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Jumping Problems for Fully Nonlinear Elliptic Variational Inequalities

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By means of nonsmooth critical point theory we prove existence of at least two solutions for a general class of variational inequalities when between the obstacle and the behavior at $+\infty$ there is a situation of jumping type.

Keywords: Jumping problems, variational inequalities, nonsmooth critical point theory

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1. Introduction

Starting from the pioneering paper of Ambrosetti and Prodi [1], jumping problems for semilinear elliptic equations of the type

$$\begin{cases} -\sum\limits_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

have been extensively treated (see e.g. [14, 18, 20, 21]).

Moreover, also the case of semilinear variational inequalities with a situation of jumping type has been discussed in [12, 19]. Very recently, quasilinear inequalities of the form:

$$\begin{cases} \int_\Omega \Big\{ \sum\limits_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j(v-u) + \frac{1}{2} \sum\limits_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u (v-u) \Big\}\, dx + \\ - \int_\Omega g(x,u)(v-u)\, dx \geqslant \langle \omega, v-u \rangle \qquad \forall v \in \widetilde{K}_\vartheta, \\ u \in K_\vartheta, \end{cases}$$

where $K_\vartheta = \{u \in H_0^1(\Omega) : \ u \geqslant \vartheta \text{ a. e. in } \Omega\}$, $\widetilde{K}_\vartheta = \{v \in K_\vartheta : \ (v-u) \in L^\infty(\Omega)\}$ and $\vartheta \in H_0^1(\Omega)$, have been considered in [11].

When $\vartheta \equiv -\infty$, namely we have no obstacle and the variational inequality becomes an equation, the problem has been also studied in [5, 6] by A. Canino and has been extended in [13] by the authors to the case of fully nonlinear operators.

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The purpose of this paper is to study the more general class of nonlinear variational inequalities of the type :

$$\begin{cases} \int_\Omega \left\{ \nabla_\xi L(x,u,\nabla u)\cdot\nabla(v-u) + D_s L(x,u,\nabla u)(v-u) \right\} dx + \\ -\int_\Omega g(x,u)(v-u)\,dx \geqslant \langle \omega, v-u\rangle \qquad \forall v\in \widetilde{K}_\vartheta, \\ u\in K_\vartheta. \end{cases} \tag{1}$$

In the main result we shall prove the existence of at least two solutions of (1). The framework is the same of [13], but technical difficulties arise, mainly in the verification of the Palais–Smale condition. This is due to the fact that such condition is proved in [13] using in a crucial way test functions of exponential type. Such test functions are not admissible for the variational inequality, so that a certain number of modifications is required in particular in the proofs of Theorem 4.4 and Theorem 5.2.

As in the previous papers dealing with quasilinear equations and inequalities (see e.g. [3, 5, 6, 7, 11, 22]) we will use variational methods based on the nonsmooth critical point theory of [9, 10]. Let us mention that similar abstract techniques have been developed independently in [15, 16].

2. The main result

In the following, Ω will denote a bounded domain of $\mathbb{R}^n$, $1<p<n$, $\vartheta\in W_0^{1,p}(\Omega)$ with $\vartheta^-\in L^\infty(\Omega)$, $\omega\in W^{-1,p'}(\Omega)$ and

$$L:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$$

is measurable in x for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$ and of class C^1 in (s,ξ) a. e. in Ω. We shall assume that $L(x,s,\cdot)$ is strictly convex and for each $t\in\mathbb{R}$

$$L(x,s,t\xi)=|t|^p L(x,s,\xi) \tag{2}$$

for a. e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$. Furthermore, we assume that :

(i) there exist $\nu>0$ and $b_1\in\mathbb{R}$ such that

$$\nu|\xi|^p\leqslant L(x,s,\xi)\leqslant b_1|\xi|^p, \tag{3}$$

for a. e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$;

(ii) there exist $b_2,b_3\in\mathbb{R}$ such that

$$|D_s L(x,s,\xi)|\leqslant b_2|\xi|^p, \tag{4}$$

for a. e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$ and

$$|\nabla_\xi L(x,s,\xi)|\leqslant b_3|\xi|^{p-1}, \tag{5}$$

for a. e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$;

(*iii*) there exist $R > 0$ and a bounded Lipschitzian function $\psi : [R, +\infty[\to [0, +\infty[$ such that

$$s \geqslant R \Longrightarrow D_s L(x, s, \xi) \geqslant 0, \tag{6}$$

$$s \geqslant R \Longrightarrow D_s L(x, s, \xi) \leqslant \psi'(s) \nabla_\xi L(x, s, \xi) \cdot \xi, \tag{7}$$

for a. e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$. We denote by $\overline{\psi}$ the limit of $\psi(s)$ as $s \to +\infty$.

(*iv*) $g(x, s)$ is a Carathéodory function and $G(x, s) = \int_0^s g(x, \tau)\, d\tau$. We assume that there exist $a \in L^{\frac{np}{n(p-1)+p}}(\Omega)$ and $b \in L^{\frac{n}{p}}(\Omega)$ such that

$$|g(x, s)| \leqslant a(x) + b(x)|s|^{p-1}, \tag{8}$$

for a. e. $x \in \Omega$ and all $s \in \mathbb{R}$. Moreover, there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{s \to +\infty} \frac{g(x, s)}{s^{p-1}} = \alpha, \tag{9}$$

for a. e. $x \in \Omega$.

