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# Partial and Generalized Subconvexity in Vector Optimization Problems

**M. Adán**
*Department of Applied Mathematics,*
*U.C.L.M., Ciudad-Real, Spain*

**V. Novo**
*Department of Applied Mathematics,*
*U.N.E.D., Madrid, Spain*
*vnovo@ind.uned.es*

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This paper studies necessary conditions of weak efficiency of a constrained vector minimization problem with equality and inequality constraints in real linear spaces. These results are obtained under generalized convexity conditions through new alternative theorems and given in linear operator rules form. We present a relaxed subconvexlikeness and generalized subconvexlikeness, and likewise, define and related to this, other new concepts such as partial subconvexlikeness and partial generalized subconvexlikeness.

*Keywords:* Vector optimization, weak efficiency, partial subconvexlikeness, partial generalized subconvexlikeness

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## 1. Introduction and Preliminaries

The very important role that convexity plays in the optimization theory is well known. There are several generalized convexity concepts and useful extensions of classic theorems that have appeared in the last few years (see [13]). Particularly it has been possible to establish results in vector optimization that allow us to characterize efficient and weak efficient points under the concept of convexlikeness, which was introduced by Fan [1]. Likewise, the convexlikeness has been generalized by several authors, for example Jeyakumar [5] defined the subconvexlikeness in $\mathbb{R}^n$, Frenk and Kassay [17] studied a relaxed concept of this, and Yang [9] extend these concepts and he defined generalized convexlikeness and generalized subconvexlikeness working in normed spaces. Furthermore, all these papers present alternative theorems adapted to the introduced concepts.
On the other hand, the convexlikeness has been generalized by Borwein, Craven, Gwinner and Jeyakumar [6, 7, 8]. They consider notions such as near convexlikeness and moderate convexlikeness, and they give new separation and alternative theorems.
Later, authors such as Paeck, Yang, Illés, Kassay, Breckner, Chen and Rong [10, 11, 12, 14, 15, 16] study these concepts and obtain several relations, characterizations and alternative theorems.
In this paper we present a relaxed subconvexlikeness and generalized subconvexlikeness defined in real linear spaces. Likewise all the near convexity concepts have been adapted to this new condition. On the other hand, it is normal in the literature to consider that

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the ordering cone is solid, instead we shall give definitions and theorems without this restrictive condition, and since we don't have any topology, we shall consider the relative algebraic interior instead of topological interior. Furthermore, because we study constrained vector optimization with inequality and equality constraints, it is useful to consider that the ordering cone is a product of cones. This fact invites us to define concepts such as partial subconvexlikeness and partial generalized subconvexlikeness. The second section is devoted to introducing and relating all these concepts. In the third section we obtain new alternative theorems under generalized subconvexlikeness and partial generalized subconvexlikeness, and as a consequence of these we can give necessary conditions of weak efficiency in linear operators rule form for vector minimization problems with inequality and equality constraints.

Let us consider a real linear space $Y$, a nontrivial ($\{0\} \neq K \neq Y$) convex cone $K \subset Y$ and a nonempty subset $A \subset Y$. Sometimes we shall need to separate $Y = Y_1 \times Y_2$, with $Y_1, Y_2$ real linear spaces and $K = K_1 \times K_2$ with $K_1, K_2$ convex cones in $Y_1, Y_2$ respectively. As usually, $K$ is *pointed* if $K \cap (-K) = \{0\}$, and $\mathrm{cone}(A), \mathrm{conv}(A), \mathrm{aff}(A)$ denote the *generated cone, convex hull* and *affine manifold* of $A$ respectively.

We use the following definitions:

1. For $\lambda \in (0,1), A$ is $\lambda-$*convex* if for all $a_1, a_2 \in A, \lambda a_1 + (1-\lambda)a_2 \in A$.
2. $A$ is *nearly convex* if there exists $\lambda \in (0,1)$ such that $A$ is $\lambda-$convex.
3. The *core or algebraic interior* of $A$, and the *intrinsic core* or *relative algebraic interior* of $A$ are defined by:

$$\mathrm{cor}(A) = \{y \in A : \forall y' \in Y, \exists \lambda' > 0 \text{ such that } y + \lambda y' \in A, \forall \lambda \in [0, \lambda']\}$$

$$\mathrm{icr}(A) = \{y \in A : \forall y' \in \mathrm{span}(A-A), \exists \lambda' > 0 \text{ such that } y + \lambda y' \in A, \forall \lambda \in [0, \lambda']\}$$

   where $\mathrm{span}(A)$ is the linear hull of $A$. We note that if $A$ is a cone then $\mathrm{span}(A - A)$=$\mathrm{span}(A\backslash\{0\} - A\backslash\{0\})$=$\mathrm{aff}(A)$=$\mathrm{aff}(A\backslash\{0\})$.
4. The algebraic dual of $Y$ is named $Y'$, and $K' = \{\mu \in Y' : \forall k \in K, \mu(k) \geq 0\}$ is the positive dual cone of $K$.

It is known that if $\mathrm{cor}(K) \neq \emptyset$ then $\mathrm{cor}(K) \cup \{0\}$ is a convex cone, $\mathrm{cor}(K) + K = \mathrm{cor}(K)$ and $\mathrm{cor}(\mathrm{cor}(K)) = \mathrm{cor}(K)$ (see [4] ).

## 2. Partial and generalized subconvexity

Fan [1] gives the name convexlike to a set $A$ if for all $\lambda \in (0,1), a_1, a_2 \in A$ there exists $a_3 \in A$ such that $\lambda a_1 + (1-\lambda)a_2 - a_3 \in K$, and it is easy to see that the convexlikeness is equivalent to the convexity of $A + K$. This concept has been generalized by several authors, and in the following definition we include some known concepts such as generalized convexlikeness and we given a natural adaptation of the subconvexlikeness and generalized subconvexlikeness to real linear spaces. For this we shall consider the algebraic interior or the relative algebraic interior instead of topological interior. As we will establish later, the following definitions are equivalent to those given by other authors. Likewise, since we are interested in the particular case of $K = K_1 \times K_2$, we introduce new definitions.

