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Locally Nonconical Convexity

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There is a hierarchy of structure conditions for convex sets. In this paper we study a recently defined [3, 8, 9] condition called locally nonconical convexity (abbreviated LNC). Is is easy to show that every strictly convex set is LNC, as are half-spaces and finite intersections of sets of either of these types, but many more sets are LNC. For instance, every zonoid (the range of a nonatomic vector-valued measure) is LNC (Corollary 6.3). However, there are no infinite-dimensional compact LNC sets (Theorem 5.3).

The LNC concept originated in a search for continuous sections, and the present paper shows how it leads naturally (and constructively) to continuous sections in a variety of situations. Let Q be a compact, convex set in $\mathbb{R}^n$, and let T be a linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$. We show (Theorem 2.1) that Q is LNC if and only if the restriction of any such T to Q is an open map of Q onto $T(Q)$. This implies that if Q is LNC, then any such T has continuous sections (i.e. there are continuous right inverses of T) that map from $T(Q)$ to Q, and in fact it is possible to define continuous sections constructively in various natural ways (Theorem 2.3, Corollary 2.4, and Theorem 2.5). If Q is strictly convex and T is not 1-1, we can construct continuous sections which take values in the boundary of Q (Theorem 2.6).

When we give up compactness it is natural to consider a closed, convex, LNC subset Q of a Hilbert space X which may be infinite-dimensional. In this case we must assume that T is left Fredholm, i.e. a bounded linear map with closed range and finite-dimensional kernel. We can then prove results analogous to those mentioned in the last paragraph (Theorems 4.3–4.7). We also prove that $T(Q)$ is LNC (Theorem 5.5).

Keywords: convex set, locally nonconical, continuous section, continuous selection, strictly convex, zonoid

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1. Introduction

The concept of a locally nonconical (abbr. LNC) convex set arose in [3] and was explored further in the Ph.D. dissertation [8] of the second author; some of the results in [8] appeared in [9]. The LNC concept originated in a search for continuous sections, and the present paper shows why it is indeed the key to the existence of continuous sections in a variety of situations.

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Definition 1.1. A convex set Q in a Hausdorff topological vector space is *locally nonconical* (LNC) if for any net (x_t) in Q that converges to a point x in Q and for any other point x' in Q, the points $x_t + (x' - x)/2$ eventually lie in Q.

Equivalently, for every pair of points x and x' in Q there exists a relative neighborhood U of x in Q such that $U + (x' - x)/2 \subset Q$. Also, observe that if Q is LNC, then because $x_t + (x' - x)/2 \to (x' + x)/2$, we can apply the LNC condition again with $(x' + x)/2$ in place of x, and conclude that $x_t + \frac{3}{4}(x' - x)$ is eventually in Q, and so on; inductively, the LNC condition implies that for any $x, x', x_t \in Q$ with $x_t \to x$, we have $x_t + e(x' - x) \in Q$ eventually, for any $e \in (0, 1)$.

The name arose from the fact that no point on the "slanted" portion of an ordinary circular cone can be the point x of this definition. See Example 3.3.

In practice, to verify the LNC property it is sufficient to show that there is a subnet of $x_t + (x' - x)/2$ which eventually lies in Q. That is because if x, x', x_t falsifies LNC, then by passing to a subnet we can ensure that $x_t + (x' - x)/2$ is never in Q. We will use this observation repeatedly.

From here on we take, as part of the definition of topological vector spaces, that they are Hausdorff. We also take the scalar field to be $\mathbb{R}$.

There is a hierarchy of conditions that have been defined for convex sets. It is shown in [8] that LNC lies below "strictly convex" and above "uniformly stable" in this hierarchy. Indeed, if the net (x_t) is not required to lie in Q, or if the word "relative" is deleted from the second LNC definition, then the condition becomes equivalent to strict convexity. What sets do we get by slightly weakening the definition of "strictly convex"? Besides strictly convex sets, we get half spaces, finite intersections of LNC sets, finite Cartesian products of LNC sets and all zonoids. See Sections 3 and 6.

Since a strictly convex set must have nonvoid topological interior, it is clear that there cannot be any compact, infinite-dimensional strictly convex sets. The same holds for compact, infinite-dimensional LNC sets (see Theorem 5.3). The proof is more difficult in the LNC case because this theorem, despite its negative sounding tone, is actually an existence result! That is, we show that any infinite-dimensional, compact, convex set Q contains points x and x' and a net $\{x_t\}$ that converges to x such that $x_t + (x' - x)/2$ never lies in Q. This implies that there is a continous affine map out of Q that is not an open map (see Corollary 5.4). On the other hand, there are plenty of closed infinite-dimensional LNC sets, and some of what we know about compact LNC sets actually holds (with more difficult proofs) for closed LNC sets; see Section 4.

Here is a little more detail on our motivation for studying the LNC property. Let $T : X \to Y$ be a linear map betwen locally convex topological vector spaces. If $Q \subset X$ is convex, we regard $f = T|_Q : Q \to T(Q)$ as a covering map and ask whether it possesses a continuous section. That is, does there exist a continuous map $g : T(Q) \to Q$ such that $f \circ g = \mathrm{id}_{T(Q)}$? Another way to put this is to ask if f has a continuous right inverse. Such a map g is also sometimes called a continuous selection, because for each $y \in T(Q)$ it selects, in a continuous fashion, a point $g(y) \in f^{-1}(y)$. Continuous selections have been studied extensively (see e.g. [6]), but our approach is different. We want a condition on the set Q that will **guarantee** the existence of continuous sections for **all** maps f as described above. We show in Section 2 that for compact sets Q in $\mathbb{R}^n$ the LNC condition fills the bill, and in Section 4 we treat the case of closed sets.

The LNC condition is not equivalent to the existence of continuous sections (see Example 3.3). However, it seems that without LNC any continuous section which does exist is somehow "ad hoc," though it is difficult to make this assertion precise. The principle we have in mind is that any reasonable general construction of sections will automatically produce continuous sections if and only if Q is LNC. Again, what constitutes a "reasonable" construction is unclear, but we will give several examples in Section 2.

2. Continuous sections in finite dimensions

Since our results are simplest and most complete in the finite-dimensional case, we begin there. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. If $Q \subset \mathbb{R}^n$ is convex and compact, we define $f = T|_Q : Q \to T(Q)$ and ask when there exists a continuous map $g : T(Q) \to Q$ such that $f \circ g = \mathrm{id}_{T(Q)}$.

It follows from Michael's selection theorem [6, Examples 1.1, 1.1*, and Theorem 3.2] that a continuous section exists if $f : Q \to T(Q)$ is an open map. The LNC condition exactly guarantees that any restriction of a linear map is open. In addition, our methods prove more. Not only do continuous sections always exist when Q is LNC, but they can be defined constructively and are of a minimizing type.

Theorem 2.1. *Let Q be a compact, convex subset of $\mathbb{R}^n$. The following conditions are equivalent:*

(i) *Q is LNC;*
(ii) *for every linear map $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$, the restriction map $f = T|_Q : Q \to T(Q)$ is open;*
(iii) *for every m and every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, the restriction map $f = T|_Q : Q \to T(Q)$ is open.*

Proof. (i) $\Rightarrow$ (iii). Suppose Q is LNC but $f = T|_Q$ is not open for some $T : \mathbb{R}^n \to \mathbb{R}^m$. We shall reach a contradiction. There exists $x' \in Q$ and a relative open neighborhood U of x' in Q such that $f(U)$ is not a relative open neighborhood of $f(x')$ in $T(Q)$. Thus, there is a sequence (y_i) in $T(Q)$ which converges to $f(x')$, such that y_i is not in $f(U)$ for any i. Find $(x_i) \subset Q$ such that $f(x_i) = y_i$ and pass to a subsequence so that (x_i) converges. Say $x_i \to x$; then

$$f(x) = \lim f(x_i) = \lim y_i = f(x').$$

Choose $e \in (0, 1)$ small enough that $x' + e(x - x') \in U$; we can do this because U is relatively open and $x' + e(x - x') \in Q$ for all $e \in [0, 1]$. Since Q is LNC, $x_i + (1-e)(x' - x)$ is eventually in Q. But since $x_i+(1-e)(x'-x)$ converges to $x+(1-e)(x'-x) = x'+e(x-x')$, the point $x_i + (1-e)(x'-x)$ must be in U for all sufficiently large i. However, none of the points $x_i + (1 - e)(x' - x)$ can belong to U because $f(x_i + (1 - e)(x' - x)) = f(x_i) = y_i$ is not in $f(U)$. We have a contradiction.

