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The Category $Cv_{D_0,N}$-Cmod and MP

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This paper deals with the halfing morphism of $Cv_{D_0,N}$-convex modules, where D_0 is the semiring of non-negative dyadic rationals, and its use in the explicit construction of the left adjoint to the functor $O_{D_0,N} : D_0$-**Mod** $\longrightarrow Cv_{D_0,N}$-**Cmod** that assigns to each D_0-module its canonical $Cv_{D_0,N}$-convex module. Furthermore it is shown that $Cv_{D_0,N}$-**Cmod** is isomorphic to the category **MP** of midpoint algebras.

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1. Introduction

For the purpose of this paper C stands for a commutative cone semiring, that is a partially ordered commutative semiring with 0 as its smallest element, and with the property that $2 = 1 + 1 \in C$ has a multiplicative inverse $\frac{1}{2}$. The ring $D := \mathbb{Z}\left[\frac{1}{2}\right]$ of dyadic rationals contains the commutative cone semiring D_0 of non-negative elements.

Let N be any infinite class. By a *finitary N-convexity theory* over C is meant a class Γ of maps $\alpha_* : N \longrightarrow C$ with the following properties:

(0) every $\alpha_* \in \Gamma$ has finite support supp α_* and $\sum\{\alpha_n : n \in \text{supp } \alpha_*\} \leq 1$;
(i) for every $n \in N$ the Dirac map δ_*^n (value 1 at n and 0 elsewhere) is in Γ;
(ii) for every $\alpha_* \in \Gamma$ and $\beta_*^n \in \Gamma,\ n \in N$, the map $< \alpha_\square, \beta_*^\square >$ given by $N \ni n \mapsto \sum\{\alpha_m \beta_n^m : m \in \text{supp } \alpha_*\}$ is in Γ.

The class $Cv_{C,N}$ of all maps $\alpha_* : N \longrightarrow C$ with finite support satisfying $\sum\{\alpha_n : n \in \text{supp } \alpha_*\} = 1$ is such an N-convexity theory. It is called the *classical N-convexity theory* over C. Our main interest is in the classical convexity theory $Cv_{D_0,N}$ over D_0.

Given a (finitary) N-convexity theory Γ, a set X together with a map $\Gamma \times (N, X) \ni (\alpha_*, x^*) \mapsto < \alpha_*, x^* > \in X$ is called a Γ-*convex module* if

(i′) for every $n \in N$ and $x^* \in (N, X), < \delta_*^n, x^* > = x^n$;
(ii′) for every $\alpha_* \in \Gamma, \beta_*^\square \in (N, \Gamma)$, and $x^* \in (N, X), \ll \alpha_\square, \beta_*^\square >, x^* > =$
$< \alpha_\square, < \beta_*^\square, x^* \gg$.

Here, (N, X) stands for the conglomerate of all maps $N \longrightarrow X, (N, \Gamma)$ the conglomerate of all maps $N \longrightarrow \Gamma$, and $< \beta_*^\square, x^* >$ for the map $N \ni n \mapsto < \beta_*^n, x^* > \in X$. Sometimes it is convenient to write $\sum\{\alpha_n x^n : n \in N\}$ instead of $< \alpha_*, x^* >$. Since $< \alpha_*, x^* >$ depends

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only on the values $x^n, n \in$ supp α_*, (see [1], 3.5, [2], 4.4, and [3], §2) we may also write $\sum\{\alpha_n x^n : n \in \text{supp } \alpha_*\}$ instead of $< \alpha_*, x^* >$. In particular we use $\frac{1}{2}x' + \frac{1}{2}x''$ instead of $< \frac{1}{2}\delta_*^{n'} + \frac{1}{2}\delta_*^{n'}, x^* >$, where $x' := x^{n'}$ and $x'' := x^{n''}$.

A map $f : X \longrightarrow X'$ between two Γ-convex modules is said to be a *morphism* from X to X' if for all $\alpha_* \in \Gamma$ and $x^* \in (N, X)$

$$f(< \alpha_*, x^* >) =< \alpha_*, f^N(x^*) >$$

where $f^N := (N, f)$ is the map induced by f. Hence we have the category Γ-**Cmod** of Γ-convex modules and their morphisms, with compositions the set-theoretical ones. Γ-**Cmod** is an algebraic category. In [1], [2], and [3] various identities for Γ-convex modules can be found.

Given any homomorphism $\rho : C \to C'$ of semirings there is a functor ρ_*, called "restriction of scalars", from the category C'-**Smod** of C'-semimodules to the category C-**Smod**. Similarly if Γ is a finitary N-convexity theory over C and Γ' is a finitary N-convexity theory over C' such that for any $\alpha_* \in \Gamma$ the map $\rho^N(\alpha_*) = \rho^N \circ \alpha_*$ is in Γ', then there is the functor $\rho_* : \Gamma'$-**Cmod**$\longrightarrow \Gamma$-**Cmod** that assigns to each Γ'-convex module X' the Γ-convex module $\rho_*(X')$ given by the composition $\Gamma \times (N, X') \ni (\alpha_*, x'^*) \mapsto < \rho^N(\alpha_*), x'^* > \in X'$.

In section 2 we discuss the halfing morphism h_a for any $Cv_{C,N}$-convex module X. If h_a is an automorphism of X then X becomes a D_0-semimodule $(X, \dot{+})$. For any $Cv_{C,N}$-convex module X, h_a given rise to a congruence relation $\sim$ on $X \times \mathbb{N}_0$ and $X \times \mathbb{N}_0/\sim$ inherits the structure of a $Cv_{C,N}$-convex module such that the map $X \ni x \mapsto (x, 0)/\sim \in X \times \mathbb{N}_0/\sim$ is a morphism j_X.

In section 3 we prove that $Cv_{D_0,N}$-structure on $X \times \mathbb{N}_0/\sim$ equals that of $O_{D_0,N}(X \times \mathbb{N}_0, \dot{+})$ where $O_{D_0,N}(M)$ for any D_0-semimodulo M is the obvious $Cv_{D_0,N}$-structure on M. This observation enters into the explicit construction of the left adjoint of the functor $O_{D_0,N}$ from the category D_0-**Mod** of D_0-modules to the category $Cv_{D_0,N}$-**Cmod**. The corresponding universal arrow is

$$X \xrightarrow{j_X} O_{D_0,N}(X \times \mathbb{N}_0/\sim, \dot{+}) \to O_{D_0,N}(X \times \mathbb{N}_0/\sim, \dot{+})^2 \to O_{D_0,N}(X \times \mathbb{N}_0, \dot{+})^2/\underset{B}{\sim},$$

where the map in the middle sends each (x, n) to $((a, 0), (x, n))$ and the map on the right is the quotient map with respect to the Bourne relation $\underset{B}{\sim}$.

The last section deals with a structure that is well known in universal algebra. It is called "commutative binary mode" (see [4], p. 91), and is a special case of an "alternation groupoid" (see e.g. [5]) and of a "espace medial surcommutative" (see [6]). We prefer to call a set equipped with this structure a "midpoint algebra". These midpoint algebras and their homomorphisms from a category **MP** and we show that **MP** is isomorphic with $Cv_{D_0,N}$-**Cmod**.

I wish to thank Professor J. D. H. Smith for supplying me with the references [4], [5], and [6].

2. The Halfing Morphism

Let X be a $Cv_{C,N}$-convex module. For any $a \in X$ let $h_a : X \longrightarrow X$ be the map given by $h_a(x) := \frac{1}{2}a + \frac{1}{2}x, x \in X$. Moreover, for any $a^* \in (N, X)$ denote by h_{a^*} the map $N \ni n \mapsto h_{a^n} \in \text{Set } (X, X)$.