**Definition 2.1.** *Let $Y$ be a real linear space, a nontrivial convex cone $K \subset Y$ and a nonempty subset $A \subset Y$.*

1. *A is said to be* convexlike (CL) *if* $A + K$ *is convex, and* generalized convexlike (GCL) *if* $\text{cone}(A) + K$ *is convex.*
2. *If* $\text{icr}(K) \neq \emptyset$*, A is said to be* subconvexlike (SCL) *if* $A+\text{icr}(K)$ *is convex, and* generalized subconvexlike (GSCL) *if* $\text{cone}(A)+\text{icr}(K)$ *is convex.*
3. *If* $K = K_1 \times K_2$ *with* $\text{cor}(K_1) \neq \emptyset$*, A is said to be* $(K_1, K_2)$ partial subconvexlike (PSCL) *if* $A+\text{cor}(K_1)\times K_2$ *is convex, and* $(K_1, K_2)$ partial generalized subconvexlike (PGSCL) *if* $\text{cone}(A)+\text{cor}(K_1) \times K_2$ *is convex.*
4. *If in the above definitions we change the convexity by near convexity we shall add "nearly" and thus we shall have sets* nearly convexlike (nearly CL), nearly generalized convexlike (nearly GCL), nearly subconvexlike (nearly SCL), nearly generalized subconvexlike (nearly GSCL), nearly $(K_1, K_2)$ partial subconvexlike (nearly PSCL) and nearly $(K_1, K_2)$ partial generalized subconvexlike (nearly PGSCL).

As it is obvious all the previous concepts given in (1,2,3) are stronger that their corresponding ones in (4).
In this section we establish several characterizations and relationships for these concepts. Firstly we need the following lemmas.

**Lemma 2.2.** *Let $Y$ be a real linear space, a nontrivial convex cone $K \subset Y$ and a nonempty subset $A \subset Y$.*

1. *If* $\text{icr}(K) \neq \emptyset$*, then* $\text{icr}(K) \cup \{0\}$ *is a convex cone, furthermore* $\text{icr}(K)+K = \text{icr}(K)$*, and* $\text{icr}(\text{icr}(K)) = \text{icr}(K)$.
2. *If* $K = K_1 \times K_2$ *with* $\text{icr}(K_1) \neq \emptyset$ *and* $\text{icr}(K_2) \neq \emptyset$ *then* $\text{icr}(K) = \text{icr}(K_1) \times \text{icr}(K_2)$*. Furthermore* $L = (\text{icr}(K_1) \times K_2) \cup \{0\}$ *is also a convex cone with* $\text{icr}(L) = \text{icr}(\text{icr}(K_1) \times K_2)$.
3. $\text{cone}(A + K) \subset \text{cone}(A) + K$*. If* $0 \in A$ *then both sets are equal.*
4. $\text{cone}(A + \text{cor}(K)) = (\text{cone}(A) + \text{cor}(K)) \cup \{0\}$.
5. $\text{cone}(\text{conv}(A)) = \text{conv}(\text{cone}(A))$.
6. *$K$ is convex if and only if $K$ is nearly convex. Analogously for* $K\backslash\{0\}$.

**Proof.**

1. Since $K$ is a cone, it is sufficient to observe that $\text{aff}(\text{icr}(K))$ is a linear subspace of $Y$, and so $\text{icr}(\text{icr}(K)) = \text{cor}(\text{cor}(K)) = \text{cor}(K) = \text{icr}(K)$, since all relative algebraic interior is equal to algebraic interior in his affine hull.
2. As $K = K_1 \times \{0_{Y_2}\} + \{0_{Y_1}\} \times K_2$ then $\text{icr}(K) = \text{icr}(K_1 \times \{0_{Y_2}\} + \{0_{Y_1}\} \times K_2)$ and since both are convex cones in $Y$, it is $\text{icr}(K) = \text{icr}(K_1 \times \{0_{Y_2}\}) + \text{icr}(\{0_{Y_1}\} \times K_2)$ (see [19 ]). Then, $\text{icr}(K) = \text{icr}(K_1) \times \text{icr}(K_2)$. On the other hand, $\text{icr}(L) = \text{icr}(\text{icr}(L)) = \text{icr}(\text{icr}(K_1) \times \text{icr}(K_2)) \subset \text{icr}(\text{icr}(K_1) \times K_2) \subset \text{icr}(L)$, and so we obtain that $\text{icr}(L) = \text{icr}(\text{icr}(K_1) \times K_2)$.
3. The inclusion $\subset$ is evident. On the other hand, suppose that $0 \in A$, if $\alpha > 0, a \in A, k \in K$ then $\alpha a+k = \alpha(a+k/\alpha) \in \text{cone}(A+K)$, and if $\alpha = 0$ then $k \in \text{cone}(A+K)$. Consequently $\text{cone}(A) + K \subset\text{cone}(A + K)$.
4. Analogously the inclusion $\subset$ is easy to see. Furthermore, if $\alpha > 0, a \in A, k \in \text{cor}(K)$ then $\alpha a+k = \alpha(a+k/\alpha) \in \text{cone}(A+\text{cor}(K))$, and if $\alpha = 0$ we can see that $\text{cor}(K) \subset \text{cone}(A + \text{cor}(K))$. In fact, if $k_0 \in \text{cor}(K)$, then for any $a \in A$ there exists $\lambda' > 0$ such that for all $\lambda \in [0, \lambda'], k_0 - \lambda a \in K$. Thus there exists $k \in K$ such that

$k_0 = \lambda a + k$, and so $k_0 = \lambda a/2 + k/2 + k_0/2 \in \text{cone}(A) + K + \text{cor}(K) = \text{cone}(A) + \text{cor}(K)$.

5. If $x \in$cone(conv($A$)), then there exist $\alpha \geq 0, \lambda \in [0,1]$, and $a_1, a_2 \in A$ such that $x = \alpha(\lambda a_1 + (1-\lambda)a_2) = \lambda(\alpha a_1) + (1-\lambda)(\alpha a_2) \in \text{conv}(\text{cone}(A))$. If $x \in$conv(cone($A$)), then there exist $\lambda \in [0,1], a_1, a_2 \in A$, and $\alpha, \beta \geq 0$, such that $x = \lambda(\alpha a_1) + (1-\lambda)(\beta a_2)$. If $\alpha, \beta = 0$, then $x = 0 \in$cone(conv($A$)), and if $(\alpha, \beta) \neq 0$ and let us denote $\mu = \lambda\alpha + (1-\lambda)\beta$, we have that $x = \mu((\lambda\alpha/\mu)a_1 + ((1-\lambda)\beta/\mu)a_2) \in \text{cone}(\text{conv}(A))$.
6. If $K$ is convex also it is nearly convex. Reciprocally, if $\alpha \in (0,1), k_1, k_2 \in K$ since there exists a $\lambda \in (0,1)$ such that for $(\alpha/\lambda)k_1, ((1-\alpha)/(1-\lambda))k_2 \in K$, then it is $\lambda((\alpha/\lambda)k_1) + (1-\lambda)((1-\alpha)/(1-\lambda))k_2) \in K$, and so $\alpha k_1 + (1-\alpha)k_2 \in K$. The proof is analogous for $K\backslash\{0\}$. □