(iii) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (i). Suppose Q is not LNC and let x, x', x_i be a falsifying case. Without loss of generality suppose $x_i + (x' - x)/2$ is never in Q and $x = 0$. Thus $x_i \to 0$ and $x_i + x'/2$ is never in Q. Let T be the projection of $\mathbb{R}^n$ onto $\mathbb{R}^n/[x'] \cong \mathbb{R}^{n-1}$ (the quotient of $\mathbb{R}^n$

by the one-dimensional subspace spanned by the vector x') and let $f = T|_Q$. The next step is to find a relatively open neighborhood U of x' such that $f(x_i)$ is not eventually in $f(U)$. Then we will have that $f(x_i)$ converges to $f(x') = 0$, but is not eventually in $f(U)$, hence f is not open. To fulfill this program let ϕ be a linear functional on $\mathbb{R}^n$ such that $\phi(x') = 1$. Let $U = \{y \in Q : \phi(y) > 3/4\}$. Then U is a relatively open neighborhood of x' in Q. Suppose (by contradiction) that $f(x_i)$ were eventually in $f(U)$; then there would exist $z_i \in U$ such that $f(z_i) = f(x_i)$. By compactness, pass to a subsequence so that (z_i) converges and let $z = \lim z_i$. Then $\phi(z) = \lim \phi(z_i) \geq 3/4$. By the definition of f there are real numbers t_i such that $z_i = x_i + t_i x'$. Then $\phi(z_i) = \phi(x_i) + t_i$. Since $x_i \to 0$ and ϕ is continuous, $\phi(x_i) \to 0$, so eventually $\phi(x_i) < 1/4$ and thus $t_i > 1/2$. But we have assumed that $x_i + x'/2$ is not in Q, so by convexity $x_i + t_i x' = z_i$ cannot be in Q for $t_i > 1/2$. Since we have chosen z_i to be in U, hence in Q, we have a contradiction. □

Next, we consider the simplest situation, where T is the projection of $\mathbb{R}^n$ onto a quotient by a one-dimensional subspace. Thus, let $[v]$ be the one-dimensional subspace spanned by a nonzero vector $v \in \mathbb{R}^n$. Let $T_v : \mathbb{R}^n \to \mathbb{R}^n/[v]$ be the quotient map, let Q be a compact, convex subset of $\mathbb{R}^n$, and let $f_v = T_v|_Q : Q \to T_v(Q)$. In this case there is a simple, explicit description of a right inverse $g_v : T_v(Q) \to Q$. Namely, given $y = f_v(x)$ with $x \in Q$, let $g_v(y) = x + av$ where

$$a = \inf\{b \in \mathbb{R} : x + bv \in Q\}.$$

Intuitively, if v points "upward", then g_v maps $f_v(Q)$ onto the lower boundary of Q. (By replacing inf with sup, we could similarly map $f_v(Q)$ onto the upper boundary of Q, without essentially affecting the discussion.)

Perhaps surprisingly, the map g_v is not necessarily continuous. Our next proposition shows that it is continuous for all v precisely if Q is LNC.

Proposition 2.2. *Let Q be a compact, convex subset of $\mathbb{R}^n$. Then Q is LNC if and only if $g_v : T_v(Q) \to Q$ is continuous for all nonzero $v \in X$.*

Proof. Suppose g_v is not continuous for some v. Then there is a sequence (z_i) in Q and an element z in Q such that $y_i = f_v(z_i) \to f_v(z) = y$ but $g_v(y_i) \not\to g_v(y)$. Pass to a subsequence so that $x_i = g_v(y_i)$ converges, say to x, and note that x must be of the form $x = z + av$ since

$$f_v(x) = \lim f_v(x_i) = \lim y_i = f_v(z).$$

Define $x' = g_v(y) = z + a'v$; then $a' < a$ by the minimality of a' in the definition of $g_v(y)$. Now the LNC property fails for the points x and x' and the sequence $x_i \to x$, because $x_i + (x' - x)/2 = x_i + (a' - a)v/2$ does not belong to Q for any i, by the minimality of x_i. Thus, discontinuity of any g_v implies that Q is not LNC.

Conversely, suppose Q is not LNC and find $x, x', x_i \in Q$ such that $x_i \to x$ but $x_i + (x' - x)/2$ is not in Q for any i. Take $v = x - x'$ and define $y_i = f_v(x_i)$ and $y = f_v(x) = f_v(x')$. Then for every i, $g_v(y_i) = x_i + a_i v$ for some $a_i > -1/2$, since $x_i + av \notin Q$ for $a = -1/2$. However, $g_v(y) = x + av$ for some $a \leq -1$, and so $g_v(y_i) \not\to g_v(y)$. Thus g_v is not continuous. □

We now give two constructions of continuous sections for linear maps of LNC sets. The first involves strictly convex norms on $\mathbb{R}^n$. This means that the closed unit ball is strictly convex. For example, any euclidean norm on $\mathbb{R}^n$ is strictly convex, as is the l^p norm for $1 < p < \infty$.

Theorem 2.3. *Let Q be a compact LNC subset of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Let $f = T|_Q : Q \to T(Q)$. Fix a strictly convex norm on $\mathbb{R}^n$. Then for each $y \in T(Q)$ there is a unique element $g(y)$ of $f^{-1}(y) = T^{-1}(y) \cap Q$ with minimal norm, and the map g is a continuous section of f.*

Proof. There exists an element of $f^{-1}(y)$ with minimal norm by compactness. For uniqueness, suppose v and w are distinct and both minimize norm; then $(v + w)/2$ also belongs to the convex set $f^{-1}(y)$, and it has strictly smaller norm by strict convexity, a contradiction. So the map g is well-defined.

It is also obviously a section of f. To verify continuity, let (y_i) be a sequence in $T(Q)$ which converges to $y \in T(Q)$ and suppose $(x_i) = (g(y_i))$ fails to converge to $x' = g(y)$. By compactness, we can pass to a subsequence so that (x_i) converges to some point $x \neq x'$. By LNC, for sufficiently large i we then have $x_i + (x' - x)/2 \in Q$.

However, $f(x) = \lim f(x_i) = y$, and since $x \neq x' = g(y)$ we must have $\|x\| > \|x'\|$. Therefore

$$\lim \|x_i + (x' - x)/2\| = \|x' + x\|/2 < \|x\| = \lim \|x_i\|,$$

so that $\|x_i + (x' - x)/2\| < \|x_i\|$ for sufficiently large i. Since $f(x' - x) = 0$, this contradicts minimality of the norm of $x_i = g(y_i)$. This completes the proof. $\square$

We isolate a special case of Theorem 2.3 in the following corollary; although simple, we believe this result is new.

Corollary 2.4. *Let Q be a compact, strictly convex subset of $\mathbb{R}^n$, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and let $f = T|_Q : Q \to T(Q)$. For $y \in T(Q)$ define $g(y)$ to be the unique element of $f^{-1}(y) = T^{-1}(y) \cap Q$ with minimal euclidean norm. Then g is a continuous section of f.*

If T is a linear map of $\mathbb{R}^n$ into itself (not an essential restriction) then Theorem 2.3 and Corollary 2.4 can be modified in the following way. Instead of taking $g(y)$ to be the element of $f^{-1}(y)$ with minimal norm, choose it instead so that $||g(y) - y||$ is minimized. Trivial modifications of the proofs show that this g is also continuous, if Q is LNC.

Our final construction of continuous sections requires some preface. In previous results we chose a distinguished element of $f^{-1}(y)$ in simple ways — in one case by taking one endpoint of a line segment, and in the other case by minimizing norm. We now introduce a new, slightly more involved method of making this choice. This supports our earlier contention that LNC guarantees that any "reasonably" defined section will be continuous.

Let $\{F_r : r = 1, ..., n\}$ be a separating family of linear functionals on $\mathbb{R}^n$. Thus, for any nonzero $v \in \mathbb{R}^n$ we have $F_r(v) \neq 0$ for some r. Given a compact set K in $\mathbb{R}^n$, define a nested family of subsets K_r $(0 \leq r \leq n)$ as follows. Let $K_0 = K$. Having defined K_{r-1}, let K_r be the set

$$K_r = K_{r-1} \cap F_r^{-1}(a)$$

where $a = \inf F_r(K_{r-1})$. Geometrically, K_r is the "lowest slice" of K_{r-1}, according to F_r.

Suppose K_n contains two distinct points v and w. Then $F_r(v) \neq F_r(w)$ for some r since the F_r's are separating. So v and w cannot both belong to K_r, a contradiction. Thus $K_n = \{v\}$ for some $v \in K$. We define $\Gamma(K) = v$; this map Γ performs the desired selection of a distinguished element of K.

Theorem 2.5. *Let Q be a compact LNC subset of $\mathbb{R}^n$, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and let $f = T|_Q : Q \to T(Q)$. Then $g(y) = \Gamma(f^{-1}(y))$ defines a continuous section of f.*

Proof. It is immediate that g is a section of f. To show that it is continuous, let $y_i \to y$ be a convergent sequence in $T(Q)$. Set $x_i = g(y_i)$ and $x' = g(y)$ and suppose $x_i \not\to x'$. Pass to a subsequence so that x_i converges and let x be its limit. We will show that x, x', and x_i contradict the LNC condition on Q.

Let r be the smallest index such that $F_r(x) \neq F_r(x')$. Take $K = f^{-1}(y)$ and observe that both x and x' belong to K. Since $F_s(x) = F_s(x')$ for all $s < r$ and $K_n = \{x'\}$, it follows that K_{r-1} contains both x and x'. But K_r cannot contain them both, so $F_r(x) > F_r(x')$.