Set now :

$$\lim_{s \to +\infty} L(x, s, \xi) = L_\infty(x, \xi)$$

(this limit exists by (6)). We also assume that $L_\infty(x, \cdot)$ is strictly convex for a. e. $x \in \Omega$. Let us remark that we are not assuming the strict convexity uniformly in x so that such L_∞ is pretty general. Moreover, assume that

$$s_h \to +\infty,\ \xi_h \to \xi \Longrightarrow \nabla_\xi L(x, s_h, \xi_h) \to \nabla_\xi L_\infty(x, \xi), \tag{10}$$

for a. e. $x \in \Omega$. Let now

$$\lambda_1 = \min \left\{ p \int_\Omega L_\infty(x, \nabla u)\, dx :\ u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p\, dx = 1 \right\}, \tag{11}$$

be the first (nonlinear) eigenvalue of

$$\{u \mapsto -\mathrm{div}\ (\nabla_\xi L_\infty(x, \nabla u))\}.$$

Observe that by [2, Lemma 1.4] the first eigenfunction ϕ_1 belongs to $L^\infty(\Omega)$ and by [23, Theorem 1.1] is strictly positive.

Our purpose is to study (1) when $\omega = -t^{p-1}\phi_1^{p-1}$, namely the family of problems

$$(P_t) \qquad \begin{cases} \int_\Omega \left\{ \nabla_\xi L(x, u, \nabla u) \cdot \nabla(v - u) + D_s L(x, u, \nabla u)(v - u) \right\} dx + \\ - \int_\Omega g(x, u)(v - u)\, dx + t^{p-1} \int_\Omega \phi_1^{p-1}(v - u)\, dx \geqslant 0 \qquad \forall v \in \widetilde{K}_\vartheta, \\ u \in K_\vartheta, \end{cases}$$

where

$$K_\vartheta = \left\{ u \in W_0^{1,p}(\Omega) :\ u \geqslant \vartheta \text{ a. e. in } \Omega \right\}$$

and $\widetilde{K}_\vartheta = \{v \in K_\vartheta :\ (v - u) \in L^\infty(\Omega)\}$.

Under the previous assumptions, the following is our main result :

Theorem 2.1. *Assume that $\alpha > \lambda_1$. Then there exists $\bar{t} \in \mathbb{R}$ such that for all $t \geqslant \bar{t}$ the problem (P_t) has at least two solutions.*

This result extends [11, Theorem 2.1] dealing with Lagrangians of the type

$$L(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x, s)\xi_i\xi_j - G(x, s)$$

for a. e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

In this particular case, existence of at least three solutions has been proved in [6] for equations assuming $\alpha > \mu_2$ where μ_2 is the second eigenvalue of the operator

$$\left\{ u \mapsto - \sum_{i,j=1}^{n} D_j(A_{ij} D_i u) \right\}.$$

In our general setting, since L_∞ is not quadratic with respect to ξ, we only have the existence of the first eigenvalue λ_1 and it is not clear how to define higher order eigenvalues $\lambda_2, \lambda_3, \ldots$. Therefore in our case the comparison of α with higher eigenvalues has no obvious formulation.

3. Recalls from nonsmooth critical point theory

Let (X, d) be a metric space and let $f : X \to \overline{\mathbb{R}}$ be a function. We denote by $B_r(u)$ the open ball of center u and radius r and set $\operatorname{epi}(f) = \{(u, \lambda) \in X \times \mathbb{R} : f(u) \leqslant \lambda\}$. In the following, the space $X \times \mathbb{R}$ will be endowed with the metric

$$d((u, \lambda), (v, \mu)) = ((d(u, v))^2 + (\lambda - \mu)^2)^{\frac{1}{2}}$$

and $\operatorname{epi}(f)$ with the induced metric. Finally, we set $D(f) = \{u \in X : f(u) < +\infty\}$.

Definition 3.1. For every $u \in X$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : (B_\delta(u, f(u)) \cap \operatorname{epi}(f)) \times [0, \delta] \to X$$

satisfying

$$d(\mathcal{H}((v, \mu), t), v) \leqslant t, \quad f(\mathcal{H}((v, \mu), t)) \leqslant \mu - \sigma t,$$

whenever $(v, \mu) \in B_\delta(u, f(u)) \cap \operatorname{epi}(f)$ and $t \in [0, \delta]$. The extended real number $|df|(u)$ is called the *weak slope* of f at u.

The above notion has been introduced, in an equivalent way, independently in [10, 16], while a variant has been considered in [15]. The form mentioned here is taken from [4]. For further details see [11, Section 3].

Definition 3.2. An element $u \in X$ is said to be a *(lower) critical point* of f if $|df|(u) = 0$. A real number c is said to be a *(lower) critical value* of f if there exists a critical point $u \in X$ of f such that $f(u) = c$. Otherwise c is said to be a *regular value* of f.

Definition 3.3. Let c be a real number. The function f is said to satisfy the Palais–Smale condition at level c ($(PS)_c$ for short), if every sequence (u_h) in X with $|df|(u_h) \to 0$ and $f(u_h) \to c$ admits a subsequence converging in X.