**Remark**. The reverse inclusion in (3) does not hold, as can be seen considering $A = \{(1,1)\} \subset \mathbb{R}^2$ and $K = \mathbb{R}^2_+$.
According to the last of the preceding results it is obvious that

$$\text{GCL} \Leftrightarrow \text{nearly GCL; PGSCL} \Leftrightarrow (K_1, K_2) \text{ nearly PGSCL; GSCL} \Leftrightarrow \text{nearly GSCL.}$$

**Lemma 2.3.** *Let $A, W$ be subsets of a real linear space $Y$ with $W$ convex.*

1. *$A + W$ is convex if and only if* $\text{conv}(A) + W = A + W$.
2. *Let $W$ be a cone, then $A + W$ is convex if and only if* $\text{conv}(A) \subset A + W$.
3. *Let $W$ be a cone, then* $\text{cone}(A) + W\backslash\{0\}$ *is convex if and only if* $\text{conv}(A) + W\backslash\{0\} \subset \text{cone}(A) + W\backslash\{0\}$.
4. *Let $W$ be a cone, then* $\text{cone}(A) + W$ *is convex if and only if* $\text{conv}(A) \subset \text{cone}(A) + W$.

**Proof.**

1. ($\Rightarrow$) The inclusion $\supset$ is evident. On the other hand, the reverse inclusion holds since for all $\lambda \in (0,1), a_1, a_2 \in A, w \in W$ it is $\lambda a_1 + (1-\lambda)a_2 + w = \lambda(a_1 + w) + (1-\lambda)(a_2 + w) \in A + W$.
   ($\Leftarrow$) If $a_1, a_2 \in A, \lambda \in (0,1), w_1, w_2 \in W$ then $\lambda(a_1 + w_1) + (1-\lambda)(a_2 + w_2) = \lambda a_1 + (1-\lambda)a_2 + \lambda w_1 + (1-\lambda)w_2 \in \text{conv}(A) + W = A + W$.
2. ($\Rightarrow$) As $A + W$ is convex then $\text{conv}(A) + W = A + W$ and so the inclusion holds, since $0 \in W$.
   ($\Leftarrow$) Let $a_1, a_2 \in A, \lambda \in (0,1), w_1, w_2 \in W$, then $\lambda(a_1 + w_1) + (1-\lambda)(a_2 + w_2) = \lambda a_1 + (1-\lambda)a_2 + \lambda w_1 + (1-\lambda)w_2 \in \text{conv}(A) + W \subset A + W + W \subset A + W$.
3. ($\Rightarrow$) It is a consequence of (1) since $\text{conv}(A) + W\backslash\{0\} \subset \text{conv}(\text{cone}(A)) + W\backslash\{0\}$.
   ($\Leftarrow$) It is sufficient to see that $\text{conv}(\text{cone}(A)) + W\backslash\{0\} \subset \text{cone}(A) + W\backslash\{0\}$. Since $\text{conv}(\text{cone}(A)) = \text{cone}(\text{conv}(A))$, for all $m \in \text{conv}(\text{cone}(A)), w \in W\backslash\{0\}$ there exist $\sigma \geq 0, a_1, a_2 \in A, \lambda \in (0,1)$ such that $m + w = \sigma(\lambda a_1 + (1-\lambda)a_2) + w$. If $\sigma > 0$ we can write $\sigma(\lambda a_1 + (1-\lambda)a_2 + w/\sigma)$, and by hypothesis there exist $\alpha \geq 0, a_3 \in A, w' \in W$, such that $m + w = \sigma(\alpha a_3 + w') = \sigma\alpha a_3 + \sigma w' \in \text{cone}(A) + W\backslash\{0\}$. For the case of $\sigma = 0$ the proof it is evident.
4. ($\Rightarrow$) Since $\text{conv}(A) \subset \text{conv}(\text{cone}(A))$, this result is a consequence of (2)
   ($\Leftarrow$) We shall see that $\text{conv}(\text{cone}(A)) \subset \text{cone}(A) + W$. Since $\text{conv}(\text{cone}(A)) = \text{cone}(\text{conv}(A))$, for all $m \in \text{conv}(\text{cone}(A))$, there exist $\sigma \geq 0, a_1, a_2 \in A, \lambda \in (0,1)$ such that $m = \sigma(\lambda a_1 + (1-\lambda)a_2)$, and by hypothesis there exist $\alpha \geq 0, a_3 \in A, w \in W$, such that $m = \sigma(\alpha a_3 + w) = \sigma\alpha a_3 + \sigma w \in \text{cone}(A) + W$. □

Several authors (see [4],[9],[10],[14],[15]) have given characterizations for some relaxed convexity concepts considering a solid convex cone $K$ in topological linear spaces. Now, with the help of preceding lemmas, we show that these characterizations hold in our context. Likewise we present some new results.

**Proposition 2.4.** *Let us consider a real linear space $Y$, a nontrivial convex cone $K \subset Y$ and a nonempty subset $A \subset Y$.*

