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Lavrentieff Phenomenon and Non Standard Growth Conditions

G. Cardone
Dipartimento di Ingegneria Civile, Seconda Università di Napoli,
Real Casa dell'Annunziata, Via Roma, 29, 81031 Aversa (CE), Italy
giuseppe.cardone@unina2.it

C. D'Apice
Università di Salerno, Dipartimento di Ingegneria dell'Informazione e Matematica Applicata
Via Ponte don Melillo, 84084 Fisciano (SA), Italy
dapice@bridge.diima.unisa.it

U. De Maio
Università di Napoli, Dipartimento di Matematica e Applicazioni,
Complesso Monte S.Angelo, Via Cintia, 80126 Napoli, Italy
demaio@matna2.dma.unina.it

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The functional $F(u) = \int_B f(x, Du)dx$ is considered, where B is the unit ball in $\mathbf{R}^n$, u varies in the set of the locally Lipschitz functions on $\mathbf{R}^n$, and f belongs to a family of integrands containing, as model case, the following one

$$f : (x, z) \in \mathbf{R}^n \times \mathbf{R}^n \mapsto \frac{|< z, x >|}{|x|^n} + |z|^p, \qquad 1 < p < n.$$

The computation of the relaxed functional of F is provided. The formula obtained shows the persistence of the Lavrentieff Phenomenon. Examples of integrands not exhibiting the Lavrentieff Phenomenon are also presented, showing that this phenomenon is not linked only to the non standard growth behaviour of integrands.

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1. Introduction

In [23] and [17] the authors considered a family of integrands of functionals of Calculus of Variations exhibiting the Lavrentieff phenomenon (see for this phenomenon [3], [6], [13], [7], [8], [22], [31], [32], [33], [35]).

In a recent work (cf. [18]), the Lavrentieff phenomenon has been examined in the context of relaxation theory (see for this approach [12], [23], [17], [19], [15], [1]), and an explicit representation formula for the Lavrentieff gap has been obtained but only in the two dimensional case for this family of integrands.

In this paper we give the explicit representation formula also for the n-dimensional case and for a more general integrands family by using a very different technical approach. By this approach we are also able to prove that for some integrands, very similar to previous

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ones, the Lavrentieff Phenomenon does not occur. Then this phenomenon seems not to be depending only on the non standard conditions on integrands.

More exactly, let (U,τ) be a topological space satisfying the first countability axiom and let X be a $\tau-$dense subset of U. If F is a functional defined on U, we define $\overline{F}$ as the relaxed functional of the restriction of F to X as

$$\overline{F}\,(u) = \inf\left\{\liminf_h F\,(u_h) : \{u_h\}_h \subseteq X,\ u_h \xrightarrow{\tau} u\right\} \qquad u \in U. \tag{1}$$

In this context the Lavrentieff phenomenon occurs when

$$F\,(u) < \overline{F}\,(u)$$

for some u.

A case of particularly significant Lavrentieff phenomenon is the one in which F is a functional of the Calculus of Variations naturally defined on some class of more or less irregular functions (e.g. $W^{1,p}$) and X is a class of regular function (e.g. locally Lipschitz continuous functions).

We consider the following integrands

$$f\,(x,z) = g\left(\frac{x}{|x|}\right)\frac{|\langle z,x\rangle|}{|x|^n} + \psi\,(x,z)\,, \qquad (x,z) \in \mathbf{R}^n \times \mathbf{R}^n,$$

where $g : S^{n-1} \to \mathbf{R}$ is a nonnegative Lipschitz continuous function such that

$$\mathcal{H}^{n-1}\left(\left\{x \in S^{n-1} : g(x) = 0\right\}\right) = 0, \tag{2}$$

$\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $\mathbf{R}^n$, $\psi(x,\cdot)$ is a convex function, for every $x \in \mathbf{R}^n$, satisfying the following standard growth conditions

$$|z|^p \le \psi(x,z) \le a(x) + b\,|z|^p\,, \qquad 1 < p < n \tag{3}$$

with $a(x) \in L^1_{loc}(\mathbf{R}^n)$ and b positive constant.

Let B be the unit ball of $\mathbf{R}^n$. We pose

$$F(u) = \begin{cases} \int_B f\,(x,Du)\,dx & \text{if } u \in W^{1,p}\,(B) \\ +\infty & \text{if } u \in L^1\,(B)\setminus W^{1,p}\,(B) \end{cases}$$

and let $\overline{F}\,(u)$ be defined by (1) for every $u \in L^1\,(B)$.

For every $u \in W^{1,p}\,(B)\,$, let w and ξ be the functions obtained from u and g by passing to polar coordinates in $\mathbf{R}^n$ and let

$$Y = (0,\pi)^{n-2} \times (0,2\pi) \subset \mathbf{R}^{n-1}\,(\varphi,\theta)$$

$$\eta(\varphi_1,..,\varphi_{n-2}) = \rho^{1-n}\left|\frac{\partial(x_1,..,x_n)}{\partial(\rho,\varphi_1,..,\varphi_{n-2},\theta)}\right| = \sin^{n-2}\varphi_{n-2}\sin^{n-3}\varphi_{n-3}\cdot\cdot\sin^2\varphi_2\sin\varphi_1. \tag{4}$$

We first prove that if u is such that $\overline{F}(u) < +\infty$, then for L^{n-1}-*a.e.* $(\varphi_1,...,\varphi_{n-2},\theta) \in \mathbf{R}^{n-1}$, w has the trace w^+ for $\rho = 0$ and that $\xi\eta w^+ \in L^1(Y)$.

By virtue of this, we can give for $\overline{F}$ the following representation formula

$$\bar{F}(u) = \begin{cases} F(u) + \min\limits_{c\in\mathbf{R}} \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \xi\left(\varphi,\theta\right)\eta(\varphi)\left|w^+\left(\varphi,\theta\right)-c\right| d\varphi d\theta & \text{if } \overline{F}\left(u\right) < +\infty \\ +\infty & \text{otherwise} \end{cases}$$

Eventually we prove that for the following integrand, very similar to the previous one,

$$f\left(x,z\right) = \frac{|z|}{|x|} + |z|^p \qquad 1<p<2,\; x=\left(x_1,x_2\right),\; z=\left(z_1,z_2\right)\in\mathbf{R}^2,$$

the Lavrentieff phenomenon does not occur. We note also that in [27], it is proved that it does not occur for a class of functionals with non standard condition of the $p-q$ type, and in [25] under no coerciveness or growth conditions.

2. Notations and preliminary results

In this section we will list some notations we will keep still throughout all the paper.

In particular we need to consider a space of functions slightly more general than the space of BV functions and we just summarize some standard results for BV functions that are still valid for this space.

We will need two different copies of $\mathbf{R}^n$: $\mathbf{R}^n\left(x_1,...,x_n\right)$ and $\mathbf{R}^n\left(\rho,\varphi_1,...,\varphi_{n-2},\theta\right)$; we will ever consider a function denoted by u as a function $u=u\left(x_1,..,x_n\right)$, a function w as a function $w=w\left(\rho,\varphi,\theta\right)$, where $\varphi=\left(\varphi_1,...,\varphi_{n-2}\right)\in\mathbf{R}^{n-2}$.

We will denote by

$$\begin{array}{ll}
B=B\left(0,1\right)\subset\mathbf{R}^n\left(x_1,...,x_n\right) & \text{the unit ball,} \\
Q=\left(a,b\right)\times\left(c_1,d_1\right)\times\cdot\cdot\times\left(c_{n-1},d_{n-1}\right)\subset\mathbf{R}^n\left(\rho,\varphi,\theta\right) & \text{any open interval,} \\
\tilde{Q}=\left(2a-b,b\right)\times\left(c_1,d_1\right)\times\cdot\cdot\times\left(c_{n-1},d_{n-1}\right), & \text{for every } Q \text{ as above,} \\
R=\left(0,1\right)\times\left(0,\pi\right)^{n-2}\times\left(0,2\pi\right)\subset\mathbf{R}^n\left(\rho,\varphi,\theta\right), & \\
\tilde{R}=\left(-1,1\right)\times\left(0,\pi\right)^{n-2}\times\left(0,2\pi\right), & \\
Y=\left(0,\pi\right)^{n-2}\times\left(0,2\pi\right)\subset\mathbf{R}^{n-1}\left(\varphi,\theta\right). &
\end{array}$$

For every bounded open set Ω and $\varepsilon>0$ we define the set $\Omega_\varepsilon^-=\{x\in\Omega : \operatorname{dist}\left(x,\partial\Omega\right)>\varepsilon\}$.

