\{(1-T) - \frac{G}{2}(1+T)\}, \tag{98}$$

$$y_B = \frac{G}{4} + \frac{r(1+T)}{\sqrt{1+T^2}}\{1 - \frac{G}{2}\}, \tag{99}$$

$$x_A = \frac{1}{2} + \frac{r}{\sqrt{1+T^2}}\{\frac{2}{G}(1-T) - (1+T)\}, \tag{100}$$

$$x_B = \frac{1}{2} + \frac{r(1-T)}{\sqrt{1+T^2}}\{\frac{2}{G} + 1\} \tag{101}$$

with T given above.

For the value E of F at this point one derives from (32), (94)

$$E = \frac{1}{2}\{\frac{G}{2}(y_A + y_C) + T(y_B - y_A) + \frac{1}{T}(x_B - x_A)\} - \frac{G}{2}(y_A + y_B + x_A - x_B).$$

Using the fact that $y_C = y_B - (x_B - x_A)$ it comes

$$\begin{aligned} E &= -\frac{G}{4}(y_A + y_B - (x_B - x_A)) + \frac{1}{2}\{T(y_B - y_A) + \frac{1}{T}(x_B - x_A)\} \\ &= -\frac{G}{4}(\frac{G}{2} - \frac{Gr}{\sqrt{1+T^2}} - \frac{GrT}{\sqrt{1+T^2}}) + \frac{rT^2}{\sqrt{1+T^2}} + \frac{r}{T\sqrt{1+T^2}} \\ &= -\frac{G^2}{8} + \frac{r}{\sqrt{1+T^2}}\{\frac{G^2}{4}(1+T) + T^2 + \frac{1}{T}\}. \end{aligned} \tag{102}$$

Let us also consider the case where the infimum of F is achieved on the border of D_r - i.e. for $x_A = x_B$ or $y_A = y_B$ - since this corresponds to a square with sides parallel to the axis one can without loss of generality assume $y_A = y_B$. Thus let us consider

Case 3: A minimizer belongs to the border of D - i.e. is such that $y_A = y_B$ - since one can assume $y_B \geq 0$ one is in the case of the formula (28) and it holds

$$F(x_A, y_A, x_B, y_B) = \frac{1}{2}\{\frac{y_A^2}{x_A} + \frac{y_A^2}{1 - x_B}\} - \frac{G}{2}(2y_A + x_A - x_B).$$

Since $x_B = x_A + 2r$ we have

$$F(x_A, y_A, x_B, y_B) = \frac{y_A^2}{2}\{\frac{1}{x_A} + \frac{1}{1 - x_A - 2r}\} - Gy_A + rG.$$

One wants to minimize this on $x_A \in (0,1)$, $y_A \geq 0$. It is easy to see that for y_A fixed the minimum of this function is achieved for

$$\frac{1}{x_A} = \frac{1}{1 - x_A - 2r} \Leftrightarrow x_A = \frac{1}{2} - r.$$

Then, for this value, one has

$$F(x_A, y_A, x_B, y_B) = \frac{y_A^2}{\frac{1}{2} - r} - Gy_A + rG$$

and the minimum is achieved for $y_A = \dfrac{G}{2}(\dfrac{1}{2} - r)$.

Thus in this case one has

$$x_A = \frac{1}{2} - r, \quad x_B = \frac{1}{2} + r, \quad y_A = y_B = \frac{G}{2}(\frac{1}{2} - r) \tag{103}$$

and the value E of F at this point is

$$\begin{aligned} E &= \frac{G^2}{4}(\frac{1}{2} - r) - \frac{G^2}{2}(\frac{1}{2} - r) + rG \\ &= -\frac{G^2}{8} + r(\frac{G^2}{4} + G). \end{aligned} \tag{104}$$

After these preliminaries let us describe the situation. First one has:

Theorem 3.2. *Assume that*

$$G \leq 2(\sqrt{3} - 1). \tag{105}$$

Then the minimization problem (6) *admits a unique solution given by a square centered in the middle of the wire with* A, B *given by* (103).