1. *Equivalent are:*
   *(a) $A$ is CL (nearly CL).*
   *(b) $\forall(\exists)\lambda \in (0,1), \forall a_1, a_2 \in A, \exists a_3 \in A$ such that $\lambda a_1 + (1-\lambda)a_2 - a_3 \in K$.*
2. *Equivalent are:*
   *(a) $A$ is SCL (nearly SCL).*
   *(b) $\forall\theta \in \mathrm{icr}(K), \forall(\exists)\lambda \in (0,1), \forall a_1, a_2 \in A, \exists a_3 \in A$ such that $\theta + \lambda a_1 + (1-\lambda)a_2 - a_3 \in \mathrm{icr}(K)$.*
   *(c) $\exists\theta \in \mathrm{icr}(K)$ such that $\forall(\exists)\lambda \in (0,1), \forall a_1, a_2 \in A, \forall\epsilon > 0, \exists a_3 \in A$ such that $\epsilon\theta + \lambda a_1 + (1-\lambda)a_2 - a_3 \in K$.*
3. *Equivalent are:*
   *(a) $A$ is GCL.*
   *(b) $\forall\lambda \in (0,1), \forall a_1, a_2 \in A, \exists a_3 \in A, \exists\nu > 0$ such that $\lambda a_1 + (1-\lambda)a_2 - \nu a_3 \in K$.*
4. *Equivalent are:*
   *(a) $A$ is GSCL.*
   *(b) $\forall\theta \in \mathrm{icr}(K), \forall\lambda \in (0,1), \forall a_1, a_2 \in A, \exists a_3 \in A, \exists\nu > 0$ such that $\theta + \lambda a_1 + (1-\lambda)a_2 - \nu a_3 \in \mathrm{icr}(K)$.*
   *(c) $\exists\theta \in \mathrm{icr}(K)$ such that $\forall\lambda \in (0,1), \forall a_1, a_2 \in A, \forall\epsilon > 0, \exists a_3 \in A, \exists\nu > 0$ such that $\epsilon\theta + \lambda a_1 + (1-\lambda)a_2 - \nu a_3 \in K$.*
5. *If $Y = Y_1 \times Y_2$ with $Y_1$ and $Y_2$ real linear spaces and $K = K_1 \times K_2$ with $K_1$ and $K_2$ convex cones in $Y_1$ and $Y_2$ respectively, and such that $\mathrm{cor}(K_1) \neq \emptyset$ then the following statements are equivalent:*
   *(a) $A$ is $(K_1, K_2)$ PSCL(nearly PSCL).*
   *(b) $\forall(\theta_1, \theta_2) \in \mathrm{cor}(K_1) \times K_2, \forall(\exists)\lambda \in (0,1), \forall a_1, a_2 \in A, \exists a_3 \in A$, such that $(\theta_1, \theta_2) + \lambda a_1 + (1-\lambda)a_2 - a_3 \in \mathrm{cor}(K_1) \times K_2$.*
   *(c) $\exists\theta \in \mathrm{cor}(K_1)$ such that $\forall(\exists)\lambda \in (0,1), \forall a_1, a_2 \in A, \forall\epsilon > 0, \exists a_3 \in A$ such that $\epsilon(\theta, 0) + \lambda a_1 + (1-\lambda)a_2 - a_3 \in K$.*
6. *If $Y = Y_1 \times Y_2$ with $Y_1$ and $Y_2$ real linear spaces and $K = K_1 \times K_2$ with $K_1$ and $K_2$ convex cones in $Y_1$ and $Y_2$ respectively, and such that $\mathrm{cor}(K_1) \neq \emptyset$, then the following statements are equivalent:*
   *(a) $A$ is $(K_1, K_2)$ PGSCL.*
   *(b) $\forall(\theta_1, \theta_2) \in \mathrm{cor}(K_1) \times K_2, \forall\lambda \in (0,1), \forall a_1, a_2 \in A, \exists a_3 \in A, \exists\nu > 0$, such that $(\theta_1, \theta_2) + \lambda a_1 + (1-\lambda)a_2 - \nu a_3 \in \mathrm{cor}(K_1) \times K_2$.*
   *(c) $\exists\theta \in \mathrm{cor}(K_1)$ such that $\forall\lambda \in (0,1), \forall a_1, a_2 \in A, \forall\epsilon > 0, \exists a_3 \in A, \exists\nu > 0$, such that $\epsilon(\theta, 0) + \lambda a_1 + (1-\lambda)a_2 - \nu a_3 \in K$.*

**Proof.** We only show the proofs for convexlikeness, in case of near convexlikeness they are similar. Firstly, (1) and (3) are obvious from Lemma 2.3.2 and Lemma 2.3.4 respectively. The same way, we only prove (2) and (5), because the proofs of (4) and (6) are analogous.

2. Evidently *(a)* is equivalent to *(b)* by Lemma 2.3.1.*(b)⇒(c).* It is easy to see since if

$\theta \in \operatorname{icr}(K)$ then for all $\epsilon > 0$ it is $\epsilon\theta \in \operatorname{icr}(K)$.

*(c)⇒(b).* Let $\theta \in \operatorname{icr}(K)$, there exist $\theta' \in \operatorname{icr}(K)$ and $\lambda \in (0,1)$ such that for all $a_1, a_2 \in A, \epsilon > 0$ there exists $a_3 \in A$ with $\epsilon\theta' + \lambda a_1 + (1-\lambda)a_2 \in a_3 + K$. Also there exists $\mu' > 0$ such that for all $\mu \in [0, \mu'/2]$ is $\theta + 2\mu(-\theta') \in K$ because $-\theta' \in \operatorname{aff}(K)$. Since $\mu\theta' \in \operatorname{icr}(K)$ then $\theta + 2\mu(-\theta') + \mu\theta' \in K + \operatorname{icr}(K)$, and this implies that $\theta - \mu\theta' \in \operatorname{icr}(K)$. Hence, setting $\mu$ small enough, it is $(\theta - \mu\theta') + (\mu\theta' + \lambda a_1 + (1-\lambda)a_2 - a_3) \in \operatorname{icr}(K) + K$ and therefore $\theta + \lambda a_1 + (1-\lambda)a_2 - a_3 \in \operatorname{icr}(K)$.

5. Evidently *(a)* is equivalent to *(b)* by (1) in Lemma 2.3.1.

*(b)⇒(c)* It is sufficient to observe that if $\theta_1 \in \operatorname{cor}(K_1)$ then for all $\epsilon > 0$ it is $\epsilon(\theta_1, 0) \in \operatorname{cor}(K_1) \times K_2 \subset K_1 \times K_2$.

*(c)⇒(b).* Let $(\theta_1, \theta_2) \in \operatorname{cor}(K_1) \times K_2$. As there exist $\theta_1' \in \operatorname{cor}(K_1), \lambda \in (0,1)$ such that for all $a_1, a_2 \in A, \epsilon > 0$ there exists $a_3 \in A$ with $\epsilon(\theta_1', 0) + \lambda a_1 + (1-\lambda)a_2 \in a_3 + K_1 \times K_2$, and for $\theta_1 \in \operatorname{cor}(K_1$, there exists $\mu' > 0$ such that for all $\mu \in [0, \mu'/2]$ is $\theta_1 + 2\mu(-\theta_1') \in K_1$. Since $\mu\theta_1' \in \operatorname{cor}(K_1)$ then $\theta_1 + 2\mu(-\theta_1') + \mu\theta_1' \in K_1 + \operatorname{cor}(K_1)$, and so $\theta_1 - \mu\theta_1' \in \operatorname{cor}(K_1)$ and also $(\theta_1 - \mu\theta_1', \theta_2) \in \operatorname{cor}(K_1) \times K_2$. Hence, setting $\mu$ small enough, it is $(\theta_1 - \mu\theta_1', \theta_2) + \mu(\theta_1', 0) + \lambda a_1 + (1-\lambda)a_2 - a_3 \in \operatorname{cor}(K_1) \times K_2 + K_1 \times K_2$ and therefore $(\theta_1, \theta_2) + \lambda a_1 + (1-\lambda)a_2 - a_3 \in \operatorname{cor}(K_1) \times K_2$. □

**Lemma 2.5.** *Let $A, W$ be subsets of a real linear space $Y$, and let $W$ be convex with $\operatorname{icr}(W) \neq \emptyset$. If $A + W$ is convex (nearly convex) then $A + \operatorname{icr}(W)$ is convex (nearly convex). If $W = W_1 \times W_2$ with $\operatorname{icr}(W_1) \neq \emptyset$ then if $A + W_1 \times W_2$ is convex (nearly convex) then also $A + \operatorname{icr}(W_1) \times W_2$ is convex (nearly convex).*

**Proof.** If $\lambda \in (0,1), a_1, a_2 \in A, w_1, w_2 \in \operatorname{icr}(W)$ then $\lambda(a_1 + w_1) + (1-\lambda)(a_2 + w_2) = \lambda(a_1 + w_1/2) + (1-\lambda)(a_2 + w_2/2) + \lambda w_1 + (1-\lambda)w_2 \in A + W + \operatorname{icr}(W) = A + \operatorname{icr}(W)$. The remaining proofs are analogous. □

With the help of the preceding results, it is easy to prove that all these concepts are related as follows.