Now for all i set $K^i = f^{-1}(y_i)$, so $K^i_n = \{x_i\}$. Let $v = (x' - x)/2$ and suppose $x_i + v$ is in Q. Then $x_i + v$ is in K^i. Since $F_s(v) = 0$ for all $s < r$, it follows that $x_i + v$ is in K^i_r. But $F_r(x_i + v) < F_r(x_i)$, so provided that we insist on our original assumption that $x \neq x'$ we must then have $x_i + v \in K^i_r$ and $x_i \notin K^i_r$, contradicting the fact that $K^i_n = \{x_i\}$. So we must reject the assumption that $x_i + v$ is in Q.

Thus, we have $x_i \to x$ but $x_i + (x' - x)/2 \notin Q$ for any i. This contradicts the LNC property. We conclude that x_i must have converged to x', and this shows that g is continuous. □

It is easy to verify that the sections defined in Theorem 2.5 have the property that $g(y)$ is an extreme point of $f^{-1}(y)$, for any $y \in T(Q)$. If the range space is $\mathbb{R}^m$ and Q satisfies a simple geometric condition, this implies that $g(y)$ is actually an extreme point of Q. The condition is that Q should contain no face of dimension between 1 and m inclusive; one says that Q has *facial dimension* greater than m. (Of course, zero-dimensional faces — i.e., extreme points — cannot be forbidden.) For example, this will be true for all $m < n$ if Q is strictly convex.

Theorem 2.6. *Assume the setup of* Theorem 2.5 *and suppose that Q has facial dimension greater than m. Then the continuous section g treated in* Theorem 2.5 *takes values in the extreme points of Q. In particular, this will be true if Q is strictly convex and $m < n$.*

The same assertion holds for the construction in Theorem 2.3 *and* Corollary 2.4*, provided $0 \notin Q$.*

Proof. It is straightforward to verify that $g(y)$ is an extreme point of $T^{-1}(y) \cap Q$ for any $y \in T(Q)$. Then $g(y)$ is an extreme point of Q by [2, Theorem 1.6]. □

3. Finite-dimensional examples

We now list several elementary finite-dimensional examples and counterexamples.

A. Strictly convex sets. It is immediate from the definitions that every strictly convex set is LNC. (This is true in infinite dimensions as well.)

B. Sets in $\mathbb{R}^1$ and $\mathbb{R}^2$. Every convex set in $\mathbb{R}^1$ or $\mathbb{R}^2$ is LNC.

C. Polytopes. The following proposition is trivial.

Proposition 3.1. *Let Q and Q' be LNC sets. Then $Q \cap Q'$ is also LNC.*

As a consequence of this result and the easy observation that any half-space is LNC, it follows that convex polytopes in $\mathbb{R}^n$ are LNC. This provides us with a large class of LNC sets which are not strictly convex. Intersecting polytopes with strictly convex sets provides even more examples.

Note that the intersection of infinitely many LNC sets need not be LNC. Indeed, any closed convex set is an intersection of half-spaces, so this follows just from the fact that there exist closed convex sets which are not LNC. We give examples of such sets in Sections E and F below.

D. Images of compact LNC sets. Any linear image of a compact LNC set in $\mathbb{R}^n$ is also LNC.

Proposition 3.2. *Let Q be a compact LNC set in $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then $T(Q)$ is also a compact LNC set.*

Proof. $T(Q)$ is compact because T is continuous. To verify that $T(Q)$ is LNC, let $y, y', y_n \in T(Q)$ and suppose $y_n \to y$. Find $x', x_n \in Q$ such that $T(x') = y'$ and $T(x_n) = y_n$. By passing to a subsequence we may assume that (x_n) converges; letting $x = \lim x_n$, we have $T(x) = \lim T(x_n) = y$. Now $x_n + (x' - x)/2$ is eventually in Q because Q is LNC, and applying T shows that $y_n + (y' - y)/2$ is eventually in $T(Q)$. □

E. Cones.

Example 3.3. The simplest example of a compact, convex, non-LNC set in $\mathbb{R}^n$ is a right circular cone, explicitly given (for example) as the convex hull of the set $\{(1-\cos t, \sin t, 1) : 0 \le t \le 2\pi\}$ together with the origin in $\mathbb{R}^3$. The LNC condition is falsified by the points $z = (0,0,1)$, $z' = (0,0,0)$, and $z_n = (1 - \cos(1/n), \sin(1/n), 1)$. Taking $T : \mathbb{R}^3 \to \mathbb{R}^2$ to be the projection $T(x, y, z) = (x, y)$, the corresponding section g defined by any of the constructions in Section 2 is discontinuous. In every case, $g(0,0) = (0,0,0)$ while $g(1 - \cos t, \sin t) = (1 - \cos t, \sin t, 1)$ for $0 < t < 2\pi$.

It is worth noting that the restriction of T to the cone does have continuous right inverses, however; the simplest is the map $(x, y) \mapsto (x, y, 1)$. A closer analysis shows that in fact the restriction to the cone of any linear map T from $\mathbb{R}^3$ to $\mathbb{R}^2$ (indeed, to any $\mathbb{R}^m$) has a continuous section. The following is an example where no continuous sections exist.

Example 3.4. Let Q be the convex hull of the helix $\{(\cos t, \sin t, t) : 0 \le t \le 2\pi\}$ in $\mathbb{R}^3$ and consider the orthogonal projection $T : \mathbb{R}^3 \to \mathbb{R}^2$ onto the xy-plane. Then any right inverse g of $f = T|_Q$ must satisfy $g(\cos t, \sin t) = (\cos t, \sin t, t)$ for $0 < t < 2\pi$, and hence must have a discontinuity at $(1, 0)$.

Example 3.3 has the following generalization.

Proposition 3.5. *Let Q_0 be a compact, convex subset of $\mathbb{R}^n$ and let*

$$Q = \{(tx, t) : x \in Q_0, 0 \leq t \leq 1\}$$

be the suspension of Q_0 in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. Then Q is LNC if and only if Q_0 is a polytope.

Proof. If Q_0 is a polytope then so is Q, hence Q is LNC. Conversely, if Q_0 is not a polytope then it has infinitely many extreme points. (Any compact, convex set is the closed hull of its extreme points, so if it had only finitely many extreme points it would be a polytope.) Let (x_i) be a sequence of distinct extreme points of Q_0 which converges to a cluster point x. Then in Q we have $(x_i, 1) \to (x, 1)$; also $(0, 0)$ is in Q, so the LNC condition would require that $(x_i - x/2, 1/2)$ is eventually in Q. But this cannot be, because if $(y/2, 1/2) = (x_i - x/2, 1/2)$ is in Q then $y = 2x_i - x$ is in Q_0. Then $x_i = (y + x)/2$, contradicting the fact that x_i is extreme. $\square$

F. Matrix algebras. The unit balls of the most common finite-dimensional Banach spaces are either strictly convex or polytopes, and hence are LNC by reasons given above. However, finite-dimensional unit balls need not be LNC.

Proposition 3.6. *Neither the unit ball nor the positive part of the unit ball of the $n \times n$ matrix algebra (with operator norm or trace norm) is LNC for $n \geq 2$.*

Proof. Take $n = 2$. To show that the positive part of the unit ball is not LNC, define x' to be 0, $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and, for $t \in (0, 1)$, define

$$x_t = \begin{pmatrix} t & \sqrt{t - t^2} \\ \sqrt{t - t^2} & 1 - t \end{pmatrix}.$$

Then $x_t \to x$ as $t \to 1$, but for any $t \in (0, 1)$ the matrix $x_t + (x' - x)/2$ has negative determinant, and hence does not belong to the positive part of the unit ball.

Replacing each matrix A with $I - A$ in the above argument shows that the unit ball of the 2×2 matrix algebra is not LNC, and the same construction can be carried out in the upper left 2×2 block of any $n \times n$ matrix algebra. $\square$

G. Sets without one-dimensional faces. The failures of the LNC condition in the cone and matrix examples considered above all happen on one-dimensional faces of Q. However, this is not essential. We now describe a set in $\mathbb{R}^4$ which has no one-dimensional faces but still fails to be LNC.

Example 3.7. Let Q_0 be a cone in $\mathbb{R}^3$ and let I be a compact interval. Then $Q_0 \times I$, the Cartesian product which is contained in $\mathbb{R}^4$, is not LNC. In fact, for any $t \in I$ the intersection of $Q_0 \times I$ with $\mathbb{R}^3 \times \{t\}$ is the "slice" $Q_0 \times \{t\}$, which is isometric to Q_0, and hence is not LNC. By Proposition 3.1 we conclude that $Q_0 \times I$ cannot be LNC.

Now $Q_0 \times I$ has some one-dimensional faces: for any one-dimensional face F of Q_0, the sets $F \times \{a\}$ and $F \times \{b\}$ are one-dimensional faces of $Q_0 \times I$, where $I = [a, b]$. Also, for any extreme point x of Q_0, the set $\{x\} \times I$ is a one-dimensional face of $Q_0 \times I$. However, all of these faces can be "sliced off" to get a set with the desired properties.

For concreteness, let Q_0 be the convex hull of the point $(0, 0, 10)$ and the circle $(\cos t, \sin t, -10)$ and let I be the interval $[-10, 10]$. Let Q be the intersection of $Q_0 \times I$ with the 4-ball of radius 2 about the origin. This removes all of the one-dimensional faces of $Q_0 \times I$ — the sphere of radius 2 does not contain any lines, so there are no one-dimensional faces on the boundary, and inside the sphere Q is a part of $Q_0 \times I$ which has only two-dimensional faces. Furthermore, the intersection of Q with $\mathbb{R}^3 \times \{0\}$ is a truncation of Q_0 which is not LNC. So as before, it follows that Q is not LNC.