We will moreover denote by $\alpha=\alpha\left(x\right)$ a positive symmetric mollifier, i.e. a function that satisfies the following properties

$$\begin{array}{ll} i) & \alpha\left(x\right)\in C_c^\infty\left(B\right); \\ ii) & \displaystyle\int_{\mathbf{R}^n}\alpha\left(x\right)dx=1; \\ iii) & \alpha\left(x\right)\geq 0; \end{array} \tag{5}$$

and set, for every $\varepsilon>0$,

$$\alpha^{(\varepsilon)} : y\in\mathbf{R}^n\mapsto\frac{1}{\varepsilon^n}\alpha\left(\frac{y}{\varepsilon}\right).$$

For every $u\in L^1\left(\Omega\right)$, $\varepsilon>0$ and $x\in\Omega_\varepsilon^-$ we define the regularization u_ε of u at x by

$$u^\varepsilon\left(x\right)=\left(\alpha^{(\varepsilon)}*u\right)\left(x\right)=\int_{\mathbf{R}^n}\alpha^{(\varepsilon)}\left(x-y\right)u\left(y\right)dy. \tag{6}$$

We refer to §4.2 of [28] for the standard properties of convolutions with mollifiers.

In the sequel we assume that ε takes value in a countable set.

Proposition 2.1. *Let Ω be an open subset of $\mathbf{R}^n$, $h(x,z) : \Omega \times \mathbf{R}^n \to \mathbf{R}$ a function such that*

i) for every $z \in \mathbf{R}^n$, the function $h(\cdot, z)$ is Lebesgue measurable on Ω,

ii) for a.e. $x \in \Omega$, the function $h(x, \cdot)$ is convex on $\mathbf{R}^n$,

iii) there exist $a(x) \in L^1(\Omega)$ and $b \in \mathbf{R}$ satisfying

$$-a(x) + b\,|z|^p \leq h(x,z), \qquad p \geq 1, \tag{7}$$

for a.e. $x \in \Omega$ and for every $z \in \mathbf{R}^n$.

Then the functional

$$H(u) = \int_\Omega h(x, Du(x))dx$$

is lower semicontinuous on $W^{1,p}(\Omega)$ with respect to the weak topology.

Proof. Cf. Example 1.24 of [21]. □

Proposition 2.2. *Let Ω be an open subset of $\mathbf{R}^n$, $h(x,z) : \Omega \times \mathbf{R}^n \to \mathbf{R}$ a function such that*

i) for every $z \in \mathbf{R}^n$, the function $h(\cdot, z)$ is Lebesgue measurable on Ω,

ii) for a.e. $x \in \Omega$, the function $h(x, \cdot)$ is continuous on $\mathbf{R}^n$,

iii) there exist $a(x) \in L^1(\Omega)$ and $b \in \mathbf{R}^n$ satisfying

$$|h(x,z)| \leq a(x) + b\,|z|^p, \qquad p \geq 1,$$

for a.e. $x \in \Omega$ and for every $z \in \mathbf{R}^n$.

Then the functional

$$H(u) = \int_\Omega h(x, u(x))dx$$

is continuous on $L^p(\Omega)$ with respect to the strong topology.

Proof. Cf. Example 1.22 of [21]. □

Definition 2.3. Let Ω be an open subset of $\mathbf{R}^n$ (ρ, φ, θ) and $1 \leq p \leq +\infty$. We will say that a function w is in $W^{1,p}_\rho(\Omega)$ if $w \in L^p(\Omega)$ and the distributional derivative $\dfrac{\partial w}{\partial \rho}$ exists and belongs to $L^p(\Omega)$.

Remark 2.4. We observe that it is possible to give the following characterization of $W^{1,p}_\rho(Q)$ (see also Theorem 2, section 4.9 in [28]):

$$W^{1,p}_\rho(Q) = \left\{ w \in L^p(Q) : w(\cdot, \varphi, \theta) \in W^{1,p}((a,b)) \text{ a.e. } (\varphi, \theta) \in (c_1, d_1) \times \cdot \cdot \times (c_{n-1}, d_{n-1}) \right\}.$$

In particular for a.e. $(\varphi, \theta) \in (c_1, d_1) \times \cdot \cdot \times (c_{n-1}, d_{n-1})$, $w(\cdot, \varphi, \theta)$ is absolutely continuous in $[a, b]$.

Let w be in $W^{1,p}_\rho(\Omega)$. We define

$$\|w\|_{W^{1,p}_\rho(\Omega)} = \|w\|_{L^p(\Omega)} + \left\|\frac{\partial w}{\partial \rho}\right\|_{L^p(\Omega)}.$$

$\|\cdot\|_{W^{1,p}_\rho(\Omega)}$ is clearly a norm on $W^{1,p}_\rho(\Omega)$.

Theorem 2.5. $\left(W^{1,p}_\rho(\Omega), \|\cdot\|_{W^{1,p}_\rho(\Omega)}\right)$ *is a Banach space.*

Proof. Cf. Proposition IX.1 of [10]. □

Theorem 2.6. *Let* $Q = (a,b) \times (c_1,d_1) \times \cdot\cdot \times (c_{n-1},d_{n-1})$ *and* $1 \le p < +\infty$. *Then there exists a bounded linear operator*

$$T : W^{1,p}_\rho(Q) \to L^p\left((c_1,d_1) \times \cdot\cdot \times (c_{n-1},d_{n-1})\right).$$

such that

$$(Tw)\left((\varphi,\theta)\right) = w\left(a,\varphi,\theta\right) \quad \textit{for all} \quad (\varphi,\theta) \in (c_1,d_1)\times\cdot\cdot\times(c_{n-1},d_{n-1}), w \in W^{1,p}_\rho(Q)\cap C(\overline{Q})$$

Proof. The proof can be simply achieved by using the estimate

$$|w\left(a,\varphi,\theta\right)| \le \int_a^r \left|\frac{\partial}{\partial\rho} w\left(s,\varphi,\theta\right)\right| ds + |w\left(r,\varphi,\theta\right)|$$

for every $r \in [a,b]$, for a.e. $(\varphi,\theta) \in (c_1,d_1) \times \cdot\cdot \times (c_{n-1},d_{n-1})$ and for every $w \in W^{1,p}_\rho(Q)$. □

Definition 2.7. The function Tw is called the trace of $w \in W^{1,p}_\rho(\Omega)$ on $\partial\Omega$. In the sequel we denote the trace Tw with w^+.

Definition 2.8. Let Ω be an open subset of $\mathbf{R}^n\left(\rho,\varphi,\theta\right)$ and let w be in $L^1\left(\Omega\right)$. We will say that w is in $\mathcal{C}\left(\Omega\right)$ if

$$\sup_{\psi\in I} \int_\Omega w \frac{\partial\psi}{\partial\rho} d\rho d\varphi d\theta < +\infty \tag{8}$$

where $I = \left\{\psi \in C^1_c\left(\Omega\right) : |\psi| \le 1\right\}$. In this case we will moreover define

$$\int_\Omega \left|\frac{\partial w}{\partial\rho}\right| = \sup_{\psi\in I} \int_\Omega w \frac{\partial\psi}{\partial\rho} d\rho d\varphi d\theta . \tag{9}$$

Remark 2.9. If $w \in \mathcal{C}\left(\Omega\right)$ then $\dfrac{\partial w}{\partial\rho}$ (in the sense of distributions) is a bounded Radon measure with total variation expressed by (9). □

Remark 2.10. *If* $\dfrac{\partial w}{\partial\rho} \in L^1\left(\Omega\right)$ *then*

$$\int_\Omega \left|\frac{\partial w}{\partial\rho}\right| = \left\|\frac{\partial w}{\partial\rho}\right\|_{L^1(\Omega)}. \tag{10}$$

(cf. [30], Example 1.2 pag. 3). □

Theorem 2.11. *Let* $\{w_h\}_h$ *be a sequence of functions in* $\mathcal{C}(\Omega)$ *such that* $w_h \to w$ *in* $L^1_{loc}(\Omega)$. *Then it results*

$$\int_\Omega \left| \frac{\partial w}{\partial \rho} \right| \leq \lim_{h\to\infty} \inf \int_\Omega \left| \frac{\partial w_h}{\partial \rho} \right|. \tag{11}$$

In particular, if the $\liminf$ *in the right hand side is finite, then* $w \in \mathcal{C}(\Omega)$.