**Proposition 2.6.** *Let us consider a real linear space $Y$, a nontrivial convex cone $K \subset Y$ and a nonempty subset $A \subset Y$.*

1. *CL ⇒ GCL*
2. *If $\operatorname{icr}(K) \neq \emptyset$ then*

*CL ⇒ SCL ⇒ GSCL, nearly CL ⇒ nearly SCL, and GCL ⇒ GSCL.*

3. *If $K = K_1 \times K_2$, with $\operatorname{cor}(K_1) \neq \emptyset$ then*

*CL ⇒ $(K_1, K_2)$ PSCL ⇒ $(K_1, K_2)$ PGSCL*

*nearly CL ⇒ $(K_1, K_2)$ nearly PSCL, and GCL ⇒ $(K_1, K_2)$ PGSCL.*

4. *If $K = K_1 \times K_2$, with $\operatorname{cor}(K_1) \neq \emptyset, \operatorname{icr}(K_2) \neq \emptyset$ then*

*$(K_1, K_2)$ PSCL ⇒ SCL*

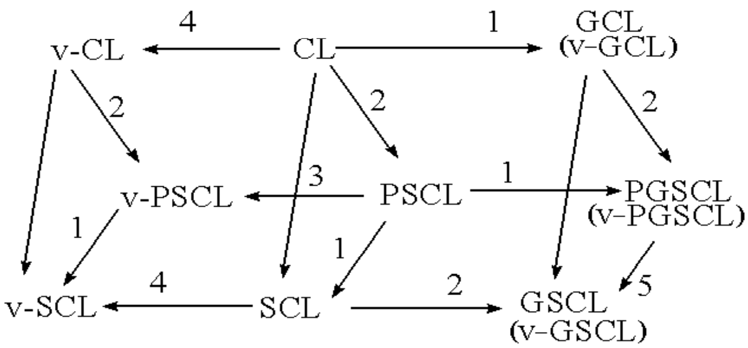
*$(K_1, K_2)$ nearly PSCL ⇒ nearly SCL, and $(K_1, K_2)$ PGSCL ⇒ GSCL.*

The following examples show us that the converses does not hold. It is worth noting that cone$(A)$ is convex for all convex set $A$, but this property does not hold for near convexity. This fact means that nearly CL does not imply GCL, nearly PSCL does not imply PGSCL, and that nearly SCL does not imply GSCL.

**Example 2.7.**

1. *Let* $K_1 = \mathbb{R}^+, K_2 = \mathbb{R}^+ \times \{0\}$, *and* $A = \{(0,0,q) : q \in \mathbb{Q}, q \neq 1\} \cup \{(x,y,z) : x > 0, y > 0\}$, *then* $A$ *is SCL and GCL but is not nearly PSCL.*
2. *Let* $K = K_1 \times K_2$ *with* $K_1 = \{(x,y) : |x| \leq y\}, K_2 = \{0\}$, *and* $M = \{(x,y,z) : 2x^2 \leq y^2, 0 \leq y, x = z\}$. *Then* $A = \{(x,0,z) : x \neq 0\}$ *is SCL with respect to* $K$ *and* $M$ *and it is PSCL with respect to* $K$, *but neither nearly CL nor GCL with respect to either. On the other hand* $B = \{(x,0,z) : z = 1/x, x \neq 1, x > 0\}$ *is PGSCL with respect to* $K$ *and GSCL with respect* $M$ *but neither GCL with respect to* $K$ *nor nearly SCL with respect to* $M$.
3. *Let* $K = K_1 \times K_2$ *with* $K_1 = \mathbb{R}^2_+, K_2 = \{0\}$, *then* $A = \{0\} \times \{0\} \times ([0,1] \cup \mathbb{Q})$ *is nearly PSCL but nor PSCL.*
4. *Let* $K = K_1 \times K_2$ *with* $K_1 = \mathbb{R}_+, K_2 = \{(0,0)\}$, *then* $A = \{(0,1,q) : q \in \mathbb{Q}\}$ *is nearly CL but neither GSCL nor PGSCL.*
5. *Let* $K = K_1 \times K_2$ *with* $K_1 = \mathbb{R}, K_2 = \{(y,z) : |y| \leq z\}$, *then* $A = \{(x,y,0) : y \neq 0\}$ *is GSCL but nor PGSCL.*

The relations between these concepts are shown in the following diagram. The number over the arrows makes a reference to the preceding counterexamples.




## 3. Necessary conditions of weak efficiency

Now, in this section, we obtain the announced results over characterization of weak efficiency. Firstly, we give two alternative theorems, and as a consequence of these we obtain necessary conditions of weak efficiency in vector optimization problems. The first result (theorem 3.5) is an alternative theorem without the restrictive condition about the ordering cone being solid, and so, it is a generalization of several theorems given by Chen and Rong (see theorem 3.1 in [15]), Yang (see theorem 1 in [9]) and Adán and Novo (see theorem 3.1 in [18]). Later, the theorem 3.7 is a generalization of theorem 6.1 of Illés and Kassay in [16], which is given for nearly SCL. As an outcome we obtain a linear operators rule as necessary condition of weak efficiency under partial generalized subconvexlikeness (theorem 3.8). Lastly, we give an analogous necessary condition of weak efficiency under generalized subconvexlikeness (corollary 3.9), which generalizes the theorems 3.2 and 3.3 in [18].