Lemma 2.1. *Let X be a $Cv_{C,N}$-convex module. Then for any $a \in X$ and $A^* \in (N, X)$*

(i) $h_a^k(x) = \frac{2^k-1}{2^k}\, a + \frac{1}{2^k}x$, $k \in \mathbb{N}$ *and* $x \in X$;
(ii) h_a *is a morphism* $X \longrightarrow X$ *of* $Cv_{C,N}$- *convex mdoules;*
(iii) $< \alpha_*, ha_{a^*} > = h_{<\alpha_*, a^*>}$.

Proof.
(i). $k = 1$ is clear by definition. Suppose we have (i) for k. Choose $x^* \in (N, X)$ such that $x^{n_1} = a$ and $x^n = x, n \in N\backslash\{n_1\}$. Then $< \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n_1}, x^* > = a$ and $< \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n}, x^* > = h_a(x), n \in N\backslash\{n_1\}$. Hence

$$\begin{aligned} h_a^{k+1}(x) = h_a^k(h_a(x)) &= < \frac{2^k-1}{2^k}\delta_\square^{n_1} + \frac{1}{2^k}\delta_\square^{n_2}, < \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^\square, x^* \gg = \\ &= \ll \frac{2^k-1}{2^k}\delta_\square^{n_1} + \frac{1}{2^k}\delta_\square^{n_2}, \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^\square >, x^* > = \\ &= < \frac{2^{k+1}-1}{2^{k+1}}\ \delta_*^{n_1} + \frac{1}{2^{k+1}}\ \delta_*^{n_2}, x^* > = \\ &= \frac{2^{k+1}-1}{2^{k+1}}a + \frac{1}{2^{k+1}}x. \end{aligned}$$

(ii). Let $\alpha_* \in Cv_{C,n}$ and $x^* \in (N, X)$. Choose a bijection $\varphi : N \longrightarrow N\backslash\{n_1\}$ and let $z^* \in (n, X)$ be such that $z^{n_1} := a$ and $z^{\varphi(n)} := x^n, n \in N$. Let furthermore $\beta_*^{n_1} := \delta_*^{n_1}$ and $\beta_*^n := \delta_*^{\varphi(n)}$ for $n \in N\backslash\{n_1, n_2\}$, and define $\beta_*^{n_2}$ by $\beta_{n_1}^{n_2} := 0$ and $\beta_{\varphi(m)}^{n_2} := \alpha_m, m \in N$. Then $\beta_*^\square$ is in $(N, Cv_{C,N})$ and we have

$$< \beta_*^n, z^* > = \begin{cases} a & , n = n_1, \\ < \alpha_*, x^* > & , n = n_2, \\ x^n & , n \in N\backslash\{n_1, n_2\}. \end{cases}$$

Hence

$$h_a(< \alpha_*, x^* >) = < \frac{1}{2}\delta_\square^{n_1} + \frac{1}{2}\delta_\square^{n_2}, < \beta_*^\square, z^* \gg = \ll \frac{1}{2}\delta_\square^{n_1} + \frac{1}{2}\delta_\square^{n_2}, \beta_*^\square >, z^* > .$$

Next let $\gamma_*^n := \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{\varphi(n)} \in Cv_{C,N}$. Then $h_a(x^n) = < \delta_*^n, z^* >, n \in N$, and thus

$$< \alpha_*, h_a^N(x^*) > = < \alpha_\square, < \gamma_*^\square, z^* \gg = \ll \alpha_\square, \gamma_*^\square >, z^* > .$$

But for any $m \in N$,

$$< \frac{1}{2}\delta_\square^{n_1} + \frac{1}{2}\delta_\square^{n_2}, \beta_m^\square > = \begin{cases} \frac{1}{2} & , \text{if } \ m = n_1 \\ \alpha_{m'} & , \text{if } \ m = \varphi(m') \end{cases}$$

and

$$< \alpha_\square, \gamma_m^\square > = \sum\{\alpha_n(\frac{1}{2}\gamma_m^{n_1} + \frac{1}{2}\gamma_m^{\varphi(n)}) : n \in N\} = \begin{cases} \frac{1}{2} & , \text{if } \ m = n_1 \\ \alpha_{m'} & , \text{if } \ m = \varphi(m'), \end{cases}$$

whence $h_a(< \alpha_*, x^* >) = < \alpha_*, h_a^N(x^*) >$. So h_a is a morphism.

(iii). This follows similarly to (ii) with these choices: $\varphi : N \longrightarrow N\backslash\{n_2\}$ a bijection; $z^* \in (N, X)$ given by $z^{n_2} := x$ and $x^{\varphi(n)} := a^n, n \in N; \beta_*^n := \frac{1}{2}\delta_*^{\varphi(n)} + \frac{1}{2}\delta_*^{n_2}, n \in N; \gamma_*^n := \delta_*^n, n \in N\backslash\{n_1\}$, and $\gamma_{n_2}^{n_1} = 0$ and $\gamma_m^{n_1} = a_{m'}, m = \varphi(m')$. □

Proposition 2.2. *Let X be any $Cv_{C,N}$-convex module and let $a \in X$ be such that h_a is a bijection. Define the binary composition $\dot{+} : X \times X \longrightarrow X$ by $h_a(x\dot{+}x') := \frac{1}{2}x + \frac{1}{2}x'$. Then $(X, \dot{+})$ is a commutative monoid (with neutral element a) that is uniquely divisible by 2 and hence is a D_0-semimodule. Moreover for any $k \in \mathbb{N}$ and $x_1, \ldots, x_2 \in X$*

$$h_a^*(x_1\dot{+}\ldots\dot{+}x_{2^k}) = \frac{1}{2^k}x_1 + \cdots + \frac{1}{2^k}x^{2^k}.$$

Proof.
(0). $h_a(a\dot{+}x) = \frac{1}{2}a + \frac{1}{2}x = h_a(x)$ and thus $a\dot{+}x = x$.
(i). $h_a(x\dot{+}x') = \frac{1}{2}x + \frac{1}{2}x' = \frac{1}{2}x' + \frac{1}{2}x = h_a(x'\dot{+}x)$ and thus $x\dot{+}x' = x'\dot{+}x$.
(ii). Due to (2.1) and [1], 3.6, resp. [2], 4.5,

$$\begin{aligned} h_a^2((x\dot{+}x')\dot{+}x'') &= h_a(\frac{1}{2}(x\dot{+}x') + \frac{1}{2}x'') = \frac{1}{2}h_a(x\dot{+}x') + \frac{1}{2}h_a(x'') = \\ &= \frac{1}{2}(\frac{1}{2}x + \frac{1}{2}x') + \frac{1}{2}(\frac{1}{2}a + \frac{1}{2}x'') = \frac{1}{2}(\frac{1}{2}a + \frac{1}{2}x) + \frac{1}{2}(\frac{1}{2}x' + \frac{1}{2}x'') = \\ &= \frac{1}{2}h_a(x) + \frac{1}{2}h_a(x'\dot{+}x'') = h_a(\frac{1}{2}x + \frac{1}{2}(x'\dot{+}x'')) = h_a^2(x\dot{+}(x'\dot{+}x'')). \end{aligned}$$

(iii). $h_a(x\dot{+}x) = \frac{1}{2}x + \frac{1}{2}x = x$. So each $x \in X$ is uniquely divisible by 2.
The final formula follows by an obvious induction argument. □

Let X be any $Cv_{C,N}$-convex module. On $X \times \mathbb{N}_0$ define the relation "$\sim$" by "$(x, n) \sim (x', n')$" if and only if "there is a $p \in \mathbb{N}_0$ with $h_a^{n'+p}(x) = h_a^{n+p}(x')$".