4. Closed LNC sets

The results in Section 2 all trivially have infinite-dimensional analogs in any TVS (just replace sequences by nets). However, this is not very interesting because infinite-dimensional compact LNC sets do not exist (Theorem 5.3). Thus, the consequences for compact sets in infinite dimensions are negative: there always exist linear maps whose restrictions are not open, and sections defined in various ways are not continuous in general.

But LNC sets which are merely closed are easy to construct in infinite dimensions (see the first paragraph of Section 5). And while the proofs are more difficult, we do have analogs of the results in Section 2, provided that T is left Fredholm, i.e. T is a bounded operator with closed range and finite-dimensional kernel. Of course, if the TVS X is already finite-dimensional then this is no restriction at all. Thus one could say that in finite dimensions closed LNC sets are practically as well-behaved as compact LNC sets.

Our fundamental tool is Lemma 4.2. In its proof we need to use Hilbert space techniques. This is irrelevant to the finite-dimensional case because without loss of generality we can always equip $\mathbb{R}^n$ with a Euclidean norm. We believe that in infinite dimensions, the Hilbert space condition on X can be weakened to X being a uniformly convex Banach space. This issue will be addressed in a future paper.

It is worth noting that if the closed LNC set in question is bounded, then the results in this section are easy variations on the results in Section 2. The idea is that if (x_n) is a bounded sequence in a Hilbert space and (Px_n) converges, where P is an orthogonal projection with finite-dimensional kernel, then we can pass to a subsequence so that $(P^\perp x_n)$ also converges, and then (x_n) converges because $x_n = Px_n + P^\perp x_n$. This technique is used in the proof of Theorem 4.7.

Our proof of Lemma 4.2 requires the following fundamental result from [8] and [9]. For the reader's convenience we include an easy proof here.

Theorem 4.1. *Let Q be an LNC set in a TVS and let $x, y \in Q$ be distinct. Then $p = \frac{1}{2}(x + y)$ has a relative neighborhood U in Q such that any $q \in U$ lies in the interior of a line segment which is contained in Q and parallel to $[x, y]$.*

Proof. Suppose the conclusion fails. Then there is a net $(q_t) \subset Q$ which converges to p such that no q_t is in the interior of a line segment which is contained in Q and parallel to $[x, y]$. Thus, setting $v = \frac{1}{4}(y - x)$, for each t the points $q_t - v$ and $q_t + v$ cannot both belong to Q. By passing to a subnet, without loss of generality we may suppose that

$q_t + v \notin Q$ for all t. Then the substitutions $x = p$, $x_t = q_t$, and $y = y$ contradict the fact that Q is LNC. □

In any Hilbert space (and hence in any finite-dimensional space) the assertion of Theorem 4.1 is actually equivalent to the LNC property [9, Theorem 2.2].

We use the notation $[X]_\epsilon$ for the closed ϵ-ball about the origin in X. We use the term "relative interior" to mean the interior of a convex set relative to the affine subspace of X that it spans.

Lemma 4.2. *Let X be a Hilbert space, let $Q \subset X$ be a closed LNC set, and let $P : X \to X$ be an orthogonal projection with finite-dimensional kernel. Suppose $0 \in Q$ and 0 is in the relative interior of $Q \cap \ker P$. Let $(x_i) \subset Q$ and suppose $P(x_i) \to 0$. Then $P'(x_i) \to 0$, $(P'(x_i))$ is eventually in Q, and $P' \geq P$, where P' is the orthogonal projection onto $(\mathrm{span}(Q \cap \ker P))^\perp$.*

Proof. Let $W = \mathrm{span}(Q \cap \ker P)$. By hypothesis, for every $v \in W$ there is a line segment in the direction of v which is contained in W and contains 0 in its interior. Fixing an orthonormal basis $\{v_1, \ldots, v_n\}$ of W, it follows from n applications of Theorem 4.1 that we can choose $\epsilon' > 0$ and $\delta' > 0$ such that for any $q \in Q \cap [X]_{\epsilon'}$ we have $[q + \delta' v_i, q - \delta' v_i] \subset Q$ for $1 \leq i \leq n$. Letting $\delta = \delta'/(\dim W)^{1/2}$, the ball of radius δ about q is contained in the convex hull of the line segments $[q + \delta' v_i, q - \delta' v_i]$, so this implies that for every $q \in Q \cap [X]_{\epsilon'}$ we have $q + [W]_\delta \subset Q$. Let $\epsilon = \min(\epsilon', 1)$.

Let $Q' = Q \cap [X]_\epsilon$ and define H to be the intersection of all closed half-spaces H_β in X such that $Q \subset H_\beta$ and $Q' \cap \partial H_\beta \neq \emptyset$. If $p \notin Q$ and $\|p\| \leq \epsilon$, we claim that there exists an H_β which excludes p. To see this, let q be the unique element of Q such that $\|q - p\|$ is minimized [7, Theorem 12.3]. Notice that if $\|q\| > \|p\|$ then $q' = (\langle p, q\rangle/\|q\|^2)q$ is in Q' (since $\|q'\| \leq \|p\| \leq \epsilon$, hence $\langle p, q\rangle/\|q\|^2 \leq \epsilon$, and $0, q \in Q$). Also $\|q' - p\| < \|q - p\|$ since q' is the projection of p onto the one-dimensional subspace $[q]$; this contradicts the minimality of $\|q - p\|$, and so we must have $\|q\| \leq \|p\|$. Thus $q \in Q'$. Now define

$$\begin{aligned} H_\beta &= \{x \in X : \langle x - p, q - p\rangle \geq \langle q - p, q - p\rangle\} \\ &= \{x \in X : \langle x, q - p\rangle \geq \langle q, q - p\rangle\}. \end{aligned}$$

We have $Q \subset H_\beta$. To see this, let $x \in Q$. Since q is also in Q, so is $tx + (1 - t)q$ for any t in [0,1]. Since q minimizes distance to p, we therefore have

$$\|t(x - q) + q - p\| = \|tx + (1 - t)q - p\| \geq \|q - p\|$$

for all t in [0,1]. Define $f(t) = \|t(x - q) + q - p\|^2$. By the above we must have $f'(0) \geq 0$. But $f'(t) = 2t\|x - q\|^2 + 2\langle x - q, q - p\rangle$. So $f'(0) = 2\langle x - q, q - p\rangle \geq 0$. That is, $\langle x, q - p\rangle \geq \langle q, q - p\rangle$. Thus x is in H_β, and we have shown $Q \subset H_\beta$. It is straightforward to verify that $q \in Q' \cap \partial H_\beta$ and $p \notin H_\beta$. This proves the claim.

Thus $Q' = H \cap [X]_\epsilon$. Now each ∂H_β contains an element q of Q', and such an element is in the relative interior of $Q \cap (q + W)$. Therefore each ∂H_β contains a translate of W and from this it follows that $H = H + W$. Also, $H \cap \ker P = W$: for any $p \in \ker P$, $p \notin W$, $\|p\| \leq \min(\epsilon, \delta)$, we must have $p \notin Q$ (since $p \notin W$), therefore $p \notin Q'$, and this implies that $p \notin H$ because $Q' = H \cap [X]_\epsilon$. This proves that $H \cap \ker P \subset W$; the reverse inclusion is trivial.

Now let P' be the orthogonal projection of X onto $W^{\perp}$. Note immediately that $P' \geq P$, that is, $P \circ P' = P' \circ P = P$ because $W \subset \ker P$. If $x_i \to 0$ we are done, because then $\|P'(x_i)\| \leq \|x_i\| \to 0$ and $\|x_i\| \leq \min(\epsilon, \delta)$ implies that $x_i \in Q'$ and $\|x_i - P'(x_i)\| \leq \delta$, hence $P'(x_i) \in Q$. Otherwise, pass to a subsequence so that $\|x_i\| \geq \gamma > 0$ for all i. Let $y_i = (P' - P)(\gamma_i x_i)$ where $\gamma_i = \min(\epsilon/\|x_i\|, 1)$ (and thus $\gamma_i \leq \epsilon/\gamma$). Observe that $P(y_i) = 0$ and $P'(y_i) = y_i$. Pass to another subsequence to ensure that (y_i) converges to some point y (this can be done because $\|y_i\| \leq \epsilon$ for all i and $\ker P$ is finite-dimensional) and note that $P'(\gamma_i x_i) \to y$ since $P(\gamma_i x_i) \to 0$. Since $\gamma_i x_i \in Q' \subset H$ for all i, we also have $P'(\gamma_i x_i) \in H + W = H$. It follows that $y \in H$ as well. Also, $P(y_i) = 0$ implies that $P(y) = 0$, so by the result of the last paragraph we have $y \in H \cap \ker P = W$. Yet $P'(y_i) = y_i$ implies $P'(y) = y$, so $y \in W^{\perp}$, and therefore $y = 0$. Therefore $P'(\gamma_i x_i) \to 0$, and this implies that $P'(x_i) \to 0$.