Proof. One knows that the problem admits a minimizer. Due to the constraints (73), (86), (90) the only possibility is to be in the case 3 above. This completes the proof of the theorem. □

When (105) holds, the minimal position in energy is described by the figure below:

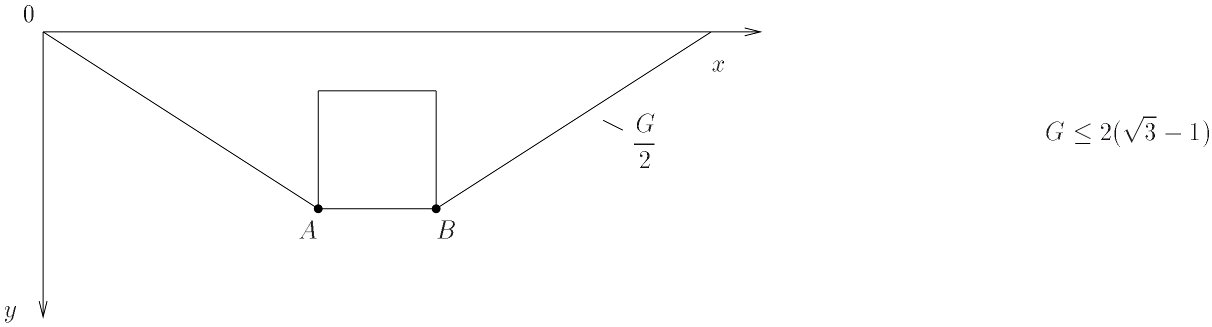

Figure 5

Let us increase the weight of the square. One has

Theorem 3.3. *Let us assume that*

$$2(\sqrt{3}-1) < G < 2 \tag{106}$$

Then the minimization problem (6) *admits two solutions. One is such that* A, B *are given by* (76), (77) *the other one is obtained by reflection with respect to the axis* $x = \frac{1}{2}$.

Proof. Due to (86), (90) a minimizer can only occur in the case 1 or in the case 3. Considering (78) and (104) one remarks that

$$\sqrt{G^2 + 4G - 4} \leq \frac{G^2}{4} + G$$

$$\begin{aligned} &\Leftrightarrow \sqrt{u-4} \leq \frac{1}{4}u \quad \text{where we have set } u = G^2 + 4G \\ &\Leftrightarrow u^2 - 16u + 64 = (u-8)^2 \geq 0. \end{aligned} \tag{107}$$

This is always true - with equality only for $u = 8$ - i.e. $G = 2(\sqrt{3}-1)$.

Thus the energy given by (78) is the smallest and A, B are given by (76), (77). this completes the proof of the theorem. □

When (106) holds then the minimum of the energy is achieved for the two symmetric configurations displayed on the figure below:

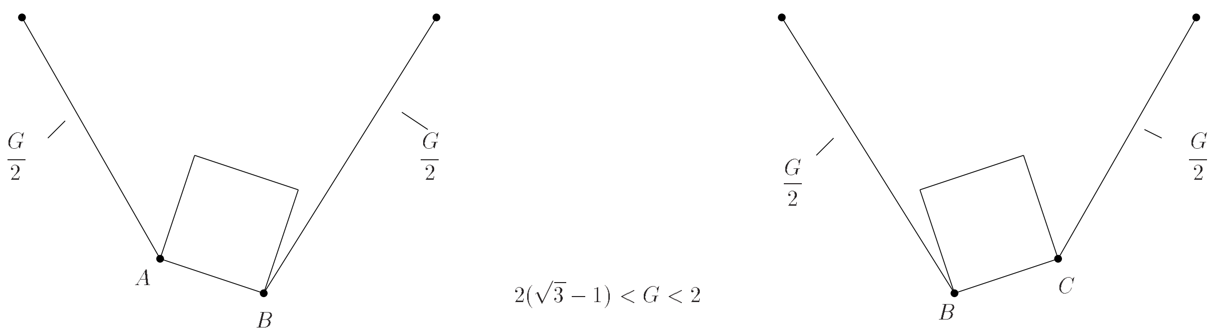

Figure 6

Let us pass to heigher weight:

Theorem 3.4. *Suppose that*

$$2 \leq G \leq 2\sqrt{7}. \tag{108}$$

Then the problem (6) *admits a unique minimizer given by a square centered in the middle of the wire, having its sides making a* 45^0 *angle with the coordinates axis and where* A*,* B *are the points given by* (92)*.*

Proof. Due to the remark 3.1, the only possibility for a minimizer is to be in the case 2 - with $T = 1$ - or 3 (note that for $G = 2\sqrt{7}$ there is only the solution $T = 1$ in case 2). So, one is lead to compare the energies given by (93) and (104).