Lemma 2.3. *The relation $\sim$ on $X \times \mathbb{N}_0$ is an equivalence relation.*

Proof. Straight forward. □

Given the $Cv_{C,N}$-convex module X we define the composition $< \; , \; >. \; Cv_{C,N} \times (N, X \times \mathbb{N}_0) \longrightarrow X \times \mathbb{N}_0$ as follows. Let $(x, m)^* \in (N, X \times \mathbb{N}_0)$ and denote by $x^* \in (N, X)$ resp. $m^* \in (N, \mathbb{N}_0)$ the composite of $(x, m)^*$ with the appropriate projection of $X \times \mathbb{N}_0$ to its factors. Let furthermore $\alpha_* \in Cv_{C,N}$, put $s := s_{a_*,m^*} := \sum\{m^n : n \in \text{supp } \alpha_*\}$ and $s^n := s - m^n, n \in \text{supp } \alpha_*$, resp. $s^n = 0, n \notin \text{supp } \alpha_*$. Denote by $h_a^{\alpha_*,m^*}(x^*)$ the map $N \ni n \mapsto h_a^{s^n}(x^n) \in X$ and let

$$< \alpha_*, (x, m)^* > := (< \alpha_*, h_a^{\alpha_*,m^*}(x^*) >, s_{\alpha_*,m^*}).$$

Lemma 2.4. *Let $\alpha_* \in Cv_{C,N}$ and $\beta_*^\square \in (N, Cv_{C,N})$. The for any $Cv_{C,N}$-convex module X and any $(x, m)^* \in (N, X \times \mathbb{N}_))$*

(i) $\quad < \delta_*^n, (x, m)^* > = (x^n, m^n), \qquad n \in N,$
(ii) $\quad < \alpha_\square, < \beta_*^\square, (x, m)^* \gg \sim \ll \alpha_\square, \beta_*^\square >, (x, m)^* > .$

Proof.
(i). This is an immediate consequence of the definitions involved.
(ii). Put

$$s^q := \sum\{m^n : n \in \text{supp } \beta_*^q\} \quad \text{and} \quad s := \sum\{s^q : q \in \text{supp } \alpha_*\}.$$

Then for any $q \in N$

$$< \beta_*^q, (x,m)^* > = (< \beta_*^q, h_a^{\beta_*^q, m^*}(x^*) >, s^q >$$

and by (2.1), (ii),

$$\begin{aligned} < \alpha_\square, < \beta_*^\square, (x,m)^* \gg &= (< \alpha_\square, h_a^{\alpha_*, s^*}(< \beta_*^\square, h_a^{\beta_*^\square, m^*}(x^*) >) >, s) = \\ &= (< \alpha_\square, < \beta_*^\square, h_a^{\alpha_*, s^*}(h_a^{\beta_*^\square, m^*}(x^*)) \gg, s) = \\ &= (\ll \alpha_\square, \beta_*^\square >, h_a^{\alpha_*, s^*}(h_a^{\beta_*^\square, m^*}(x^*)) >, s). \end{aligned}$$

On the other hand, when putting $\bar{s} := \sum\{m^n : n \in \text{supp } < \alpha_\square, \beta_*^\square >\}$,

$$\ll \alpha_\square, \beta_*^\square >, (x,m)^* > = (\ll \alpha_\square, \beta_*^\square >, h_a^{<\alpha_\square, \beta_*^\square >, m^*}(x^*) >, \bar{s}).$$

Since supp $< \alpha_\square, \beta_*^\square > = \cup\{\text{supp } \beta_*^q : q \in \text{supp } \alpha_*\}$ we have $n \in$ supp $<\alpha_\square, \beta_*^\square >$ if an only if there is a $q \in$ supp α_* with $n \in$ supp β_*^q. For such an n the value of $h_a^{\alpha_*, s^*}(h_a^{\beta_*^\square, m^*}(x^*))$ at n equals $h_a^{s-s^q}(h_a^{s^q - m^n}(x^n)) = h_a^{s-m^n}(x^n)$, while the value of $h_a^{<\alpha_\square, \beta_*^\square > m^*}(x^*)$ equals $h_a^{\bar{s}-m^n}(x^n)$. Hence (2.1), (ii), implies our assertion (ii). □

Lemma 2.5. *Let $\alpha_* \in Cv_{C,N}$ and let X be any $Cv_{C,N}$-convex module. If $(x,m)^* \in (N, X \times \mathbb{N}_0)$ and $(\bar{x}, \overline{m})^* \in (N, X \times \mathbb{N}_0)$ satisfy $(x,m)^* \sim (\bar{x}, \overline{m})^*$, which by definition means $(x^n, m^n) \sim (\bar{x}^n, \overline{m}^n)$ for all $n \in N$, then $< \alpha_*, (x,m)^* > \sim < \alpha_*, (\bar{x}, \overline{m})^* >$.*

Proof. Since supp α_* is finite there is a $p \in \mathbb{N}_0$ with $h_a^{\overline{m}^n + p}(x^n) = h_a^{m^n + p}(\bar{x}^n)$ for all $n \in$ supp α_*. Put $s := \sum\{m^n : n \in \text{supp } \alpha_*\}$ and $\bar{s} := \sum\{\overline{m}^n : n \in \text{supp } \alpha_*\}$. Then

$$\begin{aligned} & h_a^{p+\bar{s}}(h_a^{s-m^n}(x^n)) = h_a^{p+\bar{s}+s-m^n}(x^n) = h_a^{p+\overline{m}^n+\bar{s}-\overline{m}^n+s-m^n}(x^n) = \\ & = h_a^{p+m^n+\bar{s}-\overline{m}^n+s-m^n}(\bar{x}^n) = h_a^{p+s+\bar{s}-\overline{m}^n}(\bar{x}^n) = h_a^{p+s}(h_a^{\bar{s}-\overline{m}^n}(\bar{x}^n)). \end{aligned}$$

Due to (2.1), (ii), we obtain

$$\begin{aligned} & h_a^{p+\bar{s}}(< \alpha_*, h_a^{\alpha_*, m^*}(x^*) >) = < \alpha_*, h_a^{p+\bar{s}}(h_a^{\alpha_*, m^*}(x^*)) > = \\ & = < \alpha_*, h_a^{p+s}(h_a^{\alpha_*, \overline{m}^*}(\bar{x}^*)) > = h_a^{p+s}(\alpha_*, h_a^{\alpha_*, \overline{m}^*}(\bar{x}^*) >) \end{aligned}$$

and our claim follows. □

Proposition 2.6. *There is a unique composition $< \quad , \quad >: Cv_{C,N} \times (N, X \times \mathbb{N}_0/\sim) \longrightarrow X \times \mathbb{N}_0/\sim$ that makes $X \times \mathbb{N}_0/\sim$ a $Cv_{C,N}$-convex module and satisfies*

$$\pi(< \alpha_*, (x,m)^* >) = < \alpha_*, \pi^N((x,m)^*) >$$

$$, \text{ for all } \alpha_* \in Dv_{C,N}, (x,m)^* \in (N, X \times \mathbb{N}_0),$$

where π is the quotient map $X \times \mathbb{N}_0 \longrightarrow X \times \mathbb{N}_0/\sim$.