Proof. Adapt the proof of Theorem 1.9 of [30], pag. 7. □

Let w be in $\mathcal{C}(\Omega)$. We define

$$\|w\|_{\mathcal{C}(\Omega)} = \|w\|_{L^1(\Omega)} + \int_\Omega \left| \frac{\partial w}{\partial \rho} \right|. \tag{12}$$

$\|\cdot\|_{\mathcal{C}(\Omega)}$ is clearly a norm on $\mathcal{C}(\Omega)$.

Theorem 2.12. $\left(\mathcal{C}(\Omega), \|\cdot\|_{\mathcal{C}(\Omega)}\right)$ *is a Banach space.*

Proof. Cf. [30], Remark 1.12, pag. 9. □

Lemma 2.13. *Let* $Q = (a,b) \times (c_1, d_1) \times \cdot\cdot \times (c_{n-1}, d_{n-1})$ *and* $w \in \mathcal{C}(Q)$. *Then there exists a (unique) function* $w^+ \in L^1((c_1,d_1) \times \cdot\cdot \times (c_{n-1}, d_{n-1}))$ *such that*

$$\lim_{\eta\to 0^+} \tfrac{1}{\eta^n} \int_{\sigma_1-\eta}^{\sigma_1+\eta} \cdot\cdot \int_{\sigma_{n-1}-\eta}^{\sigma_{n-1}+\eta} \int_a^{a+\eta} |w(\rho,\varphi,\theta) - w^+(\varphi,\theta)|\, d\rho d\varphi d\theta = 0$$
$$a.e.\ (\sigma_1, .., \sigma_{n-1}) \in (c_1,d_1) \times \cdot\cdot \times (c_{n-1}, d_{n-1}).$$

Moreover if $\psi \in C^1_c\left(\tilde{Q}\right)$ *we have (in distributional sense)*

$$\int_Q w \frac{\partial \psi}{\partial \rho} d\rho d\varphi d\theta = -\int_Q \psi \frac{\partial w}{\partial \rho} - \int_{c_1}^{d_1} \cdot\cdot \int_{c_{n-1}}^{d_{n-1}} w^+(\varphi,\theta)\, \psi(a,\varphi,\theta)\, d\varphi d\theta.$$

Proof. Adapt the proof of Lemma 2.4 of [30], pag. 32. □

The function w^+ defined on $(c_1,d_1) \times \cdot\cdot \times (c_{n-1}, d_{n-1})$ is the trace of w on the left side of Q; in the same way we define the trace w^- on the right side.

Proposition 2.14. *Let* $Q_1 = (r,s) \times (c_1,d_1) \times \cdot\cdot \times (c_{n-1}, d_{n-1})$, $Q_2 = (s,t) \times (c_1,d_1) \times \cdot\cdot \times (c_{n-1}, d_{n-1})$ *and let* $w_1 \in \mathcal{C}(Q_1)$, $w_2 \in \mathcal{C}(Q_2)$. *Let* $Q = (r,t) \times (c_1,d_1) \times \cdot\cdot \times (c_{n-1}, d_{n-1})$ *and let* $w : Q \longrightarrow \mathbf{R}$ *defined by*

$$w = \begin{cases} w_1 & \text{in } Q_1 \\ w_2 & \text{in } Q_2. \end{cases}$$

Then $w \in \mathcal{C}(Q)$ *and*

$$\int_Q \left|\frac{\partial w}{\partial \rho}\right| = \int_{Q_1} \left|\frac{\partial w_1}{\partial \rho}\right| + \int_{Q_2} \left|\frac{\partial w_2}{\partial \rho}\right| + \int_{c_1}^{d_1} \cdot\cdot \int_{c_{n-1}}^{d_{n-1}} \left|w_1^-(\varphi,\theta) - w_2^+(\varphi,\theta)\right| d\varphi d\theta.$$

Proof. Cf. [30], Proposition 2.8, pag. 36. □

Lemma 2.15. *If* $w \in \mathcal{C}(R)$ *and* $\xi(\varphi,\theta)$ *is a continuous function on* $\overline{Y}$ *then* $w\xi \in \mathcal{C}(R)$. *Moreover*

$$\frac{\partial (w\xi)}{\partial \rho} = \xi \frac{\partial w}{\partial \rho}$$

as measures on R *and, if given a measure* ν *we denote by* $|\nu|$ *its total variation measure, we have*

$$\left|\frac{\partial (w\xi)}{\partial \rho}\right| = |\xi| \left|\frac{\partial w}{\partial \rho}\right|.$$

Proof. Cf. Lemma 8 of [18]. □

In the sequel we will denote by $|\cdot|_{n-1}$ is the $(n-1)$-dimensional Lebesgue measure.

Definition 2.16. Let w be in $L^1(R)$ and let $\xi(\varphi,\theta)$ a continuous function on $\overline{Y}$ such that $w\xi \in \mathcal{C}(R)$. Let $Z = \left\{(\varphi,\theta) \in \overline{Y} : \xi(\varphi,\theta) = 0\right\}$ and assume that $|Z|_{n-1} = 0$. We define the trace w^+ of w on the left side of R as

$$w^+ = \frac{(w\xi)^+}{\xi} \qquad \text{a.e. in } Y. \tag{13}$$

Now let us introduce the following transformation into polar coordinates in $\mathbf{R}^n$, defined by the diffeomorfism

$$J : (\rho,\varphi,\theta) \in (0,+\infty) \times Y \to (x_1,..,x_n) \in J((0,+\infty) \times Y)$$

such that

$$\begin{aligned}
x_1 &= \rho \sin\varphi_{n-2} \sin\varphi_{n-3} \cdot\cdot \sin\varphi_2 \sin\varphi_1 \cos\theta \qquad (14)\\
x_2 &= \rho \sin\varphi_{n-2} \sin\varphi_{n-3} \cdot\cdot \sin\varphi_2 \sin\varphi_1 \sin\theta \\
x_3 &= \rho \sin\varphi_{n-2} \sin\varphi_{n-3} \cdot\cdot \sin\varphi_2 \cos\varphi_1 \\
x_4 &= \rho \sin\varphi_{n-2} \sin\varphi_{n-3} \cdot\cdot \cos\varphi_2 \\
& \quad .... \\
x_{n-2} &= \rho \sin\varphi_{n-2} \sin\varphi_{n-3} \cos\varphi_{n-4} \\
x_{n-1} &= \rho \sin\varphi_{n-2} \cos\varphi_{n-3} \\
x_n &= \rho \cos\varphi_{n-2}
\end{aligned}$$

where $\varphi = (\varphi_1,...,\varphi_{n-2}) \in (0,\pi)^{n-2}$.