One notices that

$$\sqrt{2}(\frac{G^2}{4} + 1) \leq \frac{G^2}{4} + G \tag{109}$$

$$\Leftrightarrow \frac{G^2}{4}(\sqrt{2} - 1) - G + \sqrt{2} \leq 0.$$

The minimum of this parabola is achieved for $G = \dfrac{2}{\sqrt{2} - 1} \in (2, 2\sqrt{7})$.

Since for $G = 2$, $G = 2\sqrt{7}$ this quadratic expression is negative, it follows that (109) holds and the theorem is proved. □

In the case where (108) holds the configuration of minimal energy is given by the following picture:

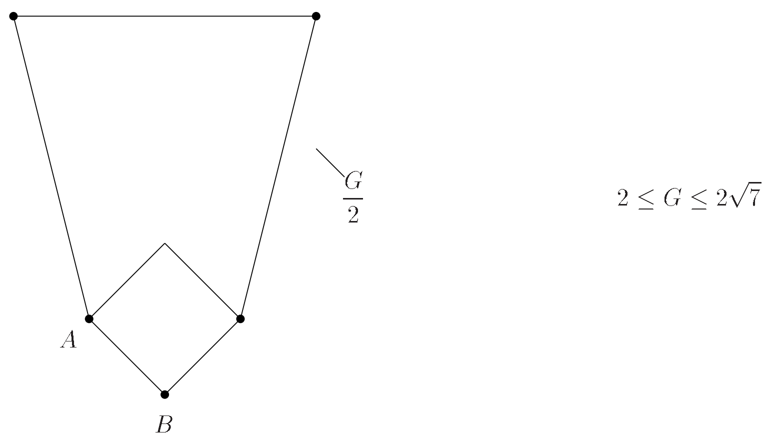

Figure 7

Finally let us assume that G passes the level $2\sqrt{7}$. One has

Theorem 3.5. *Let us assume that*

$$G > 2\sqrt{7}. \tag{110}$$

Then the problem (6) *admits two symmetric solutions one of which is given by* A*,* B *defined in* (98)–(101)*.*

Proof. The possibility for a minimizer is to be in case 2 or 3. So, one has to compare the energies given by (93), (102), (104). Let us show that the one in (102) is the smallest. This will complete the proof of the theorem.

- Comparison of (93), (102)

We would like to show that

$$\sqrt{2}(\frac{G^2}{4}+1) > \frac{1}{\sqrt{1+T^2}}\{\frac{G^2}{4}(1+T)+T^2+\frac{1}{T}\}. \tag{111}$$

As below (93) we set

$$S = \frac{\sqrt{G^2-3}-1}{4} \tag{112}$$

so that

$$T = S + \sqrt{S^2-1}. \tag{113}$$

One has also

$$T = S + \sqrt{S^2-1}.\frac{S-\sqrt{S^2-1}}{S-\sqrt{S^2-1}} = \frac{1}{S-\sqrt{S^2-1}}$$

so that

$$T + \frac{1}{T} = 2S \Leftrightarrow T^2+1 = 2ST. \tag{114}$$

With this equality (111) reduces to show

$$\begin{aligned}
4ST(\frac{G^2}{4}+1)^2 &> \{\frac{G^2}{4}(1+T)+2ST-1+2S-T\}^2 \\
&= (1+T)^2\{\frac{G^2}{4}+2S-1\}^2 \\
\Leftrightarrow 4S(\frac{G^2}{4}+1)^2 &> \frac{1+2T+T^2}{T}\{\frac{G^2}{4}+2S-1\}^2 \\
&= (\frac{1}{T}+2+T)\{\frac{G^2}{4}+2S-1\}^2 = (2S+2)\{\frac{G^2}{4}+2S-1\}^2.
\end{aligned}$$

Thus one would like to show that

$$2S(\frac{G^2}{4}+1)^2 > (S+1)\{\frac{G^2}{4}+2S-1\}^2. \tag{115}$$

One has $4S = \sqrt{G^2-3}-1$ hence

$$16S^2 = G^2-2-2\sqrt{G^2-3} = G^2-4-2(\sqrt{G^2-3}-1) = G^2-4-8S, \tag{116}$$

i.e. one has a relation between S^2 and S. Expanding (115) and using the expression of