Finally, since $P'(x_i) \to 0$ this sequence eventually lies in $[X]_\epsilon$. It also belongs to H because $x_i \in Q \subset H$ and $H + W = H$. Since $Q' = H \cap [X]_\epsilon$, we conclude that $P'(x_i) \in Q'$ eventually. In particular, $P'(x_i)$ is eventually in Q. □

Now we proceed to the promised analogs of the results in Section 2.

Theorem 4.3. *Let Q be a closed, convex subset of a Hilbert space X. The following conditions are equivalent:*

(i) *Q is LNC;*
(ii) *for every Hilbert space Y and every left Fredholm map $T : X \to Y$, the restriction map $f = T|_Q : Q \to T(Q)$ is open.*

Proof. (i) $\Rightarrow$ (ii). Suppose Q is LNC and let $T : X \to Y$ be left Fredholm. Let P be the orthogonal projection with $\ker P = \ker T$. Then $T = T' \circ P$ where T' is a linear homeomorphism from $\operatorname{ran} P$ onto $\operatorname{ran} T$. So we only need to show that $P|_Q : Q \to P(Q)$ is open.

Suppose $P|_Q$ is not open; then there exist $x \in Q$, $(x_i) \subset Q$, and a relatively open set $U \subset Q$ containing x such that $P(x_i) \to P(x)$ but $P(x_i) \notin P(U)$ for all i. Let $V = \ker P$; then $(x + V) \cap Q$ is a closed convex set, and there exist points in its relative interior arbitrarily close to x. Thus we can find a point x' belonging to the intersection of U with the relative interior of $(x + V) \cap Q$. Note that $x' - x \in V$, so $P(x') = P(x)$. Thus, replacing x with x', we may assume that x belongs to the relative interior of $(x + V) \cap Q$. Translating Q by x, we can further assume that $x = 0$. The hypotheses of Lemma 4.2 are now satisfied, so we have $P'(x_i) \to 0$ and $P'(x_i) \in Q$ eventually. Therefore $P'(x_i) \in U$ eventually, so that $P(x_i) = P(P'(x_i)) \in P(U)$ eventually, contradicting the choice of U. So $P|_Q$ must be open.

(ii) $\Rightarrow$ (i). The proof of Theorem 2.1 (ii) $\Rightarrow$ (i) works for any Hilbert space X in place of $\mathbb{R}^n$, modulo one minor modification. Instead of setting $U = \{y \in Q : \phi(y) > 3/4\}$, define $U = \{y \in Q : 5/4 > \phi(y) > 3/4\}$. Then regardless of whether Q is compact, we can pass to a subsequence of (z_i) so that $\phi(z_i)$ converges, and this is sufficient to complete the proof. □

For any nonzero $v \in X$, let T_v be the orthogonal projection onto the orthocomplement of v. Let Q be a closed, convex subset of X and let $f_v = T_v|_Q : Q \to T_v(Q)$. For any

$x \in T_v(Q)$, the set $\{b \in \mathbb{R} : x + bv \in Q\}$ is a closed interval in $\mathbb{R}$ and so, although it may not have a smallest element, it does contain a unique element of minimal absolute value. Let a be this number, and define $g_v(x) = x + av$. This is a slightly modified version of the map g_v defined in Section 2 which is necessary if Q is closed and unbounded. The original definition would work if Q were assumed to be bounded, but it is ill-defined in general because $\inf\{b \in \mathbb{R} : x + bv \in Q\}$ may not exist. The disadvantage of our new definition of g_v is that if Q is translated in X the corresponding g_v may not be a translate of the original g_v.

Proposition 4.4. *Let Q be a closed, convex subset of a Hilbert space X. Then Q is LNC if and only if $g_v : T_v(Q') \to Q'$ is continuous for all nonzero $v \in X$ and all translates Q' of Q, where g_v and T_v are as defined above.*

Proof. We can assume $Q' = Q$ in the forward direction of the proof. Thus, suppose Q is LNC and g_v is not continuous for some $v \in X$. Then there is a sequence (y_i) in $T_v(Q)$ and an element z in Q such that $y_i \to f_v(z) = y$ but $g_v(y_i) \not\to g_v(y)$. We may assume that z is in the relative interior of $(z + \ker T_v) \cap Q$.

Let $x_i = g_v(y_i)$. Applying Lemma 4.2 to the sequence $(x_i - z)$, the LNC set $Q - z$, and the projection $P = T_v$, we conclude that $P'(x_i) \to P'(z)$ and $P'(x_i) \in Q$ eventually. But since $\ker T_v$ is one-dimensional and $P' \geq P$, either P' is the identity operator or $P' = P$. In the latter case, $y_i = P'(x_i) \in Q$ eventually, so that $g_v(y_i) = y_i$ eventually and $y \in Q$, hence $g_v(y) = y$, and therefore $g_v(y_i) \to g_v(y)$, contradicting the choice of (y_i) and y. So we can assume that P' is the identity operator and so $x_i \to z$. Since P' is the identity, it follows that $(z + \ker T_v) \cap Q = \{z\}$, so we must have $g_v(y) = z$. Therefore $g_v(y_i) = x_i \to z = g_v(y)$, again contradicting the choice of (y_i) and y. We conclude that if Q is LNC then g_v is continuous.

Conversely, suppose Q is not LNC and find $x, x', x_i \in Q$ such that $x_i \to x$ but $x_i + (x' - x)/2$ is not in Q for any i. Let $Q' = Q - x'$ and define $z = v = x - x'$, $z' = 0$, and $z_i = x_i - x'$. Then define $y_i = f_v(z_i)$ and $y = 0$. Since $z_i \to z = v$, it follows that $y_i = T_v(z_i) \to 0 = y$. Also $g_v(y) = 0$, whereas $g_v(y_i) = z_i + a_i v$ where $a_i > -1/2$ since $z_i \in Q'$ but $z_i - v/2 = x_i + (x' - x)/2 - x' \notin Q'$. But convergence of $g_v(y_i)$ to $0 = z - v$, together with the fact that $z_i \to z$, would imply that $a_i \to -1$, a contradiction. Thus $g_v(y_i) \not\to g_v(y)$, and so g_v is not continuous. □

Theorem 4.5. *Let Q be a closed LNC subset of a Hilbert space X and let T be a left Fredholm map from X into another Hilbert space Y. Let $f = T|_Q : Q \to T(Q)$. Then for each $y \in T(Q)$ there is a unique element $g(y)$ of $f^{-1}(y) = T^{-1}(y) \cap Q$ with minimal norm, and the map g defined in this way is a continuous section of f.*

Proof. As in the proof of Theorem 4.3 (i) $\Rightarrow$ (ii), we may assume that T is the orthogonal projection P of X onto a subspace.

Existence and uniqueness of $g(y)$ is Theorem 12.3 of [7]. To verify continuity, let (y_i) be a sequence in $T(Q)$ which converges to $y \in T(Q)$ and suppose $(x_i) = (g(y_i))$ fails to converge to $x' = g(y)$. Fix $\epsilon > 0$ and find x in the relative interior of $(x' + \ker T) \cap Q$ such that $\|x' - x\| \leq \epsilon$. Then apply Lemma 4.2 to $Q' = Q - x$ and the sequence $(x_i') = (x_i - x)$; we get $P'(x_i) \to P'(x)$ and $P'(x_i - x) \in Q'$ eventually, hence $w_i = P'(x_i) + x - P'(x) \in Q$ eventually. Note that $w_i \to x$ and $T(w_i) = y_i$, so $\|w_i\| \geq \|x_i\|$ eventually. Since (w_i) is convergent and hence bounded, we can pass to a subsequence to ensure that the norms $\|x_i\|$ converge. Then

$$\|x'\| + \epsilon \geq \|x\| = \lim \|w_i\| \geq \lim \|x_i\|.$$

But Q is closed and convex, hence it is weakly closed, so there is a weak cluster point $x'' \in Q$ of (x_i); and

$$T(x'') = \lim T(x_i) = \lim y_i = y.$$

So we also have

$$\|x'\| \leq \|x''\| \leq \lim \|x_i\|,$$

which together with the above (and the fact that ϵ is arbitrary) establishes that $\|x'\| = \lim \|x_i\|$.