Let us remember that the jacobian of J is

$$\left|\frac{\partial(x_1,..,x_n)}{\partial(\rho,\varphi,\theta)}\right| = \rho^{n-1} \sin^{n-2}\varphi_{n-2} \sin^{n-3}\varphi_{n-3} \cdot\cdot \sin^2\varphi_2 \sin\varphi_1. \tag{15}$$

Moreover let us define

$$w(\rho,\varphi,\theta)=u(\rho\sin\varphi_{n-2}\sin\varphi_{n-3}\cdot\cdot\sin\varphi_2\sin\varphi_1\cos\theta,...,\rho\cos\varphi_{n-2}). \tag{16}$$

Lemma 2.17. *Let $u\in W^{1,p}(B)$, w defined in (16) and $\xi(\varphi,\theta)$ be a continuous function on $\overline{Y}$. Then $\xi w\in\mathcal{C}(R)$ if and only if $\xi w\in W^{1,1}_{\rho}(R)$.*

Proof. Let us suppose that $\xi w\in\mathcal{C}(R)$. Then

$$\sup_{\psi\in I}\int_R \xi w\frac{\partial\psi}{\partial\rho}d\rho d\varphi d\theta<+\infty$$

where $I=\{\psi\in C^1_c(R):|\psi|\le 1\}$. Then there exists a constant $k>0$ such that

$$\int_R\frac{\partial(\xi w)}{\partial\rho}\psi d\rho d\varphi d\theta<k,\quad \text{for every } \psi\in I. \tag{17}$$

Now we observe that the distributional derivative $\dfrac{\partial(\xi w)}{\partial\rho}$ is of function type, because $w\in W^{1,p}_{loc,\rho}(R)$ being defined as in (16). So by (17) we obtain the thesis. □

3. The computation of the relaxed functional

Let B the unit ball in $\mathbf{R}^n$, $(U,\tau)=L^1(B)$ endowed with the strong topology, $X=Lip_{loc}$ the set of locally Lipschitz functions on $\mathbf{R}^n$; let

$$f(x,z)=g\left(\frac{x}{|x|}\right)\frac{|\langle z,x\rangle|}{|x|^n}+\psi(x,z)\qquad x=(x_1,...,x_n)\,,\ z=(z_1,...,z_n)\in\mathbf{R}^n \tag{18}$$

where $g(x)$ and $\psi(x,z)$ are the functions given respectively in (2) and (3).

We can observe that, by (3), $f(x,z)$ only verifies a non standard growth condition:

$$\begin{aligned}|z|^p &\le f(x,z)\le\frac{1}{|x|^{n-1}}|z|+\psi(x,z)\le\\ &\le \frac{q-1}{q}|x|^{-q(n-1)/(q-1)}+\frac{1}{q}|z|^q+\psi(x,z)\end{aligned}$$

where $\frac{q-1}{q}|x|^{-q(n-1)/(q-1)}\in L^1_{loc}(\mathbf{R}^n)$ if $q>n$.

Let ξ is the unique Y-periodic function such that, passing to polar coordinates in $\mathbf{R}^n$,

$$g\left(\frac{x}{|x|}\right)=\xi(\varphi,\theta)\qquad\text{for every } (\varphi,\theta)\in\mathbf{R}^{n-1}. \tag{19}$$

Let

$$G(u)=\begin{cases}\int_B f(x,Du)\,dx & u\in W^{1,p}(B)\\ +\infty & u\in L^1(B)\backslash W^{1,p}(B)\end{cases},$$

$$F=G\,|_X, \tag{20}$$

and $\overline{F}(u)$ be defined by (1) for every $u\in L^1(B)$.

We will hold these notations still troughout all the section.

Lemma 3.1. *If $u \notin W^{1,p}(B)$ then $\overline{F}(u) = +\infty$.*

Proof. By contradiction, let $u \notin W^{1,p}(B)$ and $\bar{F}(u) < +\infty$.

Then there exist a sequence $\{u_h\}_h \subset Lip_{loc}$, $m > 0$:

$$\begin{cases} i) \quad u_h \longrightarrow u & \text{in } L^1(B) \text{ as } h \to +\infty \\ ii) \quad \int_B |Du_h|^p \, dx \le F(u_h) \le m & \text{for every } h \in \mathbf{N}. \end{cases}$$

Since $u_h \to u$ in $L^1(B)$ we have

$$\overline{u}_h = \frac{1}{|B|} \int_B |u_h| \, dx \longrightarrow \overline{u} = \frac{1}{|B|} \int_B |u| \, dx. \tag{21}$$

By the Poincaré-Wirtinger inequality there exists $m_1 \in \mathbf{R}$:

$$\|u_h - \overline{u}_h\|_{W^{1,p}(B)} \le m_1 \int_B |Du_h|^p \, dx \le m_1 m. \tag{22}$$

By (21) and (22) we can suppose $\{u_h\}_h$ bounded in $W^{1,p}(B)$, so that by the reflexivity of this space we can assume that

$$u_h \rightharpoonup u \qquad \text{weakly in } W^{1,p}(B). \tag{23}$$

Then we have $u \in W^{1,p}(B)$, that is a contradiction. $\square$

Let $w \in L^\infty((0,1) \times Y)$, Y periodic, $0 \le \varepsilon < 1$ and set

$$w^{\varepsilon,c}(\rho, \varphi, \theta) = \begin{cases} c & \rho \le \varepsilon, \ (\varphi, \theta) \in \mathbf{R}^{n-1}, \\ w(\rho, \varphi, \theta) & \varepsilon < \rho \le 1, \ (\varphi, \theta) \in \mathbf{R}^{n-1}, \\ w(2 - \rho, \varphi, \theta) & 1 < \rho \le 2, \ (\varphi, \theta) \in \mathbf{R}^{n-1}. \end{cases} \tag{24}$$

Proposition 3.2. *Let $u \in W^{1,p}(B) \cap L^\infty(B)$, w be defined by (16), ξ by (19) and η by (4). If $\bar{F}(u) < +\infty$ then $\xi\eta w \in W^{1,1}_\rho(R)$ and*

$$\bar{F}(u) \ge \int_B f(x, Du) dx + \min_{c \in \mathbf{R}} \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \xi(\varphi, \theta) \eta(\varphi) \left| w^+(\varphi, \theta) - c \right| d\varphi d\theta.$$

Proof. Let $\{u_h\}_h \subset Lip_{loc}$ be such that $u_h \longrightarrow u$ in $L^1(B)$ and $\liminf_h F(u_h) < +\infty$.

Let first prove that

$$\int_B \psi(x, Du) dx \le \liminf_h \int_B \psi(x, Du_h) dx. \tag{25}$$

Obviously we can assume that $\liminf_h \int_B \psi(x, Du_h) dx < +\infty$. As in Lemma 3.1, there exists a constant k such that $\|u_h\|_{W^{1,p}(B)} \le k$ and we can take $u_h \to u$ in the weak topology of $W^{1,p}(B)$. Then by Proposition 2.1 we obtain (25).

Moreover we have for every $h \in \mathbf{N}$

$$\int_B g\left(\frac{x}{|x|}\right)\frac{|\langle Du_h, x\rangle|}{|x|^n}dx = \int_R \xi(\varphi,\theta)\,\eta(\varphi)\left|\frac{\partial w_h}{\partial \rho}\right| d\rho d\varphi d\theta \tag{26}$$

where

$$w_h(\rho,\varphi,\theta) = u_h(\rho \sin\varphi_{n-2}\sin\varphi_{n-3}\cdot\cdot\sin\varphi_2\sin\varphi_1\cos\theta, ..., \rho\cos\varphi_{n-2}), \quad \text{for } (\rho,\varphi,\theta) \in (0,1)\times \mathbf{R}^{n-1}.$$

Let $c_h = u_h(0) = w_h(0,\varphi,\theta)$, for every $(\varphi,\theta) \in \mathbf{R}^{n-1}$.

By Theorem 2.6, we have that $\{c_h\}_h$ has a converging subsequence to $\bar{c}$.

If we set $y_h(\rho,\varphi,\theta) = w_h^{0,c_h} + \bar{c} - c_h$ we have that $\xi\eta y_h \to \xi\eta w^{0,\bar{c}}$ in $L^1\left(\tilde{R}\right)$, where w_h^{0,c_h} and $w^{0,\bar{c}}$ are given in (24).