Proof. (2.4) and (2.5). □

Denote the map $X \ni x \mapsto \pi(x,0) \in X \times \mathbb{N}_0/\sim$ by j_X.

Proposition 2.7. *For any $Cv_{C,N}$-convex module X the map $j_X : X \longrightarrow X \times \mathbb{N}_0/\!\sim$ is a morphism of $Cv_{C,N}$-convex modules.*

Proof. Let $\alpha_* \in Cv_{C,N}$ and $x^* \in (N,X)$. Denote by $(x,0)^*$ the map $N \ni n \mapsto (x^n, 0) \in X \times \mathbb{N}_0$. Then by (2.6)

$$\begin{aligned} <\alpha_*, j_X^N(x^*)> &= <\alpha_*, \pi^N((x,0)^*)> = \pi(<\alpha_*, (x,0)^*>) \\ &= \pi(<\alpha_*, x^*>, 0) = j_X(<\alpha_*, x^*>). \end{aligned}$$

□

In the setting of (2.7) we denote, for any $a \in X$, the map $h_{j_X(a)} : X \times \mathbb{N}_0/\!\sim \longrightarrow X \times \mathbb{N}_0/\!\sim$ by H_a. With this notation we have

Corollary 2.8. *For any $Cv_{C,N}$-convex module X and any $a \in X$ the diagram*

$$\begin{array}{ccc} X & \xrightarrow{h_a} & x \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_X} \\ X \times \mathbb{N}_0/\!\sim & \xrightarrow{H_a} & X \times \mathbb{N}_0/\!\sim \end{array}$$

commutes.

Proof. Since j_X is a morphism by (2.7) we obtain

$$j_X \circ h_a(x) = j_X(\tfrac{1}{2}a + \tfrac{1}{2}x) = \tfrac{1}{2}j_X(a) + \tfrac{1}{2}j_X(x) = H_a \circ j_X(x).$$ □

Proposition 2.9. *For any $Cv_{C,N}$-convex module X and any $a \in X, H_a$ is a bijection. In particular, $H_a(\pi(x,n)) = \pi(x, n-1)$ for all $x \in X$ and $n \in \mathbb{N}$.*

Proof. Let $x \in X$ and $n \in \mathbb{N}$. Then by (2.6)

$$\begin{aligned} H_a(\pi(x,n)) &= \frac{1}{2}\pi(a,0) + \frac{1}{2}\pi(x,n) = \pi(\frac{1}{2}(a,0) + \frac{1}{2}(x,n)) = \\ &= \pi(\frac{1}{2}h_a^n(a) + \frac{1}{2}(x,n)) = \pi(\frac{1}{2}a + \frac{1}{2}x, n) = \pi(h_a(x), n) = \pi(x, n-1). \end{aligned}$$

This formula proves that H_a is a surjection. Next let $H_a(\pi(x,n)) = H_a(\pi(\bar{x},\bar{n}))$. Then the preceding formulae show that $H_a(\pi(x,n)) = \pi(h_a(x), n), x \in X$ and $n \in \mathbb{N}_0$, whence we have $\pi(h_a(x), n) = \pi(h_a(\bar{x}), \bar{n})$ that is $h_a^{\bar{n}+p}(h_a(x)) = h_a^{n+p}(h_a(\bar{x}))$ for some $p \in \mathbb{N}_0$. Thus $h_a^{\bar{n}+p+1}(x) = h_a^{n+p+1}(\bar{x})$ or $\pi(x,n) = \pi(\bar{x},\bar{n})$. So H_a is also an injection and therefore a bijection. □

Due to (2.2) and (2.9), $(X \times \mathbb{N}_0/\sim, \dot{+})$ is a D_0-semimodule. This D_0-semimodule is cancellable under certain hypotheses involving X. They are stated in Proposition 2.11.

Definition 2.10. Let X be any $Cv_{C,N}$-convex module. Then X is called

(i) *cancellable* at a if for all $x', x'' \in X$, $\frac{1}{2}a + \frac{1}{2}x' = \frac{1}{2}a + \frac{1}{2}x''$ implies $x' = x''$;
(ii) *weakly cancellable* at a if for all $x, x', x'' \in X$ and $n, n', n'', p \in \mathbb{N}_0$, $\frac{1}{2}h_a^{n+n'+n''+p}(x) + \frac{1}{2}h_a^{2n+n''+p}(x') = \frac{1}{2}h_a^{n+n'+n''+p}(x) + \frac{1}{2}h_a^{2n+n'+p}(x'')$ implies $h_a^{n''+q}(x') = h_a^{n'+q}(x'')$ for some $q \in \mathbb{N}_0$.

X is said to be *cancellable* if it is cancellable at any $a \in X$. □

Obviously cancellability at a of X implies weak cancellability at a of X.

Proposition 2.11. *Let X be any $Cv_{C,N}$-convex module. Then*

(i) *the D_0-semimodule $(X \times \mathbb{N}_0/\sim, \dot{+})$ is cancellable if and only if X is weakly cancellable at a;*
(ii) *X is cancellable at a if and only if $j_X; X \longrightarrow X \times \mathbb{N}_0/\sim$ is an injection.*

Proof. (i). $\pi(x, n)\dot{+}\pi(x', n') = \pi(x, n)\dot{+}\pi(x'', n'')$ is equivalent with $H_a(\pi(x, n)\dot{+}\pi(x', n')) = H_a(\pi(x, n)\dot{+}\pi(x'', n''))$ due to (2.9). Since

$$H_a(\pi(x, n)\dot{+}\pi(x', n')) = \pi(\frac{1}{2}h_a^{n'}(x) + \frac{1}{2}h_a^{n}(x'), n + n')$$

due to (2.6), the initial equation is equivalent with

$$h_a^{n+n''+p}(\frac{1}{2}h_a^{n'}(x) + \frac{1}{2}h_a^{n}(x')) = h_a^{n+n'+p}(\frac{1}{2}h_a^{n''}(x) + \frac{1}{2}h_a^{n}(x'')),$$

for some $p \in \mathbb{N}_)$, which by (2.1), (ii), is the same as

$$(*) \qquad \frac{1}{2}h_a^{n+n'+n''+p}(x) + \frac{1}{2}h_a^{2n+n''+p'}(x') = \frac{1}{2}h_a^{n+n'+n''+p}(x) + \frac{1}{2}h_a^{2n+n'+p}(x'').$$

Hence weak cancellability leads to $h_a^{n''+q}(x') = h_a^{n'+q}(x'')$ for some $g \in \mathbb{N}_0$ and thus to $(x', n') \sim (x'', n'')$ or $\pi(x', n') = \pi(x'', n'')$. Conversely, cancellability of the D_0-semimodule $X \times \mathbb{N}_0/\sim$ means that $(*)$ implies $\pi(x', n') = \pi(x'', n'')$ and thus $h_a^{n''+q}(x') = h_a^{n'+q}(x'')$ for some $q \in \mathbb{N}_0$. Hence X is weakly cancellable.
(ii). $j_X(x') = j_X(x'')$ is equivalent with $(x', 0) \sim (x'', 0)$ and hence with $h_a^p(x') = h_a^p(x'')$ for some $p \in \mathbb{N}_0$. Since $h_a^p(x') = \frac{1}{2}a + \frac{1}{2}h_a^{p-1}(x')$, cancellability at a of X implies $h_a^{p-1}(x') = h_a^{p-1}(x'')$ and thus $x' = x''$ by induction, which means that j_X is an injection. Conversely if j_X is an injection and $\frac{1}{2}a + \frac{1}{2}x' = \frac{1}{2}a + \frac{1}{2}x''$ then we have $h_a(x') = h_a(x'')$, that is $(x', 0) \sim (x'', 0)$ and thus $j_X(x') = j_X(x'')$, which leads to $x' = x''$ and therefore to the cancellability at a of X. □

3. The Functor $O_{D_0,N} : D_0$-Mod $\longrightarrow Cv_{D_0,N}$-Cmod

Let M be a C-semimodule. For $\alpha_* \in Cv_{C,N}$ and $m^* \in (N, M)$ let

$$< \alpha_*, m^* >:= \sum\{\alpha_n m^n : n \in \text{supp } \alpha_*\}.$$

Note that in the finite sum on the right side supp α_* can be replaced by any finite set containing supp α_*.