Let $X$ be a nonempty subset of a real linear space $E$. Let $Y, Z, Z_1$, and $Z_2$ be real linear spaces with ordering convex cones $K, M, M_1$, and $M_2$ respectively. Let $g : X \longrightarrow Z$, $g_1 : X \longrightarrow Z_1$, $g_2 : X \longrightarrow Z_2$, and $f : X \longrightarrow Y$ be mappings. We consider the constrained vector minimization problems:

$$K - \mathrm{Min}\{f(x) : x \in X, g(x) \in -M\} \tag{3.1}$$

$$K - \mathrm{Min}\{f(x) : x \in X, g_1(x) \in -M_1, g_2(x) \in -M_2\} \tag{3.2}$$

It is necessary to consider the problems (3.1) and (3.2) separately because we study the cones $M$ such that can be represented as a product of two cones: $M = M_1 \times M_2$. Furthermore, in (3.1) we suppose that the vector minimization problems have equality constraints, nevertheless in (3.2) we consider inequality and equality constraints. The feasible sets in (3.1) and (3.2) they are respectively:

$$\Omega = \{x \in X : g(x) \in -M\}$$

$$\Omega = \{x \in X : g_1(x) \in -M_1, g_2(x) \in -M_2\}$$

**Definition 3.1.** *A point $x_0 \in X$ is called an* efficient solution *of* (3.1) *or* (3.2) *with respect to $K$, if $x_0 \in \Omega$ and if doesn't exist $x$ in $\Omega$ such that $f(x_0) \in f(x) + K$. If* $\mathrm{cor}(K) \neq \emptyset$*, a point is called a* weakly efficient solution *of* (3.1) *or* (3.2) *with respect to $K$, if $x_0 \in \Omega$ and if doesn't exist $x$ in $\Omega$ such that $f(x_0) \in f(x)+\mathrm{cor}(K)$.*

These conditions are respectively equivalent to:

$$f(\Omega) \cap (f(x_0) - K) = \{f(x_0)\}$$

$$f(\Omega) \cap (f(x_0) - \mathrm{cor}(K)) = \emptyset$$

Since $K \times M$ and $K \times M_1 \times M_2$ are ordering convex cones in the real linear space $Y \times Z$ and $Y \times Z_1 \times Z_2$ respectively, we can establish the next definitions.

**Definition 3.2.** *If* $\mathrm{icr}(K) \neq \emptyset, \mathrm{icr}(M) \neq \emptyset$, *$(f, g)$ is said to be* generalized subconvexlike (GSCL) *on $X$ with respect to $(K, M)$ if the image set $(f, g)(X)$ is GSCL with respect to $K \times M$. If* $\mathrm{cor}(K) \neq \emptyset$, *$(f, g)$ is said to be* partial generalized subconvexlike (PGSCL) *on $X$ with respect to $(K, M)$ if the image set $(f, g)(X)$ is PGSCL with respect to $(K, M)$.*

**Definition 3.3.** *It is said that* (3.1) *satisfies the* Slater type constraint qualification *if there exists $x \in \Omega$ such that $g(x) \in$* -$\mathrm{cor}(M)$. *It is said that* (3.2) *satisfies the* Slater type constraint qualification *if there exists $x \in \Omega$ such that $g_1(x) \in$* -$\mathrm{cor}(M_1)$ .

Let $L(Z, Y)$ be the set of linear operators from $Z$ into $Y$, we denote by $\Gamma = \{T \in L(Z, Y) : T(M) \subset K\}$.
Because it will be useful, we show a know result (see [2]), with a little adaptation for our needs.

**Theorem 3.4.** *Let $S$ be a convex cone of a real linear space $Y$ with* $\mathrm{icr}(S) \neq \emptyset$ *and let $y_0 \in Y$, then $y_0 \notin$* $\mathrm{icr}(S)$ *if and only if there exists a linear functional $l \in Y' \setminus \{0\}$ such that $l(s) \leq 0 \leq l(y_0)$ for all $s \in S$, and $l(s) < 0$ for all $s \in$* $\mathrm{icr}(S)$.

**Theorem 3.5.** *Let $Y$ be a real linear space and let $K \subset Y$ be a pointed convex cone with* $\operatorname{icr}(K) \neq \emptyset, 0 \notin \operatorname{icr}(K)$*. Suppose that the nonempty $A \subset Y$ is GSCL satisfying* $\operatorname{icr}(\operatorname{cone}(A) + \operatorname{icr}(K)) \neq \emptyset$*. Consider the following statements*

*(i)* $\exists a \in A, a \in$ -$\operatorname{icr}(K)$

*(ii)* $\exists \mu \in K', \mu(\operatorname{cone}(A) + \operatorname{icr}(K)) \neq \{0\}$, $\forall a \in A, \mu(a) \geq 0$

*Then not (i) implies (ii). Moreover, if* $\operatorname{aff}(\operatorname{icr}(K)) = \operatorname{aff}(\operatorname{cone}(A)+\operatorname{icr}(K))$ *then (ii) implies not (i).*

**Proof.** By hypothesis it will be $0 \notin \operatorname{cone}(A)+\operatorname{icr}(K)$, because otherwise there exist $a \in A, \alpha \geq 0, k \in \operatorname{icr}(K)$ such that $0 = \alpha a + k$, and so, for $\alpha > 0$ is $a = -k/\alpha \in \operatorname{icr}(K)$, and for $\alpha = 0$ is $0 \in \operatorname{icr}(K)$. In both cases there exists a contradiction. Furthermore, as $\operatorname{cone}(A)+\operatorname{icr}(K)$ is convex it will be $\operatorname{cone}(A)+\operatorname{icr}(K) \cup \{0\} \neq \operatorname{aff}(\operatorname{cone}(A)+\operatorname{icr}(K))$, and since $\operatorname{aff}(\operatorname{cone}(A)+\operatorname{icr}(K))=\operatorname{aff}(\operatorname{cone}(A)+\operatorname{icr}(K)\cup\{0\})$, then $0 \notin \operatorname{icr}(\operatorname{cone}(A)+\operatorname{icr}(K)\cup\{0\})$ Therefore by theorem 3.4, there exists a linear functional $\mu \in Y'\backslash\{0\}$ such that $\mu(\alpha a+k) \geq 0$ for all $\alpha \geq 0, a \in A, k \in \operatorname{icr}(K)$, being $\mu$ strictly positive in $\operatorname{icr}(\operatorname{cone}(A) + \operatorname{icr}(K))$. For $\alpha = 0$ we obtain that $\mu(k) \geq 0$ for all $k \in \operatorname{icr}(K)$ and so $\mu \in K'$. Furthermore, if $\mu(a) < 0$ for some $a \in A$, then taking any $k \in \operatorname{icr}(K)$ and a large enough $\alpha$ is $\mu(\alpha a + k) = \alpha\mu(a) + \mu(k) < 0$ but this is contradictory.
Lastly, if we assume that $\operatorname{aff}(\operatorname{icr}(K)) = \operatorname{aff}(\operatorname{cone}(A) + \operatorname{icr}(K))$, then since $\operatorname{icr}(K) \subset \operatorname{cone}(A) + \operatorname{icr}(K)$ then $\operatorname{icr}(\operatorname{icr}(K)) = \operatorname{icr}(K) \subset \operatorname{icr}(\operatorname{cone}(A) + \operatorname{icr}(K))$, and so $\mu$ is strictly positive in $\operatorname{icr}(K)$. Therefore, if there exists $a \in A$ such that $a \in -\operatorname{icr}(K)$ must be $\mu(a) < 0$ and this is contradictory. $\square$

**Remark.** The additional condition for the converse is necessary as shown in the following example: $Y = \mathbb{R}^3, K = \{(x, y, 0) : y \geq |x|\}$ and $A = \{(x, y, z) : z \geq 0\}$.