Again fix $\epsilon > 0$ and, as in the last paragraph, find a sequence (w_i) which is eventually in Q, satisfies $T(w_i) = y_i$, and converges to $x \in Q$ where $T(x) = y$ and $\|x' - x\| \leq \epsilon$. Pass to a subsequence so that $w_i \in Q$, $\|w_i - x\| \leq \epsilon$, and $|\,\|x_i\| - \|x'\|\,| \leq \epsilon$ for all i. Then $\|w_i\| \geq \|x_i\|$ and

$$\|w_i\| \leq \|x\| + \epsilon \leq \|x'\| + 2\epsilon \leq \|x_i\| + 3\epsilon$$

for all i. Also, since both w_i and x_i are in the convex set $Q \cap T^{-1}(y_i)$, and x_i is the unique element of this set with minimal norm, it follows that $\langle w_i, x_i \rangle \geq \|x_i\|^2$. Therefore

$$\begin{aligned}\|w_i - x_i\|^2 &= \|w_i\|^2 - 2\langle w_i, x_i\rangle + \|x_i\|^2 \\ &\leq (\|x_i\| + 3\epsilon)^2 - 2\|x_i\|^2 + \|x_i\|^2 \\ &= 3\epsilon(2\|x_i\| + 3\epsilon) \\ &\leq 3\epsilon(2\|x'\| + 5\epsilon)\end{aligned}$$

eventually. Thus, by choosing ϵ sufficiently small we can ensure that x_i and w_i are eventually arbitrarily close and simultaneously that $x = \lim w_i$ is arbitrarily close to x'. This implies that (x_i) converges to x'. Since we passed to a subsequence, we have really shown that every subsequence of the original sequence (x_i) has a subsequence which converges to x'. But this implies that the original sequence (x_i) converges to x', contradicting our assumption that g is not continuous. Therefore g is continuous. □

Let $T : X \to Y$ be a left Fredholm map. Let $V = \ker T$ and choose a separating family of linear functionals $\{F_r : r = 1, \ldots, n\}$ on V. Given a compact set K in V, define the sets K_r $(0 \leq r \leq n)$ and the element $\Gamma(K)$ as in Section 2, just before Theorem 2.5. For any parallel affine subspace V', write $V' = V + v$ with $v \perp V$, and for $K \subset V'$ compact define $K_r = (K - v)_r + v$ and $\Gamma(K) = \Gamma(K - v) + v$.

(Actually, the functionals F_r can be chosen independently of T; let $\{F_r\}$ be any well-ordered separating set of linear functionals on X and use them to construct $\Gamma(K)$. In effect we are using the finite set $\{F_{k_i}\}$, $1 \leq i \leq \dim V$, where $k_1 = 1$ and k_{i+1} is the first index such that $\{F_{k_1}|_V, \ldots, F_{k_{i+1}}|_V\}$ is linearly independent.)

Before proving the next result, we give a counterexample which shows that in contrast to previous results in this section, using Γ to define a continuous section of $T|_Q$ requires that Q be bounded. This is true even in finite dimensions. Of course, there is an immediate problem in the construction of K_r if K is unbounded because then there may be no "lowest slice"; even if $a = \inf F_r(K_{r-1})$ is finite, $K_{r-1} \cap F_r^{-1}(a)$ may be empty. Moreover, even if we can define a section using this procedure it need not be continuous.

Example 4.6. Define

$$Q = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, x + y \leq 1, \text{ and } z \geq \frac{(1-y)^3}{x}\},$$

where we set $(1-y)^3/x = 0$ when $x = 0$ and $y = 1$. Define $T : \mathbb{R}^3 \to \mathbb{R}$ by $T(x, y, z) = x$. One can verify that the lower boundary of Q is strictly convex, by checking that the Hessian of the function $f(x, y) = (1-y)^3/x$ is strictly positive-definite. It follows from this that Q is LNC. But $T(Q) = [0, 1]$, and if we take $F_1(y, z) = y$ and $F_2(y, z) = z$ then we have $\Gamma(f^{-1}(x)) = (x, 0, 1/x)$ for $x \in (0, 1]$ and $\Gamma(f^{-1}(0)) = (0, 1, 0)$. Thus Γ is not continuous.

Theorem 4.7. *Let Q be a closed, bounded LNC subset of a Hilbert space X, let $T : X \to Y$ be left Fredholm, and let $f = T|_Q : Q \to T(Q)$. Then $g(y) = \Gamma(f^{-1}(y))$ defines a continuous section of f.*

Proof. Observe that $f^{-1}(y) = T^{-1}(y) \cap Q$ is the intersection of a finite-dimensional affine subspace of X with a closed, bounded set, so it is compact, and therefore $\Gamma(f^{-1}(y))$ is defined for all $y \in T(Q)$. As usual, we may assume that $T = P$ is an orthogonal projection with finite-dimensional kernel. Now let $(y_i) \to y$ be a convergent sequence in $T(Q)$. Set $x_i = g(y_i)$ and $x' = g(y)$. Since $x_i = P(x_i) + P^\perp(x_i)$ and $\ker P = \operatorname{ran} P^\perp$ is finite-dimensional, we can pass to a subsequence so that $P^\perp(x_i)$ converges; then x_i also converges, say to x. Note that $T(x) = \lim T(x_i) = y = T(x')$. We will show that x, x', and x_i contradict the LNC condition on Q unless $x = x'$, and therefore g must be continuous.

Thus suppose $x \neq x'$, and let r be the smallest index such that $F_r(x) \neq F_r(x')$. Take $K = f^{-1}(y)$. Since $F_s(x) = F_s(x')$ for all $s < r$ and $K_n = \{x'\}$, it follows that K_{r-1} contains both x and x'. But K_r cannot contain them both, so $F_r(x) > F_r(x')$.

Now for all i set $K^i = f^{-1}(y_i)$, so $K^i_n = \{x_i\}$. Let $v = (x' - x)/2$ and suppose $x_i + v$ is in Q. Then $x_i + v$ is in K^i. Since $F_s(v) = 0$ for all $s < r$, it follows that $x_i + v$ is in K^i_r. But $F_r(x_i + v) < F_r(x_i)$, so provided that we insist on our original assumption that $x \neq x'$ we must then have $x_i + v \in K^i_r$ and $x_i \notin K^i_r$, contradicting the fact that $K^i_n = \{x_i\}$. So we must reject the assumption that $x_i + v$ is in Q.

Thus, we have $x_i \to x$ but $x_i + (x' - x)/2 \notin Q$ for any i. This contradicts the LNC property. We conclude that x_i must have converged to x', and this shows that g is continuous. □

Throughout this section we have insisted that T have closed range and finite-dimensional kernel. Are these assumptions necessary? The first certainly is: let Q be the unit ball of $l^2 = l^2(\mathbb{N})$ and define $T : l^2 \to l^2$ by $T((a_n)) = (a_n/(n+1))$. Then T has null kernel but does not have closed range. There is only one right inverse $g : T(Q) \to Q$ of $f = T|_Q$, and it is not continuous.

We believe that finite-dimensionality of $\ker T$ is also necessary in general, but we have not found a counterexample. The most extreme counterexample would involve a bounded, strictly convex set Q and a map T with cofinite-dimensional kernel, but even this case remains open. We pose it as a problem.

Open problem. Let Q be a closed, bounded, strictly convex subset of a Hilbert space X and let $T : X \to \mathbb{R}^n$ be a bounded linear map. Is $T|_Q : Q \to T(Q)$ necessarily open?

5. Infinite-dimensional counterexamples.

Infinite-dimensional LNC sets are abundant. As in the finite-dimensional case, strictly convex sets and half-spaces are always LNC, and so are finite intersections and Cartesian products of these sets. However, it is difficult to find other examples than these (the unit ball of c_0 is an example [9, Theorem 1.1]).

A. Compact sets in infinite dimensions. We now prove that no infinite-dimensional compact convex set is LNC. We wish to thank Stephen Simons for essential contributions to our original proof of this result.

Lemma 5.1. *Let Q be the unit ball of any infinite-dimensional dual Banach space X^*. With respect to the weak* topology, Q is not LNC.*

Proof. First, find a sequence of elements $x_n \in X$ and $f_n \in X^*$ with the properties $\|x_n\| = \|f_n\| = 1$, $f_n(x_n) = 1$, and $f_n(x_k) = 0$ if $k < n$. This can be done inductively. The base step $n = 1$ is trivial. For $n = k + 1$ let Y be the span of $x_1, \ldots, x_k$; find $x_{k+1} \in X$ such that $\|x_{k+1}\| = \|x_{k+1}/Y\| = 1$ (this is possible because Y is finite-dimensional); find $f'_{k+1} \in (X/Y)^*$ such that $\|f'_{k+1}\| = f'_{k+1}(x_{k+1}/Y) = 1$; and finally let f_{k+1} be the composition of f'_{k+1} with the natural projection of X onto X/Y. The induction can then proceed.

Next, make $\mathbb{N}$ into a graph by putting an edge between k and n, $k < n$, if $f_k(x_n) \geq 0$. By infinite Ramsey theory [4, Lemma 29.1] there is an infinite subgraph which is either complete or anti-complete. In other words, by passing to a subsequence we can ensure that either $f_k(x_n) \geq 0$ for all $k < n$ or $f_k(x_n) < 0$ for all $k < n$. In the first case define $f' = \sum f_k/2^k$ and in the second case inductively define $a_1 = -1/2$ and

$$a_{n+1} = -\min(2^{-(n+1)}, \frac{1}{2}\sum_{k=1}^{n} a_k f_k(x_{n+1})),$$

and set $f' = \sum a_k f_k$. It can be seen inductively that each a_k is negative, and therefore $\sum_1^{n-1} a_k f_k(x_n) > 0$ automatically, and a_n is chosen to be small enough that also $\sum_1^{n} a_k f_k(x_n) > 0$. Thus, in either case, for any $n > 1$ we have $f'(x_n) > 0$.