Lemma 3.1. *Let M be any C-semimodule. Then $Cv_{C,N} \times (N, M) \ni (\alpha_*, m^*) \mapsto < \alpha_*, m^* > \in M$ makes M a $Cv_{C,N}$-convex module $O_{C,N}(M)$.*

Proof. Straight forward. □

Obviously the assignment to M of $O_{C,N}(M)$ and to C-homomorphisms $f : M \longrightarrow M'$ of $O_{C,N}(f) := f$ is a functor $O_{C,N} : C$-**Smod**$\longrightarrow Cv_{C,N}$-**Cmod**. Its restriction to the full subcategory C-**Mod** of C-**Smod** generated by the class of all C-modules is also denoted by $O_{C,N}$. Note that C-**Mod** is a reflective subcategory of C-**Smod** and that the reflector R_C assigns to each C-semimodule M the C-module $M^2/\underset{B}{\sim}$, where $\underset{B}{\sim}$ is the *Bourne relation*, the C-semimodule congruence relation, "$(m,m')\underset{B}{\sim}(\overline{m},\overline{m}')$" if and only if "there is an $m_0 \in M$ with $m_0+m+\overline{m}' = m_0+\overline{m}+m'$". The reflection r_C assigns to each M the composition of $M \ni m \mapsto (0,m) \in M^2$ with the quotient map $M^2 \longrightarrow M^2/\underset{B}{\sim}$.

Lemma 3.2. *Let X be any $Cv_{C,N}$-convex module and let $a \in X$ be such that h_a is a bijection. Then $O_{D,N}((X,\dot{+})) = \rho_*(X)$.*

Proof. In this proof we shall denote the operation of $r \in D_0$ on the element x of the D_0-semimodule $(X,\dot{+})$ by $\dot{r}x$. From the formula in (2.2) we obtain for all $k \in \mathbb{N}, x_1,\dots,x_p \in X$, and $n_1,\dots,n_p \in \mathbb{N}_0$ with $n_1+\dots+n_p = 2^k$

$$\begin{aligned}\frac{\dot{n}_1}{2^k}x_1\dot{+}\dots\dot{+}\frac{\dot{n}_p}{2^k}x_p &= \frac{\dot{1}}{2k}(\dot{n}_1x_1\dot{+}\dots\dot{+}n_px_p) = h_a^k(\dot{n}_1x_1\dot{+}\dots\dot{+}\dot{n}_px_p) = \\ &= \frac{n_1}{2^k}x_1+\dots+\frac{n_p}{2^k}x_p = \rho(\frac{n_1}{2^k})x_1+\dots+\rho(\frac{n_p}{2^k})x_p,\end{aligned}$$

where the last sum is actually the composition in the $Cv_{D_0,N}$-convex module $\rho_*(X)$. □

Lemma 3.3. *Let M be any C-semimodule and let $a \in M$ be such that $h_a : M \longrightarrow M$ is a bijection. Then $(O_{C,N}(M),\dot{+}) = \rho_*(M)$.*

Proof. See proof of (3.2). □

Theorem 3.4. *The functor $O_{D_0,N} : D_0$-**Mod** $\longrightarrow Cv_{D_0,N}$-**Cmod** has a left adjoint.*

Proof. Although the existence of a left adjoint of $O_{D_0,N}$, and indeed of $O_{C,N} : C$-**Mod** $\longrightarrow Dv_{C,N}$-**Cmod**, can be obtained from general principles we wish to present an explicit construction of a left adjoint of $O_{D_0,N}$ based on the halfing morphism. Let X be a $Cv_{D_0,N}$-convex module, M a D_0-module, and $f : X \longrightarrow O_{D_0,N}(M)$ a morphism of $Cv_{D_0,N}$-convex modules. Choose $a \in X$ and put $m_0 := f(a)$. Define $\bar{f} : X \times \mathbb{N}_0/\sim \longrightarrow M$ by

$$\bar{f}(\pi(x,n)) := h_{m_0}^{-n}(f(x)) - m_0.$$

We claim that this definition makes sense. Firstly, since M is a D_0-module h_{m_0} is a bijection and indeed $h_{m_0}^{-1}(m) = 2m - m_0, m \in M$. Secondly, if $\pi(x,n) = \pi(x',n')$ then $h_a^{n'+p}(x) = h_a^{n+p}(x')$ for some $p \in \mathbb{N}_0$. Due to (2.1), (ii), we have

$$f(h_a(x)) = f(\frac{1}{2}a+\frac{1}{2}x) = \frac{1}{2}f(a)+\frac{1}{2}f(x) = h_{m_0}(f(x)).$$

Hence

$$h_{m_0}^{n'+p}(f(x)) = f(h_a^{n'+p}(x)) = f(h_a^{n+p}(x')) = h_{m_0}^{n+p}(f(x)$$

and thus $h_{m_0}^{-n}(f(x)) = h_{m_0}^{-n'}(f(x'))$. Therefore $\bar{f}(\pi(x,n)) = \bar{f}(\pi(x',n'))$. Next we show that $\bar{f}$ is a homomorphism of D_0-semimodules. Using (2.1), (ii), and the formula in (2.9) we obtain

$$\begin{aligned}
\bar{f}(\pi(x,n)\dot{+}\pi(x',n')) &= \bar{f}(H_a^{-1} \circ H_a(\pi(x,n)\dot{+}\pi(x',n'))) = \\
&= \bar{f}(H_a^{-1}(\pi(\frac{1}{2}h_a^{n'}(x) + \frac{1}{2}h_a^{n}(x), n+n') = \\
&= \bar{f}(\pi(\frac{1}{2}h_a^{n'}(x) + \frac{1}{2}h_a^{n}(x'), n+n'+1)) = \\
&= h_{m_0}^{-n-n'-1}(f(\frac{1}{2}h_a^{n'}(x) + \frac{1}{2}h_a^{n}(x'))) - m_0 = \\
&= h_{m_0}^{-n-n'-1}(\frac{1}{2}f(h_a^{n'}(x)) + \frac{1}{2}f(h_a^{n}(x'))) - m_0 = \\
&= h_{m_0}^{-n-n'-1}(\frac{1}{2}h_{m_0}^{n'}(f(x)) + \frac{1}{2}h_{m_0}^{n}(f(x'))) - m_0 = \\
&= h_{m_0}^{-1}(\frac{1}{2}h_{m_0}^{-n}(f(x)) + \frac{1}{2}h_{m_0}^{-n'}(f(x'))) - m_0 = \\
&= h_{m_0}^{-1}(\frac{1}{2}\bar{f}(\pi(x,n) + \frac{1}{2}\bar{f}(\pi(x',n') + m_0) - m_0 = \\
&= \bar{f}(\pi(x,n)) + \bar{f}(\pi(x',n')) + 2m_0 - m_0 - m_0 = \\
&= \bar{f}(\pi(x,n)) + \bar{f}(\pi(x',n')).
\end{aligned}$$