We now recall the main existence tool of the paper.

Theorem 3.4. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a function such that $D(f)$ is closed in X and $f_{|D(f)}$ is continuous. Let u_0, v_0, v_1 be in X and suppose that there exists $r > 0$ such that $\|v_0 - u_0\| < r$, $\|v_1 - u_0\| > r$, $\inf f(\overline{B_r(u_0)}) > -\infty$, and*

$$\inf\{f(u) : u \in X, \ \|u - u_0\| = r\} > \max\{f(v_0), f(v_1)\}.$$

Let

$$\Gamma = \{\gamma : [0,1] \to D(f) \text{ continuous with } \gamma(0) = v_0, \gamma(1) = v_1\}$$

and assume that $\Gamma \neq \emptyset$ and that f satisfies the Palais–Smale condition at the two levels

$$c_1 = \inf f(\overline{B_r(u_0)}), \quad c_2 = \inf_{\gamma \in \Gamma} \max_{[0,1]} (f \circ \gamma).$$

Then $-\infty < c_1 < c_2 < +\infty$ and there exist at least two critical points u_1, u_2 of f such that $f(u_i) = c_i \quad (i = 1, 2)$.

Proof. It is sufficient to combine [10, Theorem 3.12] with [11, Proposition 3.4]. □

4. The bounded Palais–Smale condition

In this section we shall consider the more general variational inequalities (1). To this aim let us now introduce the functional $f : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$

$$f(u) = \begin{cases} \int_\Omega L(x, u, \nabla u)\, dx - \int_\Omega G(x, u)\, dx - \langle \omega, u \rangle & u \in K_\vartheta \\ +\infty & u \notin K_\vartheta. \end{cases}$$

Definition 4.1. Let $c \in \mathbb{R}$. A sequence (u_h) in K_ϑ is said to be a concrete Palais–Smale sequence at level c, ($(CPS)_c$–sequence, for short) for f, if $f(u_h) \to c$ and there exists a sequence (φ_h) in $W^{-1,p'}(\Omega)$ such that $\varphi_h \to 0$ and

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(v - u_h)\, dx + \int_\Omega D_s L(x, u_h, \nabla u_h)(v - u_h)\, dx + \\ - \int_\Omega g(x, u_h)(v - u_h)\, dx - \langle \omega, v - u_h \rangle \geqslant \langle \varphi_h, v - u_h \rangle \qquad \forall v \in \widetilde{K}_\vartheta.$$

We say that f satisfies the concrete Palais–Smale condition at level c, ($(CPS)_c$, for short), if every $(CPS)_c$–sequence for f admits a strongly convergent subsequence in $W_0^{1,p}(\Omega)$.

Theorem 4.2. *Let u in K_ϑ be such that $|df|(u) < +\infty$. Then there exists φ in $W^{-1,p'}(\Omega)$ such that $\|\varphi\|_{-1,p'} \leqslant |df|(u)$ and*

$$\int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla(v - u)\, dx + \int_\Omega D_s L(x, u, \nabla u)(v - u)\, dx + \\ - \int_\Omega g(x, u)(v - u)\, dx - \langle \omega, v - u \rangle \geqslant \langle \varphi, v - u \rangle \qquad \forall v \in \widetilde{K}_\vartheta.$$

Proof. Argue as in [11, Theorem 4.6]. □

Proposition 4.3. *Let $c \in \mathbb{R}$ and assume that f satisfies the $(CPS)_c$ condition. Then f satisfies the $(PS)_c$ condition.*

Proof. It is an easy consequence of Theorem 4.2. □

Let us note that by combining (3) with the convexity of $L(x, s, \cdot)$, we get

$$\nabla_\xi L(x, s, \xi) \cdot \xi \geqslant \nu |\xi|^p \tag{12}$$

for a. e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Moreover, there exists $M > 0$ such that

$$|D_s L(x, s, \xi)| \leqslant M \nabla_\xi L(x, s, \xi) \cdot \xi \tag{13}$$

for a. e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Suppose now that $\vartheta(x) > -R$ for a. e. $x \in \Omega$, where $R > 0$ is as in (iii) and define

$$\widetilde{L}(x, s, \xi) = \begin{cases} L(x, s, \xi) & s > -R \\ L(x, -R, \xi) & s \leqslant -R. \end{cases}$$

Such $\widetilde{L}$ satisfy our assumptions. On the other hand, if u satisfies

$$(\widetilde{P}_t) \qquad \begin{cases} \int_\Omega \Big\{ \nabla_\xi \widetilde{L}(x, u, \nabla u) \cdot \nabla(v - u) + D_s \widetilde{L}(x, u, \nabla u)(v - u) \Big\} dx + \\ - \int_\Omega g(x, u)(v - u)\, dx + t^{p-1} \int_\Omega \phi_1^{p-1}(v - u)\, dx \geqslant 0 \qquad \forall v \in \widetilde{K}_\vartheta, \\ u \in K_\vartheta, \end{cases}$$

then u satisfies (P_t). Therefore, up to substituting L with $\widetilde{L}$, we can assume that L satisfies (6) for any $s \in \mathbb{R}$ with $|s| > R$. (Actually $\widetilde{L}$ is only locally Lipschitz in s but one might always define $\widetilde{L}(x, s, \xi) = L(x, \sigma(s), \xi)$ for a suitable smooth function σ).