Let f be a weak* cluster point of the sequence (f_n). Then some subnet of (f_n) converges to f, and f_n, f, and f' all belong to the unit ball of X^*. However, for any n we have

$$(f_n + (f' - f)/2)(x_n) = 1 + f'(x_n)/2 > 1,$$

and since $\|x_n\| = 1$ this implies that $\|f_n + (f' - f)/2\| > 1$, i.e. $f_n + (f' - f)/2$ does not belong to the unit ball. Thus, the unit ball of X^* is not LNC. □

Lemma 5.2. *Every compact, convex, symmetric subset of a LCS X is linearly homeomorphic to the unit ball of some dual Banach space, equipped with the weak* topology.*

Proof. Let $K \subset X$ be such a set and without loss of generality suppose K spans X. Let V denote the space of linear functionals on X whose restriction to K is continuous, and give V the sup norm it inherits as a subspace of $C(K)$. Since V is a closed subspace of $C(K)$, it is a Banach space.

Consider the map $T : K \to [V^*]_1$ from K into the unit ball of the Banach space dual of V, defined by $(Tx)(v) = v(x)$. It is easy to check that this map is continuous going into the weak* topology on V^* and that it is the restriction of a linear map from X to V^*. It is also 1-1 since local convexity of X implies that the continuous linear functionals on X separate points [5, Corollary 1.2.11]. Thus K is linearly homeomorphic to a weak* compact convex subset of $[V^*]_1$.

Now suppose $y \in V^*$ is not in the image of K. Then by a standard separation theorem [5, Theorem 1.2.10] there exists a weak* continuous linear functional F on V^* such that

$$F(Tx) \le 1 < F(y)$$

for all $x \in K$. Since F is weak* continuous, it is again standard [5, Proposition 1.3.5] that there then exists $v \in V$ such that $z(v) = F(z)$ for all $z \in V^*$, so we have

$$v(x) \le 1 < y(v)$$

for all $x \in K$. Since K is symmetric and v is linear, this implies that $\|v\| \le 1$, hence $\|y\| > 1$. We conclude that T maps K onto $[V^*]_1$, which completes the proof. □

Theorem 5.3. *Let Q be a compact, convex, infinite-dimensional set in a locally convex TVS X. Then Q is not LNC.*

Proof. Let Q be a compact, convex, infinite-dimensional set in a locally convex TVS X, and suppose Q is LNC. Let $Q' = Q - Q$. Then Q' is the image of $Q \times Q$ under the map $T : X \times X \to X$ given by $T(x, y) = x - y$. The map T is continuous and linear, so Q' is compact and convex. Also, it is easy to see that $Q \times Q$ is LNC, and a trivial variation on Proposition 3.2 (replacing sequences with nets) therefore implies that Q' is also LNC. So Q' is a compact, convex, infinite-dimensional, symmetric subset of X. By Lemma 5.2 Q' is linearly homeomorphic to the unit ball of some infinite-dimensional dual Banach space. But this contradicts Lemma 5.1. We conclude that Q cannot be LNC. □

Corollary 5.4. *Let Q be a compact, convex, infinite-dimensional set in a locally convex TVS E. Then there is a nonzero vector $v \in E$ such that the quotient map $T : E \to E/[v]$ restricts to a non-open map $T|_Q : Q \to T(Q)$ and the section g_v defined as in* Section 2 *is not continuous.*

Proof. Identical to the proofs of Theorem 2.1 (ii) $\Rightarrow$ (i) and the reverse direction of Proposition 2.2. □

B. Images of closed LNC sets. The next result shows that under familiar hypotheses a continuous linear image of a closed LNC set is LNC. In general such an image need not be closed, so in most applications this will have to be checked separately.

Theorem 5.5. *Let Q be a closed LNC subset of a Hilbert space X and let $T : X \to Y$ be a bounded linear map into another Hilbert space. Suppose T is left Fredholm. Then $T(Q)$ is LNC.*

Proof. Let $y, y', y_n \in T(Q)$ and suppose $y_n \to y$. Fix $x, x', x_n \in Q$ such that $T(x) = y$, $T(x') = y'$, and $T(x_n) = y_n$. Without loss of generality, we may assume that x belongs to the relative interior of $T^{-1}(y) \cap Q$; translating by x, we may further assume that $x = 0$. As in the proof of Theorem 4.3, we may also assume that $T = P$ is an orthogonal projection with finite-dimensional kernel.

Since $x = 0$ we also have $y = P(x) = 0$, and hence $P(x_n) = y_n \to 0$. So the hypotheses of Lemma 4.2 are satisfied, and we conclude that $P'(x_n) \to 0$ and $P'(x_n) \in Q$ eventually. Thus, since Q is LNC we must have $P'(x_n) + x'/2 \in Q$ eventually; applying P yields $y_n + y'/2 \in T(Q)$ eventually. This verifies that $T(Q)$ is LNC. □

Even in finite dimensions, one can show that if Q is not closed then $T(Q)$ need not be LNC even if Q is LNC. However, any example of such a Q must be at least four-dimensional, so the verification is bound to be tedious. Therefore we simply record an example without proving that it has the desired properties.

Example 5.6. Define

$$C = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} < z < 1\} \cup \{(0, 0, 0)\}$$

and

$$Q = \{(\lambda x, \lambda y, \lambda z, 1 - \lambda) \in \mathbb{R}^4 : (x, y, z) \in C \text{ and } \lambda \in [0, 1)\}.$$

Then Q is LNC but its image under the map $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(x, y, z, w) = (x + w, y, z + w)$ is not.

If Q is convex and open, however, then it is LNC and so is its image under any bounded linear map T with closed range. For T is an open map by the open mapping theorem, and so $T(Q)$ is also convex and open, and hence LNC.

C. Unit balls of L^p spaces. For $1 < p < \infty$, the unit ball of any L^p space is strictly convex (with respect to the norm topology) and hence LNC. However, the positive part of the unit ball is not:

Proposition 5.7. *The positive part of the unit ball of any infinite-dimensional L^p space ($1 \leq p \leq \infty$) is not LNC.*

Proof. Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of disjoint positive-measure sets. Define x' to be the origin of the L^p space and define x by $x|_{A_n} = \mu(A_n)^{-1/p} 2^{-n}$ and $x = 0$ off of $\bigcup A_n$. For each $k \in \mathbb{N}$ define

$$x_k|_{A_n} = \begin{cases} \mu(A_n)^{-1/p} 2^{-n} & \text{if } n \neq k \\ 0 & \text{if } n = k \end{cases}$$

and set $x_k = 0$ off of $\bigcup A_n$. Then $x_k \to x$, but $(x_k + (x' - x)/2)|_{A_k} = -\mu(A_n)^{-1/p} 2^{-k-1} < 0$. Thus $x_k + (x' - x)/2$ does not lie in the positive part of the unit ball. □

Likewise for the unit ball of any infinite-dimensional L^1 space:

Proposition 5.8. *The unit ball of any infinite-dimensional L^1 space is not LNC.*

Proof. Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of disjoint positive-measure sets. Define $x|_{A_n} = \mu(A_n)^{-1}2^{-n}$ and

$$x'|_{A_n} = \begin{cases} \mu(A_1)^{-1} & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Also define

$$x_k|_{A_n} = \begin{cases} \mu(A_n)^{-1}2^{-n} & \text{if } 1 \leq n < k \\ \mu(A_k)^{-1}2^{-k+1} & \text{if } n = k \\ 0 & \text{if } n > k. \end{cases}$$

A short computation shows that $x_k \to x$ but $\|x_k + (x' - x)/2\|_1 = 1 + 2^{-k} > 1$. □

The corresponding fact for L^∞ spaces will be given in Corollary 5.10.

D. Unit balls of $C(K)$ spaces. For these spaces, unit balls and their positive parts can be treated simultaneously.

Proposition 5.9. *Let K be a compact Hausdorff space and suppose $C(K)$ is infinite-dimensional. Then neither the unit ball nor the positive part of the unit ball of $C(K)$ is LNC. For K locally compact and $C_0(K)$ infinite-dimensional, the positive part of the unit ball of $C_0(K)$ is not LNC.*

Proof. First consider the case that K is locally compact. Let (U_n) be a sequence of disjoint, nonvoid open subsets of K. For each n let f_n be a Urysohn function supported in U_n. Define a function $g = \sum f_n/2^n$. Define $g_k = \sum_{n \neq k} f_n/2^n$ and $g' = 0$. Then for x in U_k such that $f_k(x) = 1$,

$$g_k(x) + (g'(x) - g(x))/2 = -2^{-k-1} < 0.$$

This shows that the positive part of the unit ball of $C_0(K)$ is not LNC.

Specializing to the compact case, we find that the positive part of the unit ball of $C(K)$ is not LNC; taking $h = 1 - g$, $h_k = 1 - g_k$, and $h' = 1 - g'$ verifies that the entire unit ball is not LNC. □

Corollary 5.10. *The unit ball of any infinite-dimensional L^∞ space is not LNC.*

Corollary 5.11. *The positive part of the unit ball of any infinite-dimensional C*-algebra is not LNC; if the C*-algebra is unital, its unit ball is also not LNC.*

Proof. By [1, p. 314] every infinite-dimensional C*-algebra $\mathcal{A}$ contains an infinite-dimensional abelian subalgebra, i.e. a copy of some $C_0(K)$. The positive part of the unit ball of $C_0(K)$ is the intersection of the positive part of the unit ball of $\mathcal{A}$ with the subalgebra. Since the intersection of two LNC sets is LNC, and the subalgebra is clearly LNC because it is a subspace, it follows that the positive part of the unit ball of $\mathcal{A}$ cannot be LNC. Likewise for the unit ball of $\mathcal{A}$ if $\mathcal{A}$ has a unit. □

The unit ball of the sequence space c_0 is LNC [9, Theorem 1.1], so Proposition 5.9 is sharp.