Moreover, from the proof of (2.9),

$$\begin{aligned}
&\bar{f}(H_a(\pi(x,n) = \bar{f}(\pi(h_a(x),n)) = h_{m_0}^{-n}f(h_a(x)) - m_0 = \\
&= h_{m_0}(h_{m_0}^{-n}(f(x))) - m_0 = \frac{1}{2}h_{m_0}^{-n}(f(x)) - \frac{1}{2}m_0 = \frac{1}{2}\bar{f}(\pi(x,n)).
\end{aligned}$$

Put $\bar{g}(\pi(x,n)) := \bar{f}(\pi(x,n)) + m_0$. Since $\bar{f}$ is a morphism $O_{D_0,N}(X \times \mathbb{N}_0/\sim, \dot{+}) \longrightarrow O_{D_0,N}(M)$ of $Cv_{D_0,N}$-convex modules, denoted by $O_{D_0,N}(\bar{f}), \bar{g}$ is also a morphism $O_{D_0,N}(X \times \mathbb{N}_0/\sim, \dot{+}) \longrightarrow O_{D_0,N}(M)$ of $Cv_{D_0,N}$-convex modules. In addition it satisfies $f = \bar{g} \circ j_X$. Denote the reflection $(X \times \mathbb{N}_0/\sim, \dot{+}) \longrightarrow (X \times \mathbb{N}_0(\sim, \dot{+})^2/\underset{B}{\sim}$ by r'_X. Then there is a unique homomorphism $f' : (X \times \mathbb{N}_0/\sim, \dot{+})^2/\underset{B}{\sim} \longrightarrow M$ of D_0-modules with $\bar{f} = f' \circ r'_X$. Put $r'_X \circ j_X(a,0) =: a'$ and let $g' := f' + c_{m_0}$, where c_{m_0} stands for the constant map with value m_0 and some appropriate domain. Clearly g' is a morphism of $Cv_{D_0,N}$-convex modules from $O_{D_0,N}((X \times \mathbb{N}_0/\sim, \dot{+})^2/\underset{B}{\sim}) \longrightarrow O_{D_0,N}(M)$ and we obtain (on the set-level)

$$\begin{aligned}
f &= \bar{g} \circ j_X = (\bar{f} + c_{m_0}) \circ j_X = (f' \circ r'_X + c_{m_0}) \circ j_X = f' \circ r'_X \circ j_X + c_{m_0} = \\
&= (g' - c_{m_0}) \circ r'_X \circ j_X + m_0 = g' \circ r'_X \circ j_X.
\end{aligned}$$

$\eta_X := r'_X \circ j_X$ is a morphism of $Cv_{D_0,N}$-convex modules from X to $O_{D_0,N}((X \times \mathbb{N}_0/\sim, \dot{+})^2/\underset{B}{\sim})$. We claim that η_X is a universal arrow. Already we have the factorization of f through η_X as f', being a homomorphism of D_0-semimodules, is also a morphism $O_{D_0,N}(f')$ between the associated $Cv_{D_0,N}$-convex modules. In order to prove uniqueness of the factorization it suffices to prove the uniqueness of $\overline{g}$ in terms of f. So let $\bar{h}: O_{D_0,N}((X\times \mathbb{N}_0/\sim, \dot{+})) \longrightarrow O_{D_0,N}(M)$ be a morphism with $f = \bar{h}\circ j_X$. Then $\bar{h}(\pi(x,0)) = f(x), x \in X$, whence $\bar{h}$ is uniquely determined by f on $\{\pi(x,0) : x \in X\}$. Let $n \in N$ and let $x \in X$. Since $\pi(x,n)\dot{+}\pi(a,0) = \pi(x,n)$ and $H_a(\pi(x,n)) = \pi(x,n-1)$ we have

$$\begin{aligned}\bar{h}(\pi(x,n-1)) &= \bar{h}(H_a(\pi((x,n))) = \bar{h}(H_a(\pi(x,n)\dot{+}\pi(a,0))) = \\ &= \bar{h}(H_a(\pi(x,n))\dot{+}H_a(\pi(a,0))) = \frac{1}{2}\bar{h}(\pi(x,n)) + \frac{1}{2}\bar{h}(\pi(a,0)) = \\ &= \frac{1}{2}\bar{h}(\pi(x,n)) + \frac{1}{2}m_0\end{aligned}$$

or

$$\bar{h}(\pi(x,n)) = 2\bar{h}(\pi(x,n-1)) - m_0 = h_{m_0}^{-1}(\bar{h}(\pi(x,n-1)).$$

Hence an obvious induction argument shows that $\bar{h}$ is unique in terms of f, that is $\bar{h} = \bar{g}$. From $\bar{g}$ we recover uniquely $\bar{f}$ as $\bar{f} = \bar{g} - c_{m_0}$, and the first part of the proof shows that $\bar{f}$ is a homomorphism. Thus g' is uniquely determined by f. □

Definition 3.5. Let X be any $Cv_{D_0,N}$-convex module. Then X is said to be *imbeddable* in a D_0-module (resp. D_0-semimodule) if and only if there is a D_0-module (resp. D_0-semimodule) M and an injective morphism $X \longrightarrow O_{D_0,N}(M)$ of $Cv_{D_0,N}$-convex modules.

Proposition 3.6. *Let X be any $Cv_{D_0,N}$-convex module. Then X is imbeddable in a D_0-module if and only if X is cancellable.*

Proof. X is imbeddable in a D_0-module if and only if η_X is an injection. Let $x, x' \in X$. Then $\eta_X(x) = \eta_X(x')$ means that $(0,\pi(x,0))\underset{B}{\sim}(0,\pi(x',0))$, that is that $\pi(y,n)\dot{+}\pi(x,0) = \pi(y,n)\dot{+}\pi(x',0)$ for some $(y,n) \in X\times\mathbb{N}_0$. The latter is equivalent with $H_a(\pi(y,n)hspace* -0.3pt\dot{+}\pi(x,0)) = H_a(\pi(y,n)\dot{+}\pi(x',0))$ and hence with $\pi(\frac{1}{2}y + \frac{1}{2}h_a^n(x), n) = \pi(\frac{1}{2}y + \frac{1}{2}h_a^n(x'), n)$ and thus with $h_a^{n+p}(\frac{1}{2}y + \frac{1}{2}h_a^n(x)) = h_a^{n+p}(\frac{1}{2}y + \frac{1}{2}h_a^n(x'))$ for some $p \in \mathbb{N}_0$. By (2.1), (ii), the last equality is the same as

$$(*) \qquad \frac{1}{2}h_a^{n+p}(y) + \frac{1}{2}h_a^{2n+p}(x) = \frac{1}{2}h_a^{n+p}(y) = \frac{1}{2}h_a^{2n+p}(x').$$

So if X is cancellable and $(*)$ holds then $h_a^{2n+p}(x) = h_a^{2n+p}(x')$ and therefore $x = x'$, which means that η_X is an injection and consequently X is imbeddable in some D_0-module. Conversely if X is imbeddable in some D_0-module and we have $\frac{1}{2}y + \frac{1}{2}x = \frac{1}{2}y + \frac{1}{2}x'$ then $(*)$ is satisfied for $n = p = 0$, whence we obtain $\eta_X(x) = \eta_X(x')$ and therefore $x = x'$, which means cancellability. □

Note that (2.11), (ii), furnishes a sufficient condition for imbeddability of a $Cv_{D_0,N}$-convex module in a D_0-semimodule.