The following result is a consequence of the preceding theorem and the elementary cones properties whose algebraic cores are nonempty.

**Corollary 3.6.** *Let $Y$ be a real linear space and let $K \subset Y$ be a pointed convex cone with* $\operatorname{cor}(K) \neq \emptyset$*. Suppose that the nonempty set $A \subset Y$ is GSCL. Then exactly one of the following statements hold*

*(i)* $\exists a \in A, a \in$ -$\operatorname{cor}(K)$.

*(ii)* $\exists \mu \in K' \setminus \{0\}, \forall a \in A, \mu(a) \geq 0$.

**Theorem 3.7 (PGSCL Alternative Theorem).** *Let $Y_1, Y_2$ be real linear spaces and $K_1 \subset Y_1, K_2 \subset Y_2$ are pointed convex cones with* $\operatorname{cor}(K_1) \neq \emptyset$*. Suppose that the nonempty set $A \subset Y = Y_1 \times Y_2$ is $(K_1, K_2)$PGSCL satisfying* $\operatorname{icr}(\operatorname{cone}(A) + \operatorname{cor}(K_1) \times K_2) \neq \emptyset$*. Consider the following statements*

(i) $\exists a = (a_1, a_2) \in A, a_1 \in$ -$\operatorname{cor}(K_1), a_2 \in -K_2$

(ii) $\exists \mu = (\mu_1, \mu_2) \in K_1' \times K_2' \setminus \{(0, 0)\}, \forall (a_1, a_2) \in A, \mu_1(a_1) + \mu_2(a_2) \geq 0$.

*Then not (i) implies (ii). Moreover, if $\mu_1 \neq 0$ then (ii) implies not (i).*

**Proof.**

1. Analogously to theorem 3.5 it is easy to see that $0 \notin \operatorname{icr}(\operatorname{cone}(A) + \operatorname{cor}(K_1) \times K_2)$ and so by theorem 3.4, there exists a linear functional $\mu = (\mu_1, \mu_2) \in Y_1' \times Y_2' \setminus$

$\{(0,0)\}$ such that $(\mu_1,\mu_2)(\alpha(a_1,a_2)+(k_1,k_2)) \geq 0$ for all $\alpha \geq 0, (a_1,a_2) \in A, k_1 \in \mathrm{cor}(K_1), k_2 \in K_2$. For $\alpha = 0$ we obtain that $(\mu_1,\mu_2)(k_1,k_2) \geq 0$ for all $k_1 \in \mathrm{cor}(K_1), k_2 \in K_2$. Particularly for $k_2 = 0$ it is $\mu_1(k_1) \geq 0$ for all $k_1 \in \mathrm{cor}(K_1)$ and so $\mu_1 \in K_1'$. Furthermore, if there exists $k_2 \in K_2$ such that $\mu_2(k_2) = r < 0$ then taking any $k_1 \in K_1$ such that $\mu_1(k_1) = s < -r$ we obtain a contradiction since $(\mu_1,\mu_2)(k_1,k_2) = s+r < 0$. So $(\mu_1,\mu_2) \in K_1' \times K_2'$. Lastly, we obtain a contradiction if we suppose that there exists $(a_1,a_2) \in A$ such that $(\mu_1,\mu_2)(a_1,a_2) < 0$, since taking $k_1 \in \mathrm{cor}(K_1), k_2 = 0$ and $\alpha > 0$ large enough, it is $(\mu_1,\mu_2)(\alpha(a_1,a_2)+(k_1,0)) = \alpha(\mu_1,\mu_2)(a_1,a_2)+(\mu_1,\mu_2)(k_1,0) < 0$.

2. We obtain a contradiction if we suppose that there exists $a = (a_1,a_2) \in A \cap (-\mathrm{cor}(K_1) \times (-K_2))$, since by hypothesis it will be $\mu_1(-a_1) > 0, \mu_2(-a_2) \geq 0$, and consequently $\mu_1(a_1)+\mu_2(a_2) < 0$. □

**Remark.** It is worth noting that the hypothesis $\mathrm{icr}(\mathrm{cone}(A)+\mathrm{cor}(K_1) \times K_2) \neq \emptyset$ is weaker than the condition $\mathrm{icr}(K_2) \neq \emptyset$. To see this, let $Y = \mathbb{R} \times C_{(0,1)}$ where $C_{(0,1)}$ is the real linear space of continuous real functions in (0,1). Let $K_1 = \mathbb{R}_+, K_2 = \{f : (0,1) \to \mathbb{R}$ such that $f$ is continuous, $f(x) \geq 0$ for all $x \in (0,1)\}$ and $A = \mathbb{R}_+ \times \{f : (0,1) \to \mathbb{R}$ such that $f$ is continuous, $f(x) \leq 0$ for all $x \in (0,1)\}$. It is easy to see that $\mathrm{icr}(\mathrm{cone}(A)+\mathrm{cor}(K_1) \times K_2) = \mathbb{R}_+ \setminus \{0\} \times C_{(0,1)}$. Likewise $\mathrm{icr}(K_2) = \emptyset$, since for all $f \in K_2$ if we denote $h(x) = x$ and we take $g = -f/h \in C_{(0,1)}$, then for all $\lambda' > 0$ there exists $0 < \lambda < \lambda'$ such that $(f+\lambda g)(x) < 0$ for some $x \in (0,1)$. In effect it is sufficient to do $\lambda = \lambda'/2$ and $x = \lambda'/4 \in (0,1)$ if $\lambda' < 1$, and $\lambda = 1/2\lambda'$ and $x = 1/4\lambda' \in (0,1)$ if $\lambda' \geq 1$. Analogously we can see that $\mathrm{icr}(\mathrm{cone}(A)) = \emptyset$.