Then by Theorem 2.11, Proposition 2.14, Lemma 2.15 and Definition 2.16 we get

$$\begin{aligned}\liminf_h \int_R \xi\eta\left|\frac{\partial w_h}{\partial\rho}\right| d\rho d\varphi d\theta &= \liminf_h \int_{\tilde{R}} \xi\eta\left|\frac{\partial w_h^{0,c_h}}{\partial\rho}\right| d\rho d\varphi d\theta \\ &= \liminf_h \int_{\tilde{R}} \xi\eta\left|\frac{\partial y_h}{\partial\rho}\right| d\rho d\varphi d\theta \geq \int_{\tilde{R}}\left|\frac{\partial\left(\xi\eta w^{0,\bar{c}}\right)}{\partial\rho}\right|\end{aligned} \tag{27}$$

Then $\xi\eta w^{0,\bar{c}} \in \mathcal{C}(\widetilde{R})$ and so $\xi\eta w \in \mathcal{C}(R)$. By Lemma 2.17, $\xi\eta w \in W^{1,1}_\rho(R)$. We have

$$\begin{aligned}\int_{\tilde{R}}\left|\frac{\partial\left(\xi\eta w^{0,\bar{c}}\right)}{\partial\rho}\right| &= \int_R \xi\eta\left|\frac{\partial w}{\partial\rho}\right| + \int_{[0,\pi]^{n-2}}\int_0^{2\pi} \xi\eta\left|w^+(\varphi,\theta) - \bar{c}\right| d\varphi d\theta \\ &\geq \int_R \xi\eta\left|\frac{\partial w}{\partial\rho}\right| + \min_{c\in\mathbf{R}}\int_{[0,\pi]^{n-2}}\int_0^{2\pi} \xi\eta\left|w^+(\varphi,\theta) - c\right| d\varphi d\theta.\end{aligned} \tag{28}$$

The thesis easily follows by (25), (26), (28) and (29). □

Remark 3.3. If we set $\psi(c) = \int_{[0,\pi]^{n-2}}\int_0^{2\pi}\xi\eta\left|w^+(\varphi,\theta)-c\right| d\varphi d\theta$, then if $\psi \neq +\infty$ it can easily be seen that ψ is continuous and coercive so that there exists $\bar{c}$ such that $\psi(\bar{c}) = \min\limits_{c\in\mathbf{R}} \psi(c)$. □

In order to prove the opposite inequality, we state the following proposition.

Proposition 3.4. *Let $u \in W^{1,p}(B) \cap L^\infty(B)$, w be defined by (16), ξ by (19) and η by (4). If $\xi\eta w \in W^{1,1}_\rho(R)$, then $\overline{F}(u) < +\infty$ and*

$$\overline{F}(u) \leq \int\limits_B f(x, Du)dx + \min_{c\in\mathbf{R}} \int\limits_{[0,\pi]^{n-2}} \int\limits_0^{2\pi} \xi(\varphi,\theta)\,\eta(\varphi)\left|w^+(\varphi,\theta) - c\right| d\varphi d\theta \tag{29}$$

Proof. Let us consider the function

$$\varphi(x) = \frac{1}{|x|^\gamma} \quad \text{such that} \quad 0 < \gamma < \frac{n}{p} - 1. \tag{30}$$

Let us consider the following sequences

$$\varphi_h(x) = \begin{cases} 1 & \text{if } |x| < (h+1)^{-\frac{1}{\gamma}} \\ 0 & \text{if } |x| > (h)^{-\frac{1}{\gamma}} \\ \varphi(x) - h & \text{if } (h+1)^{-\frac{1}{\gamma}} \le |x| \le (h)^{-\frac{1}{\gamma}}. \end{cases} \tag{31}$$

and, for every $c \in \mathbf{R}$,

$$u_h(x) = c\varphi_h(x) + (1 - \varphi_h(x))u(x) \tag{32}$$

for every $h \in \mathbf{N}$ and $u \in W^{1,p}_{loc}(\mathbf{R}^n)$. Obviously

$$Du_h = (1 - \varphi_h)Du + (c - u)D\varphi_h. \tag{33}$$

Let us prove that

$$\liminf_h \int_B f(x, Du_h)dx \le \int_B f(x, Du)dx + \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \xi(\varphi, \theta)\, \eta(\varphi) \left| w^+(\varphi, \theta) - c \right| d\varphi d\theta. \tag{34}$$

Now we prove that

$$\liminf_h \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle Du_h, x\rangle|}{|x|^n} dx \le \int_R \xi\eta \left|\frac{\partial w}{\partial \rho}\right| + \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \xi(\varphi, \theta)\, \eta(\varphi) \left| w^+(\varphi, \theta) - c \right| d\varphi d\theta. \tag{35}$$

For every $h \in \mathbf{N}$, we have

$$\begin{aligned} \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle Du_h, x\rangle|}{|x|^n} dx \;\le\; & \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle (1-\varphi_h) Du, x\rangle|}{|x|^n} dx + \\ & + \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle (c-u) D\varphi_h, x\rangle|}{|x|^n} dx. \end{aligned} \tag{36}$$

We have

$$\begin{aligned} \int_{B-B_h} g\left(\frac{x}{|x|}\right) \frac{|\langle Du, x\rangle|}{|x|^n} dx \;\le\; & \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle (1-\varphi_h) Du, x\rangle|}{|x|^n} dx \le \\ \le\; & \int_{B-B_{h+1}} g\left(\frac{x}{|x|}\right) \frac{|\langle Du, x\rangle|}{|x|^n} dx, \end{aligned} \tag{37}$$

where $B_h = B\left(0, \frac{1}{(h)^{\frac{1}{\gamma}}}\right)$.

We observe that, by Lemma 2.17, we have that

$$\int_B g\left(\frac{x}{|x|}\right) \frac{|\langle Du, x\rangle|}{|x|^n} dx = \int_R \xi\eta \left|\frac{\partial w}{\partial \rho}\right| d\rho d\varphi d\theta < +\infty.$$

Then, by B.Levi Theorem we have

$$\lim_h \int_{B-B_h} g\left(\frac{x}{|x|}\right)\frac{|\langle Du,x\rangle|}{|x|^n}dx = \int_B g\left(\frac{x}{|x|}\right)\frac{|\langle Du,x\rangle|}{|x|^n}dx$$

and so

$$\lim_h \int_B g\left(\frac{x}{|x|}\right)\frac{|\langle(1-\varphi_h)Du,x\rangle|}{|x|^n}dx = \int_B g\left(\frac{x}{|x|}\right)\frac{|\langle Du,x\rangle|}{|x|^n}dx. \tag{38}$$

Moreover

$$\begin{aligned}\int_B g\left(\frac{x}{|x|}\right)&\frac{|\langle(c-u)D\varphi_h,x\rangle|}{|x|^n}dx =\\ &= \int_{\left\{x\in B:(h+1)^{-\frac{1}{\gamma}}\le|x|\le(h)^{-\frac{1}{\gamma}}\right\}} g\left(\frac{x}{|x|}\right)\frac{|\langle(c-u)D\varphi,x\rangle|}{|x|^n}dx =\\ &= \int_{\left\{x\in B:(h+1)^{-\frac{1}{\gamma}}\le|x|\le(h)^{-\frac{1}{\gamma}}\right\}} g\left(\frac{x}{|x|}\right)\frac{\gamma}{|x|^{\gamma+n}}|c-u|\,dx.\end{aligned} \tag{39}$$

Passing to polar coordinates, setting

$$w(\rho,\varphi,\theta) = u(\rho\sin\varphi_{n-2}\sin\varphi_{n-3}\cdot\cdot\sin\varphi_2\sin\varphi_1\cos\theta,...,\rho\cos\varphi_{n-2}).$$

it results

$$\begin{aligned}\int_{\left\{x\in B:(h+1)^{-\frac{1}{\gamma}}\le|x|\le(h)^{-\frac{1}{\gamma}}\right\}} &g\left(\frac{x}{|x|}\right)\frac{\gamma}{|x|^{\gamma+n}}|c-u|\,dx =\\ &= \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}}\int_{[0,\pi]^{n-2}}\int_0^{2\pi}\frac{\gamma}{\rho^{\gamma+1}}\xi(\varphi,\theta)\,\eta(\varphi)\,|c-w|\,d\rho d\varphi d\theta.\end{aligned} \tag{40}$$