4. MP and the Category of $Cv_{D_0,N}$-convex Modules

Definition 4.1. A *midpoint adgebra* with underlying set X is this set together with a single binary composition $X \times X \ni (x,y) \mapsto xy \in X$ satisfying

(i) $\qquad x^2 = x \qquad$, for all $x \in X$,

(ii) $xy = yx$, for all $x, y \in X$,

(ii) $(xy)(uv) = (xu)(yv)$, for all $x, y, y, v \in X$.

A morphism $f : X \longrightarrow X'$ of midpoint algebras is a map f from X to X' that takes products to products.

The category of midpoint algebras and their morphisms, composition being the set-theoretical one, is denoted by **MP**.

Let X be a midpoint algebra and let $x_1, \ldots, x_n \in X$, where $n = 2^k$ and $k \in \mathbb{N}_0$. Define inductively

$$(x) := x \qquad (x_1 \ldots x_n) := (x_1 \ldots x_{\frac{n}{2}})(x_{\frac{n}{s}+1} \ldots x_n).$$

Lemma 4.2. *Let $n = 2^k, k \in \mathbb{N}$, and let π be a permutation of $\{1, \ldots, n\}$. Let furthermore X be a midpoint algebra and $x_1, \ldots, x_n \in X$. Then*

$$(x_1 \ldots x_n) = (x_{\pi(1)} \ldots x_{\pi(n)}).$$

Proof. By definition this is valid for $k = 1$. Suppose that statement is valid for k. Let $m := 2^{k+1}$ and let $x_1, \ldots, x_n \in X$. Suppose that τ is the transposition of $\{1, \ldots, m\}$ that interchanges p and q. We wish to prove the formula in (4.2) for τ rather than π. We may assume $p < q$. If $q \leq \frac{m}{2}$ or $\frac{m}{2} + 1 \leq p$ then the formula is valid by induction hypothesis. So let $p \leq \frac{m}{2}$ and $\frac{m}{2} + 1 \leq q$. Denote $(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{\frac{m}{2}})$ by $(y_1, \ldots, y_{\frac{m}{2}-1})$ and $(x_{\frac{m}{2}+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_m)$ by $(z_1, \ldots, z_{\frac{m}{2}-1})$. By induction hypothesis

$$(x_1 \ldots x_{\frac{m}{2}}) = (x_p y_1 \ldots y_{\frac{m}{2}-1}) \quad \text{and} \quad (x_{\frac{m}{2}+1} \ldots x_m) = (x_q z_1 \ldots z_{\frac{m}{2}-1}).$$

Hence

$$\begin{aligned}
(x_1 \ldots x_m) &= (x_p y_1 \ldots y_{\frac{m}{2}-1})(x_q z_1 \ldots z_{\frac{m}{2}-1}) = \\
&= ((x_p y_1 \ldots y_{\frac{m}{4}-1})(y_{\frac{m}{4}} \ldots y_{\frac{m}{2}-1}))((x_q z_1 \ldots z_{\frac{m}{4}-1})(z_{\frac{m}{4}} \ldots z_{\frac{m}{2}-1})) = \\
&= ((x_p y_1 \ldots y_{\frac{m}{4}-1})(x_q z_1 \ldots z_{\frac{m}{4}-1}))((y_{\frac{m}{4}} \ldots y_{\frac{m}{2}-1})(z_{\frac{m}{4}} \ldots z_{\frac{m}{2}-1})) = \\
&= ((x_q y_1 \ldots y_{\frac{m}{4}-1})(x_p z_1 \ldots z_{\frac{m}{4}-1}))((y_{\frac{m}{4}} \ldots y_{\frac{m}{2}-1})(z_{\frac{m}{4}} \ldots z_{\frac{m}{2}-1})) = \\
&= ((x_p y_1 \ldots y_{\frac{m}{4}-1})(y_{\frac{m}{4}} \ldots y_{\frac{m}{2}-1}))((x_p z_1 \ldots z_{\frac{m}{4}-1})(z_{\frac{m}{4}} \ldots z_{\frac{m}{2}-1})) = \\
&= (x_1 \ldots x_{p-1} x_q x_{p+1} \ldots x_{\frac{m}{2}})(x_{\frac{m}{2}+1} \ldots x_{q-1} x_p x_{q+1} \ldots x_m) = \\
&= (x_{\tau(1)} \ldots x_{\tau(\frac{m}{2})} \ldots x_{\tau(m)}).
\end{aligned}$$

Since every permutation is a composition of transpositions the general formula follows. □

Lemma 4.3. *Let $f : X \longrightarrow X'$ be a morphism of midpoint algebras and let $x_1, \ldots, x_n \in X$, with $n = 2^k$ and $k \in \mathbb{N}$. Then*

$$f(x_1 \ldots x_n) = (f(x_1) \ldots f(x_n)).$$

Proof. By obvious induction. □

Let $\alpha_* \in Cv_{D_0,N}$. Since α_* has finite support there is an $\ell \in \mathbb{N}_0$ and $a_n \in \mathbb{N}_0, n \in N$ with

4.4. $\alpha_n = \frac{a_n}{2^\ell} \qquad , n \in N.$
Obviously we have $\sum a_n = 2^\ell$. Suppose we are also given $x^* \in (N, X)$. Put supp $\alpha_* = \{k_1, \dots, k_q\}$ and form

$$(< a_{k+_1} x^{k_1} > \cdots < a_{k_q} x^{k_q} >) := (\underbrace{x^{k_1} \dots x^{k_1}}_{a_{k_1}-\text{times}} \dots (\underbrace{x^{k_q} \dots x^{k_q}}_{a_{k_q}-\text{times}}).$$

Lemma 4.5. *Let $\alpha_* \in Cv_{D_0,N}$, let X be midpoint algebra and $x^* \in (N, X)$. Then $(< a_{k_1} x^{k_1} > \cdots < a_{k_q} x^{k_q} >)$ is independent of the presentation* (4.4) *of α_*.*

Proof. We have to show the independence of the choice of ℓ in (4.4). It suffices to show that the presentation

(4.4′) $\quad a_n = \frac{a_n}{2^\ell} = \frac{a'_n}{2^{\ell+1}} \qquad , n \in N.$

leads to $(< a'_{k_1} x^{k_1} > \cdots < a'_{k_q} x^{k_q} >) = (< a_{k_1} x^{k_1} > \cdots < a_{k_q} x^{k_q} >)$. Since $a'_n = 2a_n, n \in N$, we have by (4.1), (i),

$$\begin{aligned} &(< a'_{k_1} x^{k_1} > \cdots < a'_{k_q} x^{k_q} >) = \\ &(< a_{k_1} x^{k_1} > \cdots < a_{k_q} x^{k_q} >)(< a_{k_1} x^{k_1} \cdots < a_{k_q} x^{k_q} >) = \\ &(< a_{k_1} x^{k_1} > \cdots < a_{k_q} x^{k_q} >). \end{aligned}$$

□

Theorem 4.6. *The category* **MP** *is isomorphic to the category $Cv_{D_0,N}$-**Cmod** of $Cv_{D_0,N}$-convex modules.*

Proof. Let X be a $Cv_{D_0,N}$-convex module. Given any two distinct elements n_1, n_2 of N we have $\frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n_1} \in Cv_{D_0,N}$. Let $x, y \in X$ and choose $x^* \in (N, X)$ such that $x^{n_1} = x$ and $x^{n_2} = y$. Since $< \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n_1}, x^* >$ depends on x and y only, due to [1], 3.5 , and [2], 4.4 , we put

$$(*) \qquad xy := < \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n_2}, x^* > .$$