Now, we want to provide in Theorem 4.5 a very useful criterion for the verification of $(CPS)_c$ condition. Let us first prove a local compactness property for $(CPS)_c$−sequences.

Theorem 4.4. *Let (u_h) be a sequence in K_ϑ and (φ_h) a sequence in $W^{-1,p'}(\Omega)$ such that (u_h) is bounded in $W_0^{1,p}(\Omega)$, $\varphi_h \to \varphi$ and*

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(v - u_h)\, dx + \\ + \int_\Omega D_s L(x, u_h, \nabla u_h)(v - u_h)\, dx \geqslant \langle \varphi_h, v - u_h \rangle \qquad \forall v \in \widetilde{K}_\vartheta. \tag{14}$$

Then it is possible to extract a subsequence (u_{h_k}) strongly convergent in $W_0^{1,p}(\Omega)$.

Proof. Up to a subsequence, (u_h) converges to some u weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a. e. in Ω. Moreover, arguing as in step I of [11, Theorem 4.18] it follows that

$$\nabla u_h(x) \to \nabla u(x) \quad \text{for a. e. } x \in \Omega.$$

We divide the proof into several steps.

I) Let us prove that

$$\limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(-u_h^-) \exp\{-M(u_h - R)^-\}\, dx \leqslant \leqslant \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla(-u^-) \exp\{-M(u - R)^-\}\, dx \tag{15}$$

where $M > 0$ is defined in (13) and $R > 0$ has been introduced in hypothesis (6).

Consider the test functions

$$v = u_h + \zeta \exp\{-M(u_h + R)^+\}$$

in (14) where $\zeta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\zeta \geqslant 0$. Then

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla\zeta \exp\{-M(u_h + R)^+\}\, dx + \\ + \int_\Omega [D_s L(x, u_h, \nabla u_h) - M\nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+]\zeta \exp\{-M(u_h + R)^+\}\, dx \\ \geqslant \langle \varphi_h, \zeta \exp\{-M(u_h + R)^+\}\rangle.$$

From (6) and (13) we deduce that

$$\left[D_s L(x, u_h, \nabla u_h) - M\nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\right] \zeta \exp\{-M(u_h + R)^+\} \leqslant 0,$$

so that by the Fatou's Lemma we get

$$\int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla\zeta \exp\{-M(u + R)^+\}\, dx + \\ + \int_\Omega [D_s L(x, u, \nabla u) - M\nabla_\xi L(x, u, \nabla u) \cdot \nabla(u + R)^+]\zeta \exp\{-M(u + R)^+\}\, dx \geqslant \\ \geqslant \langle \varphi, \zeta \exp\{-M(u + R)^+\}\rangle \qquad \forall \zeta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ \zeta \geqslant 0. \tag{16}$$

Now, let us consider the functions

$$\eta_k = \eta \exp\{M(u + R)^+\}\vartheta_k(u),$$

where $\eta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\eta \geqslant 0$ and $\vartheta_k \in C^\infty(\mathbb{R})$ is such that $0 \leqslant \vartheta_k(s) \leqslant 1$, $\vartheta_k = 1$ on $[-k, k]$, $\vartheta_k = 0$ outside $[-2k, 2k]$ and $|\vartheta_k'| \leqslant c/k$ for some $c > 0$.

Putting them in (16), for each $k \geqslant 1$ we obtain

$$\int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla(\eta\vartheta_k(u))\, dx + \int_\Omega D_s L(x, u, \nabla u)\eta\vartheta_k(u)\, dx \geqslant \\ \geqslant \langle \varphi, \eta\vartheta_k(u)\rangle \qquad \forall \eta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ \eta \geqslant 0.$$

Passing to the limit as $k \to +\infty$ we obtain

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot \nabla\eta\, dx + \int_\Omega D_s L(x,u,\nabla u)\eta\, dx \geqslant$$
$$\geqslant \langle \varphi, \eta\rangle \qquad \forall \eta \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \eta \geqslant 0. \tag{17}$$

Taking $\eta = (\vartheta^- - u^-)\exp\{-M(u-R)^-\} \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ in (17) we get

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot \nabla(\vartheta^- - u^-)\exp\{-M(u-R)^-\}\, dx \geqslant$$
$$\geqslant -\int_\Omega \left[D_s L(x,u,\nabla u) - M\nabla_\xi L(x,u,\nabla u)\cdot\nabla(u-R)^-\right]$$
$$(\vartheta^- - u^-)\exp\{-M(u-R)^-\}dx + \langle \varphi, (\vartheta^- - u^-)\exp\{-M(u-R)^-\}\rangle. \tag{18}$$

On the other hand, taking

$$v = u_h - (\vartheta^- - u_h^-)\exp\{-M(u_h-R)^-\} \geqslant u_h - (\vartheta^- - u_h^-) = u_h^+ - \vartheta^-$$

in (14) we obtain

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla(\vartheta^- - u_h^-)\exp\{-M(u_h-R)^-\}\, dx +$$
$$\int_\Omega \left[D_s L(x,u_h,\nabla u_h) - M\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla(u_h-R)^-\right](\vartheta^- - u_h^-)\exp\{-M(u_h-R)^-\}dx$$
$$\leqslant \langle \varphi_h, (\vartheta^- - u_h^-)\exp\{-M(u_h-R)^-\}\rangle. \tag{19}$$