Since $\xi\eta w \in W^{1,1}_\rho(R)$, we can consider $\xi\eta w^+$ for $\rho = 0$ (given by definition 2.16)

$$\begin{aligned}&\int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}}\int_{[0,\pi]^{n-2}}\int_0^{2\pi}\frac{\gamma}{\rho^{\gamma+1}}\xi(\varphi,\theta)\,\eta(\varphi)\,|c-w|\,d\rho d\varphi d\theta \le\\ &\le \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}}\int_{[0,\pi]^{n-2}}\int_0^{2\pi}\frac{\gamma}{\rho^{\gamma+1}}\xi(\varphi,\theta)\,\eta(\varphi)\,\left|c-w^+\right|d\rho d\varphi d\theta+\\ &+\int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}}\int_{[0,\pi]^{n-2}}\int_0^{2\pi}\frac{\gamma}{\rho^{\gamma+1}}\xi(\varphi,\theta)\,\eta(\varphi)\,\left|w^+-w\right|d\rho d\varphi d\theta.\end{aligned} \tag{41}$$

Then, since $\int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}}\frac{\gamma}{\rho^{\gamma+1}}d\rho = 1$, by Fubini Theorem

$$\begin{aligned}\int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}}&\int_{[0,\pi]^{n-2}}\int_0^{2\pi}\frac{\gamma}{\rho^{\gamma+1}}\xi(\varphi,\theta)\,\eta(\varphi)\,\left|c-w^+\right|d\rho d\varphi d\theta =\\ &= \int_{[0,\pi]^{n-2}}\int_0^{2\pi}\xi(\varphi,\theta)\,\eta(\varphi)\,\left|c-w^+\right|d\varphi d\theta\int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}}\frac{\gamma}{\rho^{\gamma+1}}d\rho =\\ &= \int_{[0,\pi]^{n-2}}\int_0^{2\pi}\xi(\varphi,\theta)\,\eta(\varphi)\,\left|c-w^+\right|d\varphi d\theta,\end{aligned} \tag{42}$$

We have, by Fubini Theorem and the first two lines of Proposition 2.6 of [G],

$$\begin{aligned}
&\int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \frac{\gamma}{\rho^{\gamma+1}} \xi(\varphi,\theta)\,\eta(\varphi)\left|w^+ - w\right| d\rho d\varphi d\theta = \\
&= \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \frac{\gamma}{\rho^{\gamma+1}} \xi(\varphi,\theta)\,\eta(\varphi)\left|w^+ - w\right| d\rho d\varphi d\theta \leq \\
&\leq \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \frac{\gamma}{\rho^{\gamma+1}} \int_0^{\rho} \left|\frac{\partial(\xi\eta w)}{\partial y}(y,\varphi,\theta)\right| dy d\rho d\varphi d\theta \leq \\
&\leq \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \int_0^{(h)^{-\frac{1}{\gamma}}} \left|\frac{\partial(\xi\eta w)}{\partial y}(y,\varphi,\theta)\right| dy d\varphi d\theta \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \frac{\gamma}{\rho^{\gamma+1}} d\rho = \\
&= \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \int_0^{(h)^{-\frac{1}{\gamma}}} \left|\frac{\partial(\xi\eta w)}{\partial \rho}(\rho,\varphi,\theta)\right| d\rho d\varphi d\theta
\end{aligned} \tag{43}$$

So by (43)

$$\int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \frac{\gamma}{\rho^{\gamma+1}} \xi(\varphi,\theta)\,\eta(\varphi)\left|w^+ - w\right| d\rho d\varphi d\theta \leq \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \int_0^{(h)^{-\frac{1}{\gamma}}} \left|\frac{\partial(\xi\eta w)}{\partial \rho}\right| d\rho d\varphi d\theta. \tag{44}$$

Moreover by Lemma 2.17, then

$$\lim_h \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \int_0^{(h)^{-\frac{1}{\gamma}}} \left|\frac{\partial(\xi\eta w)}{\partial \rho}\right| d\rho d\varphi d\theta = 0 \tag{45}$$

So by (39)÷(42), (44) and (45)

$$\lim_h \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle (c-u) D\varphi_h, x\rangle|}{|x|^n} dx \leq \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \xi(\varphi,\theta)\,\eta(\varphi)\left|c - w^+\right| d\varphi d\theta. \tag{46}$$

By (36), (38) and (46) we have

$$\begin{aligned}
&\liminf_h \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle Du_h, x\rangle|}{|x|^n} dx \leq \\
&\leq \int_R \xi\eta \left|\frac{\partial w}{\partial \rho}\right| + \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \xi(\varphi,\theta)\,\eta(\varphi)\left|c - w^+\right| d\varphi d\theta.
\end{aligned} \tag{47}$$

Now let us prove that

$$\lim_{h\to+\infty} \int_B \psi(x, Du_h) dx = \int_B \psi(x, Du) dx. \tag{48}$$

where $(u_h)_h$ is given by (32).

We have that

$$\lim_h \int_B |Du_h|^p \, dx = \int_B |Du|^p \, dx. \tag{49}$$

In fact, by (33) and convexity, for every $\lambda \in (0,1)$

$$|Du_h|^p \leq \lambda^{1-p} |c-u|^p |D\varphi_h|^p + (1-\lambda)^{1-p} |1-\varphi_h|^p |Du|^p . \tag{50}$$

Let us observe that $(\varphi_h)_h$ is a decreasing sequence such that

$$\lim_h \varphi_h(x) = 0, \text{ for every } x \in B \backslash \{0\}. \tag{51}$$

By (51) and B.Levi Theorem, we obtain that

$$\lim_h (1-\lambda)^{1-p} \int_B |1-\varphi_h|^p |Du|^p \, dx = (1-\lambda)^{1-p} \int_B |Du|^p \, dx. \tag{52}$$

Moreover, taking $E_h = \left\{ x \in B : (h+1)^{-\frac{1}{\gamma}} \leq |x| \leq (h)^{-\frac{1}{\gamma}} \right\}$,

$$0 \leq \lambda^{1-p} \int_B |c-u|^p |D\varphi_h|^p \, dx \leq \lambda^{1-p} \left(c + \|u\|_{L^\infty(B)} \right)^p \int_{E_h} |D\varphi|^p \, dx. \tag{53}$$

We have $|D\varphi|^p \in L^1(B)$, because $|D\varphi|^p = \dfrac{\gamma}{|x|^{p(\gamma+1)}}$ and $0 < \gamma < \dfrac{n}{p} - 1$; since $\lim_h |E_h|_n = 0$, we obtain by absolute continuity of the integral

$$\lim_h \int_{E_h} |D\varphi|^p \, dx = 0. \tag{54}$$

By (50), (52), (53) and (54) we obtain

$$\limsup_h \int_B |Du_h|^p \, dx \leq (1-\lambda)^{1-p} \int_B |Du|^p \, dx.$$

Passing to the limit on $\lambda \to 0^+$, we have

$$\limsup_h \int_B |Du_h|^p \, dx \leq \int_B |Du|^p \, dx. \tag{55}$$

Moreover by (55), we have $u_h \to u$ weakly in $W^{1,p}(B)$, and so, by Proposition 2.1, with $h(x,z) = |z|^p$, we obtain

$$\int_B |Du|^p \, dx \leq \liminf_h \int_B |Du_h|^p \, dx. \tag{56}$$

So by (55) and (56), we have (49).

Moreover, since $Du_h \to Du$ weakly in $L^p(B)$, by (49), we obtain that

$$Du_h \to Du \text{ strongly in } L^p(B). \tag{57}$$

So by (57) and Proposition 2.2, we have (48).

Now we prove that there exists a sequence v_h in Lip_{loc} such that $v_h \to u$ in $L^1(B)$ and

$$\liminf_h \int_B f(x, Dv_h)dx \leq \int_B f(x, Du)dx + \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \xi(\varphi, \theta)\, \eta(\varphi) \left| w^+(\varphi, \theta) - c \right| d\varphi d\theta. \quad (58)$$

Let us define u_h^ε the regularization of u_h. Let us prove that

$$\lim_{\varepsilon \to 0} \int_B f(x, Du_h^\varepsilon)dx = \int_B f(x, Du_h)dx. \quad (59)$$

In fact by (31), since $Du_h^\varepsilon = 0$ on B_{h+2} and ε small enough

$$\int_B g\left(\frac{x}{|x|}\right) \frac{|\langle Du_h^\varepsilon, x\rangle|}{|x|^n} dx = \int_{B - B_{h+2}} g\left(\frac{x}{|x|}\right) \frac{|\langle Du_h^\varepsilon, x\rangle|}{|x|^n} dx, \quad \varepsilon \text{ small enough} \quad (60)$$

where $B_h = B\left(0, \frac{1}{(h)^{\frac{1}{\gamma}}}\right)$.