S^2 in terms of S it comes after some manipulations:

$$\begin{aligned}
2S(\frac{G^2}{4}+1)^2 &> (S+1)\{(\frac{G^2}{4}-1)^2+4S(\frac{G^2}{4}-1)+4S^2\} \\
&= (S+1)\{(\frac{G^2}{4}-1)^2+4S(\frac{G^2}{4}-1)+\frac{G^2}{4}-1-2S\} \\
&= (S+1)\{(\frac{G^2}{4}-1)\frac{G^2}{4}+S(G^2-6)\} \\
&= S(\frac{G^2}{4}-1)\frac{G^2}{4}+S^2(G^2-6)+(\frac{G^2}{4}-1)\frac{G^2}{4}+S(G^2-6) \\
&= S(\frac{G^2}{4}-1)\frac{G^2}{4}+(\frac{1}{4}(\frac{G^2}{4}-1)-\frac{S}{2})(G^2-6)+(\frac{G^2}{4}-1)\frac{G^2}{4}+S(G^2-6) \\
&= S\{(\frac{G^2}{4}-1)\frac{G^2}{4}+\frac{1}{2}(G^2-6)\}+(\frac{G^2}{4}-1)(\frac{G^2}{2}-\frac{3}{2}) \\
&= S\{\frac{G^4}{16}+\frac{G^2}{4}-3\}+(\frac{G^2}{4}-1)\frac{G^2-3}{2}. \qquad (117)
\end{aligned}$$

Thus (117) reduces to

$$2S(1+\frac{G^2}{2}+\frac{G^4}{16})-S(\frac{G^4}{16}+\frac{G^2}{4}-3) > \frac{(G^2-4)(G^2-3)}{8}$$

$\Leftrightarrow$

$$S(G^4+12G^2+80) > 2(G^2-4)(G^2-3). \qquad (118)$$

We set $X = G^2 > 28$. Then (118) is equivalent to

$$(\sqrt{X-3}-1)(X^2+12X+80) > 8(X-4)(X-3)$$

$\Leftrightarrow$

$$\sqrt{X-3}(X^2+12X+80) > 9X^2-44X+176$$

$\Leftrightarrow$

$$\begin{aligned}
\sqrt{X-3}-5 &> \frac{9X^2-44X+176-5(X^2+12X+80)}{X^2+12X+80} \\
&= \frac{4X^2-104X-224}{X^2+12X+80} = \frac{4(X-28)(X+2)}{X^2+12X+80}.
\end{aligned}$$

Since

$$\sqrt{X-3}-5 = \frac{X-28}{\sqrt{X-3}+5},$$

the above inequality reduces to show - after division by $X-28$ -

$$X^2+12X+80 > 4(X+2)(\sqrt{X-3}+5)$$

$\Leftrightarrow$

$$X^2 - 8X + 40 > 4(X+2)\sqrt{X-3}$$

$\Leftrightarrow$

$$(X^2 - 8X + 40)^2 > 16(X-3)(X^2 + 4X + 4)$$

$\Leftrightarrow$

$$X^4 - 32X^3 + 128X^2 - 512X + 1792 = (X-28)(X^3 - 4X^2 + 6X - 64) > 0.$$

Thus the whole problem reduces to show that for $X > 28$

$$P(X) = X^3 - 4X^2 + 16X - 64 > 0$$

One has $P'(X) = 3X^2 - 8X + 16 = 3\{(X - \frac{4}{3})^2 + \frac{32}{9}\} > 0$ and $P(28) > 0$. This completes the proof of (111).