From (6) and (13) we deduce that

$$D_s L(x,u_h,\nabla u_h) - M\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla(u_h-R)^- \geqslant 0.$$

From (19), using Fatou's Lemma and (18) we obtain

$$\limsup_h \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla(\vartheta^- - u_h^-)\exp\{-M(u_h-R)^-\}\, dx \leqslant$$
$$\leqslant \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot \nabla(\vartheta^- - u^-)\exp\{-M(u-R)^-\}\, dx. \tag{20}$$

Since

$$\lim_h \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla\vartheta^- \exp\{-M(u_h-R)^-\}\, dx =$$
$$= \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot \nabla\vartheta^- \exp\{-M(u-R)^-\}\, dx,$$

then from (20) we deduce (15).

II) Let us now prove that

$$\limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h^+ \exp\{-M(u_h - R)^-\}\, dx \leqslant$$
$$\leqslant \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u^+ \exp\{-M(u - R)^-\}\, dx. \tag{21}$$

We consider the test functions

$$v = u_h - \left[(u_h^+ - \vartheta^+) \wedge k\right] \exp\{-M(u_h - R)^-\} \geqslant \vartheta + (\vartheta^- - u_h^-)$$

in (14). By Fatou's Lemma, we get

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h^+ - \vartheta^+) \exp\{-M(u_h - R)^-\}\, dx +$$
$$\int_\Omega \left[D_s L(x, u_h, \nabla u_h) - M \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h - R)^-\right] (u_h^+ - \vartheta^+) \exp\{-M(u_h - R)^-\}\, dx$$
$$\leqslant \langle \varphi_h, (u_h^+ - \vartheta^+) \exp\{-M(u_h - R)^-\}\rangle \tag{22}$$

from which we deduce that

$$\left[D_s L(x, u_h, \nabla u_h) - M \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h - R)^-\right] (u_h^+ - \vartheta^+) \exp\{-M(u_h - R)^-\}$$

belongs to $L^1(\Omega)$. Using Fatou's Lemma in (22) we obtain

$$\limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h^+ - \vartheta^+) \exp\{-M(u_h - R)^-\}\, dx \leqslant$$
$$\leqslant -\int_\Omega \left[D_s L(x, u, \nabla u) - M \nabla_\xi L(x, u, \nabla u) \cdot \nabla(u - R)^-\right] \tag{23}$$
$$(u^+ - \vartheta^+) \exp\{-M(u - R)^-\} dx + \langle \varphi, (u^+ - \vartheta^+) \exp\{-M(u - R)^-\}\rangle,$$

from which we also deduce that

$$\left[D_s L(x, u, \nabla u) - M \nabla_\xi L(x, u, \nabla u) \cdot \nabla(u - R)^-\right] (u^+ - \vartheta^+) \exp\{-M(u - R)^-\} \tag{24}$$

belongs to $L^1(\Omega)$. Now, taking $\eta_k = [(u^+ - \vartheta^+) \wedge k] \exp\{-M(u - R)^-\}$ in (17), we have

$$\int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla \left[(u^+ - \vartheta^+) \wedge k\right] \exp\{-M(u - R)^-\}\, dx +$$
$$+ \int_\Omega \left[D_s L(x, u, \nabla u) - M \nabla_\xi L(x, u, \nabla u) \cdot \nabla(u - R)^-\right] \left[(u^+ - \vartheta^+) \wedge k\right] \exp\{-M(u - R)^-\} dx$$
$$\geqslant \left\langle \varphi, \left[(u^+ - \vartheta^+) \wedge k\right] \exp\{-M(u - R)^-\}\right\rangle. \tag{25}$$

Using (24) and passing to the limit as $k \to +\infty$ in (25), it results

$$\int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla(u^+ - \vartheta^+) \exp\{-M(u - R)^-\}\, dx +$$
$$+ \int_\Omega \left[D_s L(x, u, \nabla u) - M \nabla_\xi L(x, u, \nabla u) \cdot \nabla(u - R)^-\right] (u^+ - \vartheta^+) \exp\{-M(u - R)^-\} dx$$
$$\geqslant \langle \varphi, (u^+ - \vartheta^+) \exp\{-M(u - R)^-\}\rangle. \tag{26}$$

Combining (26) with (23) we obtain

$$\limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h^+ - \vartheta^+) \exp\{-M(u_h - R)^-\}\, dx \leqslant$$
$$\leqslant \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla(u^+ - \vartheta^+) \exp\{-M(u - R)^-\}\, dx \tag{27}$$

Since

$$\lim_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla\vartheta^+ \exp\{-M(u_h - R)^-\}\, dx =$$
$$= \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla\vartheta^+ \exp\{-M(u - R)^-\}\, dx$$

from (27) we deduce (21).