By the equiabsolute continuity of Du_h^ε respect to ε, we have the equiabsolute continuity of the integrands in left side of (60). Then by Vitali's convergence Theorem, we have

$$\lim_{\varepsilon \to 0} \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle Du_h^\varepsilon, x\rangle|}{|x|^n} dx = \int_B g\left(\frac{x}{|x|}\right) \frac{|\langle Du_h, x\rangle|}{|x|^n} dx. \quad (61)$$

Moreover, since $Du_h^\varepsilon \to Du_h$ in $L^p(B)$, for $\varepsilon \to 0$, by Proposition 2.2 it results

$$\lim_{\varepsilon \to 0} \int_B \psi(x, Du_h^\varepsilon)dx = \int_B \psi(x, Du_h)dx. \quad (62)$$

and so by (61) and (62), we obtain (59).

By (59), for every $h \in \mathbf{N}$, there exists ε_h such that

$$\int_B f(x, Du_h^{\varepsilon_h})dx \leq \int_B f(x, Du_h)dx + \frac{1}{h}.$$

So

$$\liminf_h \int_B f(x, Du_h^{\varepsilon_h})dx \leq \liminf_h \int_B f(x, Du_h)dx.$$

By (34), we obtain (58) with $v_h = u_h^{\varepsilon_h}$. For the arbitrariety of c by (58) we obtain the thesis. □

Theorem 3.5. *Let $u \in W^{1,p}(B)$. Then $\bar{F}(u)$ is finite if and only if $\xi\eta w \in W_\rho^{1,1}(R)$; moreover*

$$\overline{F}(u) = \int_B f(x, Du)dx + \min_{c \in \mathbf{R}} \int_{[0,\pi]^{n-2}} \int_0^{2\pi} \xi(\varphi, \theta)\, \eta(\varphi) \left| w^+(\varphi, \theta) - c \right| d\varphi d\theta \quad (63)$$

Proof. If $u \in W^{1,p}(B) \cap L^\infty(B)$, (63) follows by Propositions 3.2 and 3.4; if $u \in W^{1,p}(B) \backslash L^\infty(B)$, (63) follows by Lemma 2.2 of [16]. □

4. An example in which Lavrentieff phenomenon does not occur

Now we study the case in which instead of the function given in (18), we consider, for $n = 2$,

$$\widetilde{f}(x, z) = \frac{|z|}{|x|} + |z|^p \qquad 1 < p < 2,\ x = (x_1, x_2),\ z = (z_1, z_2) \in \mathbf{R}^2.$$

We show that the presence of the Lavrentieff phenomenon, while depending on the non standard growth condition of the integrand, is not implied by this.

Let us consider the functional

$$\widetilde{G}(u) = \begin{cases} \displaystyle\int_B \widetilde{f}(x, Du)\, dx & u \in W^{1,p}(B) \\ +\infty & u \in L^1(B) \backslash W^{1,p}(B) \end{cases},$$

Let us define $\widetilde{F} = \widetilde{G}\,|_X$, where $X = Lip_{loc}(B)$.

Let us first state the following lemma:

Lemma 4.1. *Let $(a_h)_h$ a non negative sequence such that there exists $\gamma > 0$ such that*

$$\sum_{h=j}^{+\infty} a_h \geq \frac{1}{j}\gamma, \qquad \forall j \in \mathbf{N}.$$

Then

$$\sum_{h=1}^{+\infty} h a_h = +\infty.$$

Proof. It results

$$\sum_{h=1}^{+\infty} h a_h = \sum_{j=1}^{+\infty} \left(\sum_{h=j}^{+\infty} a_h \right) \geq \sum_{j=1}^{+\infty} \frac{1}{j}\gamma = +\infty.$$

and so the thesis □

We have the following result

Proposition 4.2. *Let $u \in W^{1,p}(B) \cap L^\infty(B)$, w be defined by $w(\rho, \theta) = u(\rho\cos\theta, \rho\sin\theta)$, $\rho \in (0, 1)$, $\theta \in \mathbf{R}$.*

If

$$\int_B \frac{|Du|}{|x|} dx < +\infty.$$

then $w \in \mathcal{C}(R)$ and $w^+(\theta)$ is a.e. constant in $[0, 2\pi]$.

Proof. We have

$$\int_B \frac{|Du|}{|x|} dx = \int_R \sqrt{\left|\frac{\partial w}{\partial \rho}\right|^2 + \frac{1}{\rho^2}\left|\frac{\partial w}{\partial \theta}\right|^2}\, d\rho d\theta. \tag{64}$$

By (64) and the boundedness of w we have that $w \in W^{1,1}(R)$, and so by lemma 2.17 we have that $w \in C(R)$

Let us suppose that w^+ is not constant. Then there exist θ_1 and $\theta_2 \in \mathbf{R}$ such that $\theta_1 < \theta_2$, $w^+(\theta_2) = \Lambda$, $w^+(\theta_1) = \lambda$ and $\Lambda > \lambda$.

By definition of trace, fixed $\sigma > 0$ such that $\Lambda - \lambda - 2\sigma > 0$, there exists $\varepsilon > 0$ such that, for $\alpha < \varepsilon$,

$$\frac{1}{2\alpha^2} \int_{R_2^\alpha} w(\rho,\theta)\, d\rho d\theta \geq \Lambda - \sigma \qquad \text{and} \qquad \frac{1}{2\alpha^2} \int_{R_1^\alpha} w(\rho,\theta)\, d\rho d\theta \leq \lambda + \sigma \tag{65}$$

where $R_i^\alpha = (0,\alpha) \times (\theta_i - \alpha, \theta_i + \alpha)$, $i = 1, 2$. Let us observe that $(\rho,\theta) \in R_1^\alpha$ if, and only if, $(\rho, \theta + \theta_2 - \theta_1) \in R_2^\alpha$. We have

$$|w(\rho, \theta + \theta_2 - \theta_1) - w(\rho,\theta)| \leq \int_\theta^{\theta+\theta_2-\theta_1} \left| \frac{\partial w(\rho,\eta)}{\partial \eta} \right| d\eta.$$

Integrating, by (65), it results

$$\begin{aligned} \int_{\theta_1-\alpha}^{\theta_1+\alpha} \int_0^\alpha |w(\rho, \theta + \theta_1 - \theta_2) - w(\rho,\theta)|\, d\rho d\theta \geq \\ \geq \left| \int_{\theta_1-\alpha}^{\theta_1+\alpha} \int_0^\alpha w(\rho, \theta + \theta_1 - \theta_2) - w(\rho,\theta) d\rho d\theta \right| = \\ = \left| \int_{R_2^\alpha} w(\rho,\theta)\, d\rho d\theta - \int_{R_1^\alpha} w(\rho,\theta)\, d\rho d\theta \right| \geq \\ \geq 2\alpha^2(\Lambda - \lambda - 2\sigma). \end{aligned} \tag{66}$$

On the other hand

$$\int_{\theta_1-\alpha}^{\theta_1+\alpha} \int_0^\alpha \int_\theta^{\theta+\theta_2-\theta_1} \left| \frac{\partial w(\rho,\eta)}{\partial \eta} \right| d\eta d\rho d\theta \leq 2\alpha \int_{(0,\alpha)\times(\theta_1-\alpha,\theta_2+\alpha)} \left| \frac{\partial w(\rho,\eta)}{\partial \eta} \right| d\eta d\rho. \tag{67}$$

By (66) and (67), we obtain

$$\int_{(0,\alpha)\times(\theta_1-\alpha,\theta_2+\alpha)} \left| \frac{\partial w(\rho,\eta)}{\partial \eta} \right| d\eta d\rho \geq \alpha(\Lambda - \lambda - 2\sigma)$$

and so

$$\int_{(0,\alpha)\times(0,2\pi)} \left| \frac{\partial w(\rho,\eta)}{\partial \eta} \right| d\eta d\rho \geq \frac{\alpha}{2}(\Lambda - \lambda - 2\sigma). \tag{68}$$