**Theorem 3.8.** *For the problem* (3.2)*, let $K, M_1, M_2$ be pointed convex cones such that* $\mathrm{cor}(K) \neq \emptyset$ *and* $\mathrm{cor}(M_1) \neq \emptyset$. *Let $Z = Z_1 \times Z_2$ and let $x_0 \in X$ be a weakly efficient solution for the problem* (3.2). *If $(f(x)-f(x_0), g_1(x), g_2(x))$ is $(K \times M_1, M_2)$ PGSCL with* $\mathrm{icr}(\mathrm{cone}((f(X)-f(x_0), g_1(X), g_2(X))) + \mathrm{cor}(K \times M_1) \times M_2) \neq \emptyset$ *and if the Slater constraint qualification holds then there exists $T_0 \in \Gamma$ such that $x_0$ is a weakly efficient solution of the unconstrained problem:*

$$K - \mathrm{Min}\{f(x) + T_0(g_1(x), g_2(x)) : x \in X\} \tag{3.3}$$

*and, in addition $T_0(g_1(x_0), g_2(x_0)) = 0$.*

**Proof.** By hypothesis, $K \times M_1 \times M_2$ is a pointed convex cone with $\mathrm{cor}(K \times M_1) \neq \emptyset$. If $x_0 \in X$ is a weakly efficient solution of 3.2 then there is no $x \in X$ such that $(f(x)-f(x_0), g_1(x), g_2(x)) \in -\mathrm{cor}(K) \times (-M_1) \times (-M_2)$ and thus it doesn't exist $x \in X$ such that $(f(x)-f(x_0), g_1(x), g_2(x)) \in -\mathrm{cor}(K \times M_1) \times (-M_2)$. Applying theorem 3.7, there exists $\mu = (\mu_K, \mu_1, \mu_2) \in K' \times M_1' \times M_2', \setminus\{(0,0,0)\}$ such that $< \mu, (f(x)-f(x_0), g_1(x), g_2(x)) > \geq 0$ for all $x \in X$, and so we can write for all $x \in X$:

$$< \mu_K, f(x) > + < \mu_1, g_1(x) > + < \mu_2, g_2(x) > \quad \geq \quad < \mu_K, f(x_0) >$$

If we take $x = x_0$ then $< \mu_1, g_1(x_0) > + < \mu_2, g_2(x_0) > \geq 0$, but since $x_0 \in \Omega$ then must be $< \mu_1, g_1(x_0) > + < \mu_2, g_2(x_0) > = 0$. So we can write

$$< \mu_K, f(x) > + < \mu_1, g_1(x) > + < \mu_2, g_2(x) > \quad \geq$$

$$< \mu_K, f(x_0) > + < \mu_1, g_1(x_0) > + < \mu_2, g_2(x_0) >$$

Furthermore, $\mu_K \neq 0$, otherwise $< \mu_1, g_1(x) > + < \mu_2, g_2(x) > \geq 0$ for all $x \in X$, which contradicts the Slater constraint qualification.
Therefore there exists $k_1 \in K$ such that $\mu_K(k_1) > 0$, otherwise $\mu_K(K) = \{0\}$ and so $\mu_K(-K) = \{0\}$and then $\mu_K(K - K) = \{0\}$, but since $K$ has nonempty algebraic core, it will be $K - K = Y$ and thus $\mu_K(Y) = \{0\}$ which is a contradiction. Then for such $k_1 \in K$ let $y_1 = k_1/\mu_K(k_1) \in K$, and with it we can establish a linear mapping $h : \mathbb{R} \longrightarrow Y, h(r) =< r, y_1 >$.
If $r \geq 0$ then it is $h(r) \in K$ and $\mu_K(h(r)) = < r, \mu_K(y_1) > = < r, 1 > = r$, which means that $\mu_K \circ h$ is the identity mapping.
Let $T_0$ be the linear mapping from $Z_1 \times Z_2$ to $Y$: $T_0 = h\circ(\mu_1, \mu_2)$. For all $(q_1, q_2) \in M_1 \times M_2$, it is $T_0(q_1, q_2) = h((\mu_1, \mu_2))(q_1, q_2) \in h(\mathbb{R}_+) \subset K$. Furthermore $\mu_K \circ T_0 = \mu_K \circ h \circ (\mu_1, \mu_2) = (\mu_1, \mu_2)$, and then

$$\begin{aligned} < \mu_K, f(x) > + < \mu_1, g_1(x) > + < \mu_2, g_2(x) > &= \\ = \ < \mu_K, f(x) > + < \mu_K \circ T_0, (g_1(x), g_2(x)) > \end{aligned}$$

therefore

$$\begin{aligned} < \mu_K, f(x) > + < \mu_1, g_1(x) > + < \mu_2, g_2(x) > &= \\ = \ < \mu_K, f(x) + T_0 \circ (g_1(x), g_2(x)) > \end{aligned}$$

and consequently, $< \mu_K, f(x)+T_0\circ(g_1(x), g_2(x)) > \geq < \mu_K, f(x_0)+T_0\circ(g_1(x_0), g_2(x_0)) >$ for all $x \in X$. This is equivalent to affirming that $x_0$ is a weak efficient solution of the problem:

$$K - Min\{< \mu_K, f(x) + T_0(g_1(x), g_2(x)) >: x \in X\} \tag{3.4}$$

and so $x_0$ is a weakly efficient solution of the problem 3.3. Otherwise if there exists $x' \in X$ such that

$$f(x_0) + T_0 \circ (g_1(x_0), g_2(x_0)) \in f(x') + T_0 \circ (g_1, g_2)(x') + \mathrm{cor}(K)$$

then

$$f(x_0) + T_0 \circ (g_1, g_2)(x_0) - f(x') - T_0 \circ (g_1, g_2)(x') \in \mathrm{cor}(K)$$

and so

$$< \mu_K, f(x_0) + T_0 \circ (g_1, g_2)(x_0) - f(x') - T_0 \circ (g_1, g_2)(x') > \ > \ 0$$

and therefore

$$< \mu_K, f(x_0) + T_0 \circ (g_1, g_2)(x_0) > \ > \ < \mu_K, f(x') + T_0 \circ (g_1, g_2)(x') >$$

which is a contradiction.
Finally, since $< (\mu_1, \mu_2), (g_1(x_0), g_2(x_0)) > = 0$ we have that $T_0(g_1(x_0), g_2(x_0)) = 0$. □
The proof of the following result is similar to the preceding.

**Corollary 3.9.** *For the problem* (3.1), *let $K, M$ be nontrivial pointed convex cones such that* $\mathrm{cor}(K) \neq \emptyset$ *and* $\mathrm{cor}(M) \neq \emptyset$. *Let $x_0 \in X$ be a weakly efficient solution for the problem* (3.1). *If $(f(x)-f(x_0), g(x))$is $K \times M$ GSCL and if the Slater constraint qualification holds then there exists $T_0 \in \Gamma$ such that $x_0$ is a weakly efficient solution of the unconstrained problem:*

$$K - \mathrm{Min}\{f(x) + T_0(g(x)) : x \in X\} \tag{3.5}$$

*and, in addition $T_0(g(x_0)) = 0$.*

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