III) Let us finally prove that $u_h \to u$ strongly in $W_0^{1,p}(\Omega)$. We claim that

$$\limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{-M(u_h - R)^-\}\, dx \leqslant$$
$$\leqslant \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u \exp\{-M(u - R)^-\}\, dx$$

In fact using (15) and (21) we get

$$\limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{-M(u_h - R)^-\}\, dx \leqslant$$
$$\leqslant \limsup_h \int_{\Omega\cap\{u_h \geqslant 0\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h^+ \exp\{-M(u_h - R)^-\}\, dx +$$
$$+ \limsup_h \int_{\Omega\cap\{u_h \leqslant 0\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(-u_h^-) \exp\{-M(u_h - R)^-\}\, dx \leqslant$$
$$\leqslant \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u \exp\{-M(u - R)^-\}\, dx \tag{28}$$

From (28) using the Fatou Lemma we get

$$\lim_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{-M(u_h - R)^-\}\, dx =$$
$$= \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u \exp\{-M(u - R)^-\}\, dx.$$

Therefore, since by (12) we have

$$\nu \exp\{-M(R + \|\vartheta^-\|_\infty)\}|\nabla u_h|^p \leqslant \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{-M(u_h - R)^-\},$$

it follows that

$$\lim_h \int_\Omega |\nabla u_h|^p\, dx = \int_\Omega |\nabla u|^p\, dx,$$

namely the strong convergence of (u_h) to u in $W_0^{1,p}(\Omega)$. □

Theorem 4.5. *For every $c \in \mathbb{R}$ the following assertions are equivalent:*

(a) f satisfies the $(CPS)_c$ condition;

(b) every $(CPS)_c$-sequence for f is bounded in $W_0^{1,p}(\Omega)$.

Proof. Since the map $\{u \mapsto g(x,u)\}$ is completely continuous from $W_0^{1,p}(\Omega)$ to $L^{\frac{np'}{n+p'}}(\Omega)$, the proof goes like [11, Theorem 4.37]. □

5. The Palais–Smale condition

Let us now set

$$g_0(x,s) = g(x,s) - \alpha(s^+)^{p-1}, \quad G_0(x,s) = \int_0^s g_0(x,t)dx.$$

Of course, g_0 is a Carathéodory function satisfying

$$\lim_{s\to+\infty} \frac{g_0(x,s)}{s^{p-1}} = 0, \qquad |g_0(x,s)| \leqslant a(x) + b(x)|s|^{p-1},$$

for a. e. $x \in \Omega$ and all $s \in \mathbb{R}$ where $a \in L^{\frac{np}{n(p-1)+p}}(\Omega)$ and $b \in L^{\frac{n}{p}}(\Omega)$. Then (P_t) is equivalent to finding $u \in K_\vartheta$ such that

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla(v-u)\,dx + \int_\Omega D_s L(x,u,\nabla u)(v-u)\,dx+$$
$$-\alpha\int_\Omega (u^+)^{p-1}(v-u)\,dx - \int_\Omega g_0(x,u)(v-u)\,dx + t^{p-1}\int_\Omega \phi_1^{p-1}(v-u)\,dx \geqslant 0 \quad \forall v \in \widetilde{K}_\vartheta.$$

Let us define the functional $f : W_0^{1,p}(\Omega) \to \mathbb{R}\cup\{+\infty\}$ by setting

$$f(u) = \begin{cases} \int_\Omega L(x,u,\nabla u)\,dx - \frac{\alpha}{p}\int_\Omega (u^+)^p\,dx - \int_\Omega G_0(x,u)\,dx + t^{p-1}\int_\Omega \phi_1^{p-1}u\,dx & \text{if } u \in K_\vartheta \\ +\infty & \text{if } u \notin K_\vartheta. \end{cases}$$

In view of Theorem 4.2, any critical point of f is a weak solutions of (P_t). Let us introduce a new functional $f_t : W_0^{1,p}(\Omega) \to \mathbb{R}\cup\{+\infty\}$ by setting for each $t > 0$

$$f_t(u) = \begin{cases} \int_\Omega L(x,tu,\nabla u)\,dx - \frac{\alpha}{p}\int_\Omega (u^+)^p\,dx - \frac{1}{t^p}\int_\Omega G_0(x,tu)\,dx + \int_\Omega \phi_1^{p-1}u\,dx & \text{if } u \in K_t \\ +\infty & \text{if } u \notin K_t. \end{cases}$$

where we have set

$$K_t = \left\{u \in W_0^{1,p}(\Omega) : tu \geqslant \vartheta \text{ a. e. in } \Omega\right\}.$$

From Theorem 4.2 it follows that if u is a critical point of f_t then tu satisfies (P_t).