Due to [1], 3.8, (which remains valid for arbitrary infinite classes) we have $x^2 = x$ for all $x \in X$, and [1], 3.6, (which remains valid for arbitrary infinite classes) implies $xy = yx$ for all $x, y \in X$. In order to obtain (4.1), (iii), let $n_1, \dots, n_4 \in N$ be mutually distinct and choose $x^* \in (N, X)$ such that $x^{n_1} = x, x^{n_2} = y, x^{n_3} = u, x^{n_4} = v$. Let $\alpha_* := \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n_2}$ and choose $\beta_*^\square \in (N, Cv_{D_0,N})$ such that

$$\beta_*^{n_1} = \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n_2} \quad \text{and} \quad \beta_*^{n_2} = \frac{1}{2}\delta_*^{n_3} + \frac{1}{2}\delta_*^{n_4}.$$

Then

$$\begin{aligned} (xy)(uv) &= < \alpha_\square, < \beta_*^\square, x^* \gg = \ll \alpha_\square, \beta_*^\square >, x^* > \\ &= < \frac{1}{4}\delta_*^{n_1} + \frac{1}{4}\delta_*^{n_2} + \frac{1}{4}\delta_*^{n_3} + \frac{1}{4}\delta_*^{n_4}, x^* > . \end{aligned}$$

Let $\delta_*^\square \in (N, Cv_{D_0,N})$ satisfy

$$\gamma_*^{n_1} = \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n_3} \quad \text{and} \quad \gamma_*^{n_1} = \frac{1}{2}\delta_*^{n_2} + \frac{1}{4}\delta_*^{n_4}.$$

Then

$$\begin{aligned}(xu)(yv) &= <\alpha_\square, <\gamma_*^\square, x^* \gg = \ll \alpha_\square, \beta_*^\square >, x^* > \\ &= <\frac{1}{4}\delta_*^{n_1} + \frac{1}{4}\delta_*^{n_2} + \frac{1}{4}\delta_*^{n_3} + \frac{1}{4}\delta_*^{n_4}, x^* > .\end{aligned}$$

Hence (4.1), (iii), is valid and X equipped with the product $(*)$ is a midpoint algebra.

Conversely assume that X is a midpoint algebra. Given $\alpha_* \in Cv_{D_0,N}$ and $x^* \in (N, X)$ we define

$$(**) \qquad <\alpha_*, x^* > := (< a_{k_1} x^{k_1} > \cdots < a_{k_q} x^{k_q} >)$$

and claim that $Cv_{D_0,N} \times (N, X) \ni (\alpha_*, x^*) \mapsto <\alpha_*, x^* > \in X$ makes X a $Cv_{D_0,N}$-convex module. Obviously we have $<\delta_*^n, x^* > = x^n$ for all $n \in X$ and $x^* \in (N, X)$. Next we have to verify

$$(+) \qquad <\alpha_\square, <\beta_*^\square, x^* \gg = \ll \alpha_\square, \beta_*^\square, x^* >,$$
$$\text{for all } x^* \in (N, X), \alpha_* \in Cv_{D_0,N}, \beta_*^\square \in (N, Cv_{D_0,N}).$$

Let supp $\alpha_* = \{k_1, \ldots, k_q\}$. Then there is an $\ell \in \mathbb{N}_0$ such that for some $a_n, b_n^m \in \mathbb{N}_0, m \in$ supp α_* and $n \in N$,

$$\alpha_n = \frac{a_n}{2^\ell} \qquad \text{and} \qquad \beta_n^m = \frac{b_n^m}{2^\ell} \qquad , m \in \text{supp } \alpha_*, n \in N,$$

hold. We have $\sum a_n = 2^\ell$ and $\sum_n b_n^m = 2^\ell, m \in$ supp α_*. Denote supp β_*^m by $\{\ell_{1,m}, \ldots, \ell_{p_m,m}\}$.

Then

$$<\beta_*^m, x^* > = (< b_1^m x^{\ell_{1,m}} > \cdots < b_{p_m}^m x^{\ell_{p_m,m}} >), m \in \text{supp } \alpha_*$$

and thus

$$\begin{aligned}<\alpha_\square, <\beta_*^\square, x^* \gg = (< a_{k_1}(< b_1^{k_1} x^{\ell_{1,k_1}} > \cdots < b_{p_1}^{k_1} x^{\ell_{p_{k_1},i_1}} >) > \ldots \\ \cdots < a_{k_q}(< b_1^{k_q} x^{\ell_{1,k_q}} > \cdots < b_{p_q}^{k_q} x^{\ell_{p_{k_q},k_q}} >) >).\end{aligned}$$

By applying (4.2) and the definition of $(x_1 \ldots x_n)$ repeatedly the right side of the last equation turns out to be

$$(\underbrace{x^{\ell_{1,k_1}} \ldots x^{\ell_{1,k_1}}}_{a_{k_1} b_1^{k_1} - \text{times}} \cdots \underbrace{x^{\ell_{p_{k_q},k_q}} \ldots x^{\ell_{p_{k_q},k_q}}}_{a_{k_q} b_{k_q}^{k_q} - \text{times}}).$$

On the other hand,

$$<\alpha_\square, \beta_m^\square > = \sum_n \alpha_n \beta_m^n = 2^{-2\ell}(a_{k_1} b_m^{k_1} + \cdots + a_{k_q} b_m^{k_q}), m \in N,$$

and

$$\text{supp} < \alpha_\square, \beta_*^\square > = \cup\{\text{supp } \beta_*^m : m \in \text{supp } \alpha_*\}.$$

Therefore

$$\ll \alpha_\square, \beta_*^\square >, x^* > = (\underbrace{x^{\ell_{1,k_1}} \dots x^{\ell_{1,k_1}}}_{a_{k_1} b_1^{k_1} - \text{times}} \cdots \underbrace{x^{\ell_{p_{k_q},k_q}} \dots x^{\ell_{p_{k_q},k_q}}}_{a_{k_q} b_{k_q}^{k_q} - \text{times}}).$$

Having shown the validity of (+) we know that the composition given by (∗∗) makes X a $Cv_{D_0,N}$-convex module. A simple argument shows that both "midpoint algebra $\xrightarrow{(**)}$ $Cv_{D_0,N}$-convex module $\xrightarrow{(*)}$ midpoint algebra" and "$Cv_{D_0,N}$-convex module $\xrightarrow{(*)}$ midpoint algebra $\xrightarrow{(**)}$ $Cv_{D_0,N}$-convex module" produce the original structure.

Next let $f : X \longrightarrow X'$ be a morphism of $Cv_{D_0,N}$-convex modules. Then for any $x, y \in X$, using the previous notation,

$$f(xy) = f(< \frac{1}{2}\delta_*^{n_1} + \frac{1}{2}\delta_*^{n_2}, x^* >) = < \frac{1}{2}\delta_*^{n_2} + \frac{1}{2}\delta_*^{n_2}, f^N(x^*) >= f(x)f(y),$$

whence f is a morphism of the associated midpoint algebras.

Finally let $f : X \longrightarrow X'$ be a morphism of midpoint algebras. Due to (3.3) we have

$$f(< \alpha_*, x^* >) = f(< a_{k_1} x^{k_1} > \cdots < a_{k_q} x^{k_q} >) =$$

$$= (< a_{k_1} f(x^{k_1}) > \cdots < a_{k_q} f(x^{k_q}) >) =< \alpha_*, f^N(x^*) > .$$

□

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