Let us define, for every $h \in \mathbf{N}$

$$\alpha_h = \frac{1}{h}$$

and

$$a_h = \int_0^{2\pi} \int_{\alpha_{h+1}}^{\alpha_h} \left| \frac{\partial w(\rho,\eta)}{\partial \eta} \right| d\eta d\rho. \tag{69}$$

Then by (68),

$$\sum_{h=j}^{+\infty} a_h \geq \frac{1}{j}\left(\frac{\Lambda-\lambda-2\sigma}{2}\right), \ \forall j \in \mathbf{N}. \tag{70}$$

By (69) and (70), we have

$$\int_R \frac{1}{\rho}\left|\frac{\partial w}{\partial \eta}\right| d\rho d\eta \geq \sum_{h=1}^{\infty} h a_h.$$

By Lemma 4.1 we get a contradiction and so the thesis. □

Proposition 4.3. *Let $u \in W^{1,p}(B) \cap L^\infty(B)$. Then*

$$\overline{\overline{F}}(u) \geq \int_B |Du|^p \, dx + \int_B \frac{|Du|}{|x|} dx. \tag{71}$$

Proof. Let $\{u_h\}_h \subset Lip_{loc}(\mathbf{R}^2)$ such that $u_h \longrightarrow u$ in $L^1(B)$. Then

$$\liminf_h \int_B |Du_h|^p \, dx \leq \liminf_h \int_B \widetilde{f}(x, Du_h) \, dx.$$

We can assume that $\liminf_h \int_B \widetilde{f}(x, Du_h)\, dx < +\infty$. So, as in Lemma 3.1, $u_h \to u$ weakly in $W^{1,p}(B)$.

Then by Proposition 2.1

$$\int_B |Du|^p \, dx + \int_B \frac{|Du|}{|x|} dx \leq \liminf_h \left[\int_B |Du_h|^p \, dx + \int_B \frac{|Du_h|}{|x|} dx\right]$$

and so, by definition (1), (71) follows. □

Proposition 4.4. *Let $u \in W^{1,p}(B) \cap L^\infty(B)$. Then*

$$\overline{\overline{F}}(u) \leq \int_B |Du|^p \, dx + \int_B \frac{|Du|}{|x|} dx$$

Proof. We can assume that $\int_B \frac{|Du|}{|x|} dx < +\infty$. Then, by (64), $\int_R \left|\frac{\partial w}{\partial \rho}\right| d\rho d\varphi d\theta < +\infty$ and so $w \in \mathcal{C}(R)$. By Proposition 4.2, w^+ is constant, then let us pose $w^+ = \overline{c}$, with $\overline{c} \in \mathbf{R}$.

Let φ_h the sequence as in (31). Moreover for every $h \in \mathbf{N}$, let u_h the sequence

$$u_h = \overline{c}\varphi_h(x) + (1-\varphi_h(x))u(x)$$

for every $h \in \mathbf{N}$, and $u \in W^{1,p}_{loc}(\mathbf{R}^2)$.

Let us prove that

$$\liminf_h \int_B \frac{|Du_h|}{|x|} dx \leq \int_B \frac{|Du|}{|x|} dx \tag{72}$$

and

$$\lim_h \int_B |Du_h|^p \, dx = \int_B |Du|^p \, dx. \tag{73}$$

For every $h \in \mathbf{N}$, we have

$$\int_B \frac{|Du_h|}{|x|} dx \leq \int_B \frac{|(1-\varphi_h)Du|}{|x|} dx + \int_B \frac{|(\overline{c}-u)D\varphi_h|}{|x|} dx. \tag{74}$$

We have

$$\int_B \frac{|(1-\varphi_h)Du|}{|x|} dx \leq \int_{B-B_{h+1}} \frac{|Du|}{|x|} dx, \tag{75}$$

where $B_h = B\left(0, \frac{1}{(h)^{\frac{1}{\gamma}}}\right)$.

Then, by B.Levi Theorem we have

$$\lim_h \int_{B-B_{h+1}} \frac{|Du|}{|x|} dx = \int_B \frac{|Du|}{|x|} dx$$

and so

$$\lim_h \int_B \frac{|(1-\varphi_h)Du|}{|x|} dx \leq \int_B \frac{|Du|}{|x|} dx. \tag{76}$$

Moreover

$$\begin{aligned} \int_B \frac{|(\overline{c}-u)D\varphi_h|}{|x|} dx = \\ &= \int_{\left\{x \in B : (h+1)^{-\frac{1}{\gamma}} \leq |x| \leq (h)^{-\frac{1}{\gamma}}\right\}} \frac{|(\overline{c}-u)D\varphi|}{|x|} dx = \\ &= \int_{\left\{x \in B : (h+1)^{-\frac{1}{\gamma}} \leq |x| \leq (h)^{-\frac{1}{\gamma}}\right\}} \frac{\gamma}{|x|^{\gamma+2}} |\overline{c}-u| \, dx. \end{aligned} \tag{77}$$

Passing to polar coordinates, setting

$$w(\rho, \theta) = u(\rho\cos\theta, \rho\sin\theta), \qquad \rho \in (0,1), \ \theta \in \mathbf{R}$$

it results

$$\int_{\left\{x \in B :. (h+1)^{-\frac{1}{\gamma}} \leq |x| \leq (h)^{-\frac{1}{\gamma}}\right\}} \frac{\gamma}{|x|^{\gamma+2}} |\overline{c}-u| \, dx = \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \int_0^{2\pi} \frac{\gamma}{\rho^{\gamma+1}} |\overline{c}-w| \, d\rho d\theta \tag{78}$$

We have, by Fubini Theorem and Proposition 2.6 of [30],

$$\begin{aligned} &\int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \int_0^{2\pi} \frac{\gamma}{\rho^{\gamma+1}} |\overline{c}-w| \, d\rho d\theta = \\ &= \int_0^{2\pi} \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \frac{\gamma}{\rho^{\gamma+1}} |\overline{c}-w| \, d\rho d\theta \leq \int_0^{2\pi} \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \frac{\gamma}{\rho^{\gamma+1}} \int_0^{(h)^{-\frac{1}{\gamma}}} \left|\frac{\partial w}{\partial y}(y,\theta)\right| dy d\rho d\theta = \\ &= \int_0^{2\pi} \int_0^{(h)^{-\frac{1}{\gamma}}} \left|\frac{\partial w}{\partial y}(y,\theta)\right| dy d\theta \int_{(h+1)^{-\frac{1}{\gamma}}}^{(h)^{-\frac{1}{\gamma}}} \frac{\gamma}{\rho^{\gamma+1}} d\rho = \int_0^{2\pi} \int_0^{(h)^{-\frac{1}{\gamma}}} \left|\frac{\partial w}{\partial \rho}(\rho,\theta)\right| d\rho d\theta. \end{aligned} \tag{79}$$

Since $\dfrac{\partial w}{\partial \rho} \in L^p(R)$ and $w \in \mathcal{C}(R)$,

$$\lim_h \int_0^{2\pi} \int_0^{(h)^{-\frac{1}{\gamma}}} \left| \frac{\partial w}{\partial \rho}(\rho, \theta) \right| d\rho d\theta = 0. \tag{80}$$

So by (77)÷(80), we have

$$\lim_h \int_B \frac{|(\overline{c} - u) D\varphi_h|}{|x|} dx = 0. \tag{81}$$

By (74), (76) and (81), we have (72).

Moreover we obtain (73) as in Proposition 3.2. □

Theorem 4.5. *Let $u \in W^{1,p}(B)$. Then*

$$\overline{\overline{F}}(u) = \int_B |Du|^p \, dx + \int_B \frac{|Du|}{|x|} dx \tag{82}$$

where $\overline{\overline{F}}(u) = +\infty$ if $\int_B \frac{|Du|}{|x|} dx = +\infty$.

Proof. If $u \in W^{1,p}(B) \cap L^\infty(B)$, (82) follows by Propositions 4.3 and 4.4; if $u \in W^{1,p}(B) \setminus L^\infty(B)$, (82) follows by Lemma 2.2 of [16]. □

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