Lemma 5.1. *Let (u_h) a sequence in $W_0^{1,p}(\Omega)$ and $\varrho_h \subseteq]0,+\infty[$ with $\varrho_h \to +\infty$. Assume that the sequence $\left(\frac{u_h}{\varrho_h}\right)$ is bounded in $W_0^{1,p}(\Omega)$. Then*

$$\frac{g_0(x,u_h)}{\varrho_h^{p-1}} \to 0 \quad \text{in } L^{\frac{np'}{n+p'}}(\Omega), \qquad \frac{G_0(x,u_h)}{\varrho_h^p} \to 0 \quad \text{in } L^1(\Omega).$$

Proof. Argue as in [5, Lemma 3.3]. □

In view of (12) and (18), we can extend ψ to $[-N,+\infty[$ where N is such that $\|\vartheta^-\|_\infty \leqslant N$, so that assumption (7) becomes

$$s \geqslant -N \Longrightarrow D_s L(x,s,\xi) \leqslant \psi'(s)\nabla_\xi L(x,s,\xi)\cdot\xi. \tag{29}$$

Theorem 5.2. *Let $\alpha > \lambda_1$, $c \in \mathbb{R}$ and let (u_h) in K_ϑ be a $(CPS)_c$-sequence for f. Then (u_h) is bounded in $W_0^{1,p}(\Omega)$.*

Proof. By Definition 4.1, there exists a sequence (φ_h) in $W^{-1,p'}(\Omega)$ with $\varphi_h \to 0$ and

$$\begin{aligned}
&\int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla(v-u_h)\,dx + \int_\Omega D_s L(x,u_h,\nabla u_h)(v-u_h)\,dx + \\
&\quad -\alpha\int_\Omega (u_h^+)^{p-1}(v-u_h)\,dx - \int_\Omega g_0(x,u_h)(v-u_h)\,dx + t^{p-1}\int_\Omega \phi_1^{p-1}(v-u_h)\,dx \geqslant \\
&\quad\quad \geqslant \langle\varphi_h, v-u_h\rangle \qquad \forall v \in K_\vartheta :\ (v-u_h)\in L^\infty(\Omega).
\end{aligned} \tag{30}$$

We set now $\varrho_h = \|u_h\|_{1,p}$, and suppose by contradiction that $\varrho_h \to +\infty$. If we set $z_h = \varrho_h^{-1} u_h$, up to a subsequence, z_h converges to some z weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a. e. in Ω. Note that $z \geqslant 0$ a. e. in Ω.

We shall divide the proof into several steps.

I) We firstly prove that

$$\int_\Omega \nabla_\xi L_\infty(x,\nabla z)\cdot\nabla z\,dx \geqslant \alpha\int_\Omega z^p\,dx. \tag{31}$$

Consider the test functions $v = u_h + (z\wedge k)\exp\{-\psi(u_h)\}$, where ψ is the function defined in (7). Putting such v in (30) and dividing by ϱ_h^{p-1}, we obtain

$$\begin{aligned}
&\int_\Omega \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla(z\wedge k)\exp\{-\psi(u_h)\}\,dx + \\
&+\frac{1}{\varrho_h^{p-1}}\int_\Omega \left[D_s L(x,u_h,\nabla u_h) - \psi'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\right](z\wedge k)\exp\{-\psi(u_h)\}dx \geqslant \\
&\quad\geqslant \alpha\int_\Omega (z_h^+)^{p-1}(z\wedge k)\exp\{-\psi(u_h)\}\,dx + \int_\Omega \frac{g_0(x,u_h)}{\varrho_h^{p-1}}(z\wedge k)\exp\{-\psi(u_h)\}\,dx + \\
&\quad\quad - t^{p-1}\int_\Omega \frac{\phi_1^{p-1}}{\varrho_h^{p-1}}(z\wedge k)\exp\{-\psi(u_h)\}\,dx + \frac{1}{\varrho_h^{p-1}}\langle\varphi_h,(z\wedge k)\exp\{-\psi(u_h)\}\rangle.
\end{aligned}$$

Observe now that the first term

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla(z\wedge k)\exp\{-\psi(u_h)\}\,dx$$

passes to the limit, yielding

$$\int_\Omega \nabla_\xi L_\infty(x,\nabla z)\cdot\nabla(z\wedge k)\exp\{-\overline{\psi}\}\,dx.$$

Indeed, by taking into account assumptions (10) and (5), we may apply [8, Theorem 5] and deduce that, up to a subsequence,

$$\text{a. e. in } \Omega\setminus\{z=0\}:\ \nabla z_h(x)\to\nabla z(x).$$

Since of course, being $u_h(x)\to+\infty$ a. e. in $\Omega\setminus\{z=0\}$, again recalling (10), we have

$$\text{a. e. in } \Omega\setminus\{z=0\}:\ \nabla_\xi L(x,u_h(x),\nabla z_h(x))\to\nabla_\xi L_\infty(x,\nabla z(x)).$$

Since by (5) the sequence $(\nabla_\xi L(x,u_h(x),\nabla z_h(x)))$ is bounded in $L^{p'}(\Omega)$, the assertion follows. Note also that the term

$$\frac{1}{\varrho_h^{p-1}}\langle\varphi_h,(z\wedge k)\exp\{-\psi(u_h)\}\rangle,$$

goes to 0 even if $1<p<2$. Indeed, in this case, one could use the Cerami–Palais–Smale condition, which yields $\varrho_h\varphi_h\to 0$ in $W_0^{-1,p'}(\Omega)$.

Now, by (29) we have

$$D_sL(x,u_h,\nabla u_h)-\psi'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\leqslant 0,$$

then, passing to the limit as $h\to+\infty$, we get

$$\int_\Omega\nabla_\xi L_\infty(x,\nabla z)\cdot\nabla(z\wedge k)\exp\{-\bar\psi\}\,dx\geqslant\alpha\int_\Omega z^{p-1}(z\wedge k)\exp\{-\bar\psi\}\,dx.$$

Passing to the limit as $k\to+\infty$, we obtain (31).