- Comparison of (102) and (104)

One would like to show that

$$(\frac{G^2}{4} + G) > \frac{1}{\sqrt{1+T^2}}\{\frac{G^2}{4}(1+T) + T^2 + \frac{1}{T}\}. \tag{119}$$

One proceeds as above to get instead of (115)

$$S(\frac{G^2}{4} + G)^2 > (S+1)\{\frac{G^2}{4} + 2S - 1\}^2.$$

Then instead of (117)

$$S(\frac{G^2}{4} + G)^2 > S\{\frac{G^4}{16} + \frac{G^2}{4} - 3\} + (\frac{G^2}{4} - 1)(\frac{G^2 - 3}{2})$$

i.e.

$$S(\frac{G^4}{16} + \frac{G^3}{2} + G^2) > S(\frac{G^4}{16} + \frac{G^2}{4} - 3) + \frac{(G^2-4)(G^2-3)}{8}$$

$\Leftrightarrow$

$$S(\frac{G^3}{2} + \frac{3G^2}{4} + 3) > \frac{(G^2-4)(G^2-3)}{8}$$

$\Leftrightarrow$

$$S(4G^3 + 6G^2 + 24) > (G^2-4)(G^2-3).$$

Replacing S by its value - $S = \dfrac{\sqrt{G^2-3}-1}{4}$ - we have to show

$$(\sqrt{G^2-3}-1)(2G^3+3G^2+12) > 2G^4-14G^2+24$$

$\Leftrightarrow$

$$\sqrt{G^2-3}(2G^3+3G^2+12) > 2G^4+2G^3-11G^2+36.$$

One has $\sqrt{G^2-3}-G = -\dfrac{3}{\sqrt{G^2-3}+G}$, hence the above is equivalent to

$$G(2G^3+3G^2+12) > 2G^4+2G^3-11G^2+36+\frac{3(2G^3+3G^2+12)}{G+\sqrt{G^2-3}}$$

$\Leftrightarrow$

$$G^3+11G^2+12G-36 > \frac{3}{G+\sqrt{G^2-3}}(2G^3+3G^2+12)$$

since $\sqrt{G^2-3} > 5$ it is enough to show that

$$(G+5)(G^3+11G^2+12G-36) > 6G^3+9G^2+36$$

$\Leftrightarrow$

$$G^4+10G^3+58G^2+24G-216 > 0$$

which of course holds for $G^2 \geq 28$.

This completes the proof of the theorem. $\square$

Thus, in this case, the equilibrium positions are the following

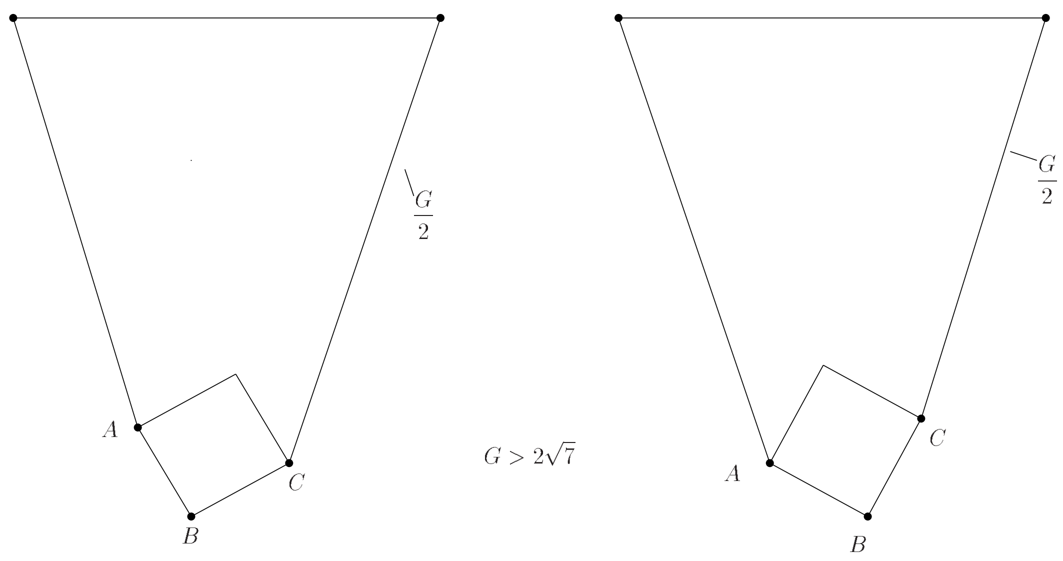

Figure 8

Remark 3.6. One sees on (100), (101) that $x_A, x_B \to \frac{1}{2} - r$ when $G \to +\infty$. In this case the square tends to have its sides parallel to the coordinates axis at infinity.

Acknowledgements. The work of the second author has been supported by the Swiss National Science Foundation under the contract $\sharp 20 - 58856.99$.

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