6. Zonoids and related sets.

A *zonoid* is the range of a nonatomic vector-valued measure. By a well-known theorem of Lyapunov (see e.g. [2]), every zonoid is compact and convex. Equivalently, zonoids are those sets which arise as images of the positive part of the unit ball of $L^\infty[0,1]$ under weak*-continuous linear maps into $\mathbb{R}^n$.

In this section we will prove that every zonoid is LNC. This is our most sophisticated construction of finite-dimensional sets with the LNC property, and it actually can be applied to a somewhat broader class of sets than zonoids. The basic theorem is the following.

Theorem 6.1. *Let Q be a compact, convex subset of a real TVS X. Suppose that for any closed face F of Q there are compact, convex sets $A, B \subset X$ such that $Q = A + B$ and A is a translate of F. Then the image of Q under any continuous map $T : X \to \mathbb{R}^n$ is LNC.*

Proof. Fix $z, z', z_n \in T(Q)$ such that $z_n \to z$ and let F' be the smallest face of $T(Q)$ which contains z and z'. Now $F = T^{-1}(F') \cap Q$ is a closed face of Q, so by hypothesis there exist compact, convex sets $A, B \subset X$ and $w \in X$ such that $Q = A + B$ and $F = A + w$.

We claim that $a \in A$, $b \in B$, $a+b \in F$ implies $b = w$. To prove this let $f : \mathbb{R}^n \to \mathbb{R}$ be any linear functional and α any real number such that $F' \subset f^{-1}(\alpha)$. Then $F \subset (f \circ T)^{-1}(\alpha)$ and

$$f(T(a)) = f(T(a + w)) - f(T(w)) = \alpha - \beta$$

for any $a \in A$, where $\beta = f(T(w))$. Therefore, if $a \in A$, $b \in B$, and $a + b \in F$ then

$$f(T(b)) = f(T(a + b)) - f(T(a)) = \beta,$$

and hence $f(T(A + b)) = \alpha$, so that $T(A + b) \subset f^{-1}(\alpha)$. Now since F' is a face of $T(Q)$, for any $u \in T(Q) - F'$ there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ and a real number α such that $F' \subset f^{-1}(\alpha)$ and $u \notin f^{-1}(\alpha)$. Therefore, by the preceding, $u \notin T(A + b)$, and we conclude that $T(A+b) \subset F'$. That is, $A+b \subset F = A+w$. But a compact set cannot contain a nonzero translate of itself, so $b = w$ as claimed.

For each n write $z_n = x_n + y_n$ where $x_n \in T(A)$ and $y_n \in T(B)$, and pass to a subsequence so that (x_n) and (y_n) converge, $x_n \to x$ and $y_n \to y$. Thus $z = x + y$. By the previous paragraph, this implies that $y = T(w)$. Hence $T(A) = F' - y$.

Set $v = (z' - z)/2$. Since F' is the smallest face which contains z and z', the point $z + v$ must belong to the interior of F'. Subtracting y, this shows that $x + v$ is in the interior of $T(A)$. Therefore $x_n + v \in T(A)$ for sufficiently large n, since the convexity notion of interior coincides with the topological notion inside the span of $T(A)$. Thus

$$z_n + v = x_n + y_n + v \in T(A) + T(B) = T(Q)$$

for sufficiently large n, as desired. $\square$

Corollary 6.2. *Let K be a compact, strictly convex subset of $\mathbb{R}^n$, let Ω be a σ-finite measure space, and let $Q = L^\infty(\Omega; K) \subset L^\infty(\Omega; \mathbb{R}^n)$. Then any finite-dimensional weak* continuous linear image of Q is LNC.*

Proof. By Theorem 6.1 it will suffice to show that Q has the decomposability property described there. We claim that for any weak* closed face F of Q there is a subset $\Omega' \subset \Omega$ and a measurable function f from Ω' into the boundary of K, such that F consists of precisely those functions in Q which agree with f almost everywhere on Ω'. From this claim it immediately follows that $Q = A + B$ where

$$A = L^\infty(\Omega - \Omega'; K)$$

is a translate of F (namely, $F = A + f$) and

$$B = L^\infty(\Omega'; K),$$

which verifies the needed decomposition property.

To prove the structure theorem, for each $g \in F$ let Ω_g be the set of points on which g takes values in the boundary of K. Then let Ω' be the set whose characteristic function $\chi_{\Omega'}$ is the infimum in $L^\infty(\Omega)$ of the characteristic functions χ_{Ω_g}. It follows that all $g \in F$ take values in the boundary of K almost everywhere on Ω', and any $g_1, g_2 \in F$ agree almost everywhere on Ω' since otherwise $(g_1 + g_2)/2$ would take values in the interior of K on a positive measure subset of Ω', contradicting its definition. Thus we define $f = g|_{\Omega'}$ where g is any function in F.

Now let h be any function in Q which agrees with f almost everywhere on Ω'. We will show that h is a weak* limit of functions in F, hence $h \in F$. This will complete the proof. For any $g \in F$ and any $\epsilon > 0$ the function h_ϵ defined by

$$h_\epsilon(x) = \begin{cases} g(x) & \text{if } d(g(x), \mathbb{R}^n - K) \leq \epsilon \\ h(x) & \text{otherwise} \end{cases}$$

belongs to F since

$$g = \lambda h_\epsilon + (1 - \lambda)(g + \frac{\lambda}{1 - \lambda}(g - h_\epsilon))$$

and $g + \frac{\lambda}{1-\lambda}(g - h_\epsilon) \in F$ for sufficiently small λ. Taking the limit of h_ϵ as $\epsilon \to 0$, we find that F contains the function h_g which agrees with g when $g(x)$ belongs to the boundary of K and agrees with h when $g(x)$ belongs to the interior of K. Now for any finite set $\{g_1, \ldots, g_n\} \subset F$ the average $g = (g_1 + \cdots + g_n)/n$ also belongs to F, and $g(x)$ belongs to the boundary of K only if $g_i(x)$ belongs to the boundary of K for all i. Thus h is a weak* cluster point of the functions h_g as g ranges over F, and we conclude that $h \in F$. □

Corollary 6.3. *Every zonoid is LNC.*

Proof. Take $\Omega = [0, 1]$ and $K = [0, 1] \subset \mathbb{R}$ in Corollary 6.2. Then Q is the positive part of the unit ball of $L^\infty[0, 1]$, and by Corollary 6.2 any weak* continuous linear image of Q is LNC. □

Example 6.4. Note that a continuous image of the positive part of the unit ball of l^1 need not be LNC. For example, let C be the convex hull of the origin in $\mathbb{R}^3$ and the points $p_n = (1/n, 1/n^2, 1)$ $(n \in \mathbb{N})$. This set is not LNC by Proposition 3.5. Define $T : l^1 \to \mathbb{R}^3$ by $T(f) = \sum f(n)p_n$. Then the image under T of the positive part of the unit ball of l^1 equals C, which is not LNC.

Example 6.5. Similarly, the positive part of the unit ball of any nonabelian von Neumann algebra has weak* continuous, finite-dimensional, linear images which are not LNC. To see this, let $\mathcal{M}$ be a nonabelian von Neumann algebra and find a subalgebra $\mathcal{A}$ which is isomorphic to $M_2(\mathbb{C})$ [10, p. 302]. Find a faithful normal representation of $\mathcal{M}$ on a Hilbert space H. Since any representation of $M_2(\mathbb{C})$ can be decomposed into an orthogonal direct sum of two-dimensional representations, we can find a two-dimensional subspace K of H such that $p\mathcal{A}p = B(K)$ where p is the orthogonal projection of H onto K. Now define $T : \mathcal{M} \to B(K)$ by $Tx = pxp$. This is a weak*-continuous linear map, and it takes the positive part of the unit ball of $\mathcal{M}$ onto the positive part of the unit ball of $B(K) \approx M_2(\mathbb{C})$, which is not LNC by Proposition 3.6.

Example 6.6. We also note that there is a finite-dimensional subspace of l^∞ whose intersection with the positive part of the unit ball is not LNC. To prove this let D be a countable set of unit vectors in $\mathbb{C}^2$, dense in the set of all unit vectors, and for each $x_n \in D$ ($n \in \mathbb{N}$) define $f_n : M_2(\mathbb{C}) \to \mathbb{C}$ by $f_n(A) = \langle Ax_n, x_n \rangle$. Now define $T : M_2(\mathbb{C}) \to l^\infty$ by $T(A) = (f_n(A))$ (the sequence whose nth term is $f_n(A)$). This map is nonexpansive because each f_n has norm one, and in fact $\|T(A)\| = \|A\|$ for all positive matrices A. Thus $T(M_2(\mathbb{C}))$ is a finite-dimensional subspace of l^∞ whose intersection with the positive part of the unit ball of l^∞ is isometric to the positive part of the unit ball of $M_2(\mathbb{C})$, which again is not LNC by Proposition 3.6.

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