II) Let us prove that $z_h\to z$ strongly in $W_0^{1,p}(\Omega)$, so that of course $\|z\|_{1,p}=1$. Consider the function $\zeta:[-R,+\infty[\to\mathbb{R}$ defined by

$$\zeta(s)=\begin{cases} MR & \text{if} \quad s\geqslant R\\ Ms & \text{if} \quad |s|<R\end{cases} \tag{32}$$

where $M\in\mathbb{R}$ is such that for a. e. $x\in\Omega$, each $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^n$

$$|D_sL(x,s,\xi)|\leqslant M\nabla_\xi L(x,s,\xi)\cdot\xi.$$

If we choose the test functions

$$v=u_h-\frac{u_h-\vartheta}{\exp(MR)}\exp(\zeta(u_h))$$

in (30), we have

$$\int_\Omega\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla(u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx+$$

$$+\int_\Omega\left[D_sL(x,u_h,\nabla u_h)+\zeta'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\right](u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx\leqslant$$

$$\leqslant \alpha \int_\Omega (u_h^+)^{p-1}(u_h - \vartheta)\exp\{\zeta(u_h)\}\,dx + \int_\Omega g_0(x,u_h)(u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx +$$
$$-t^{p-1}\int_\Omega \phi_1^{p-1}(u_h-\vartheta)\exp\{\zeta(u_h)\}\,dx + \langle \varphi_h, (u_h-\vartheta)\exp\{\zeta(u_h)\}\rangle\,.$$

Note that it results

$$[D_s L(x,u_h,\nabla u_h) + \zeta'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h]\,(u_h-\vartheta) \geqslant 0.$$

Therefore, after division by ϱ_h^p we get

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla\left(z_h - \frac{\vartheta}{\varrho_h}\right)\exp\{\zeta(u_h)\}\,dx \leqslant$$
$$\leqslant \alpha\int_\Omega (z_h^+)^{p-1}\left(z_h-\frac{\vartheta}{\varrho_h}\right)\exp\{\zeta(u_h)\}\,dx + \frac{1}{\varrho_h^{p-1}}\int_\Omega g_0(x,u_h)\left(z_h-\frac{\vartheta}{\varrho_h}\right)\exp\{\zeta(u_h)\}\,dx +$$
$$-\frac{t^{p-1}}{\varrho_h^{p-1}}\int_\Omega \phi_1^{p-1}\left(z_h-\frac{\vartheta}{\varrho_h}\right)\exp\{\zeta(u_h)\}\,dx + \frac{1}{\varrho_h^{p-1}}\left\langle \varphi_h, \left(z_h-\frac{\vartheta}{\varrho_h}\right)\exp\{\zeta(u_h)\}\right\rangle,$$

which yields

$$\limsup_h \int_\Omega \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx \leqslant \alpha\exp\{MR\}\int_\Omega z^p\,dx. \tag{33}$$

By combining (33) with (31) we get

$$\limsup_h \int_\Omega \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx \leqslant \exp\{MR\}\int_\Omega \nabla_\xi L_\infty(x,\nabla z)\cdot\nabla z\,dx.$$

In particular, by Fatou's Lemma, it results

$$\begin{aligned}
&\exp\{MR\}\int_\Omega \nabla_\xi L_\infty(x,\nabla z)\cdot\nabla z\,dx \leqslant\\
&\leqslant \liminf_h \int_\Omega \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx \leqslant\\
&\leqslant \limsup_h \int_\Omega \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx \leqslant\\
&\leqslant \exp\{MR\}\int_\Omega \nabla_\xi L_\infty(x,\nabla z)\cdot\nabla z\,dx,
\end{aligned}$$

namely, we get

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,dx \to \int_\Omega \exp\{MR\}\nabla_\xi L_\infty(x,\nabla z)\cdot\nabla z\,dx.$$

Therefore, since

$$\nu\exp\{-MR\}|\nabla z_h|^p \leqslant \nabla_\xi L(x,u_h,\nabla z_h)\cdot\nabla z_h\exp\{\zeta(u_h)\}\,,$$

thanks to the generalized Lebesgue's theorem, we conclude that

$$\lim_h \int_\Omega |\nabla z_h|^p\,dx = \int_\Omega |\nabla z|^p\,dx,$$

and z_h converges to z in $W_0^{1,p}(\Omega)$.

III) Let us consider the test functions $v = u_h + \varphi \exp\{-\psi(u_h)\}$ with $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ and $\varphi \geqslant 0$. Taking such v in (30) and dividing by ϱ_h^{p-1} we obtain

$$\begin{aligned}
&\int_\Omega \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla\varphi \exp\{-\psi(u_h)\}\, dx + \\
&\quad + \frac{1}{\varrho_h^{p-1}} \int_\Omega [D_s L(x, u_h, \nabla u_h) - \psi'(u_h) \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h]\, \varphi \exp\{-\psi(u_h)\}\, dx \geqslant \\
&\qquad \geqslant \alpha \int_\Omega (z_h^+)^{p-1} \varphi \exp\{-\psi(u_h)\}\, dx + \int_\Omega \frac{g_0(x, u_h)}{\varrho_h^{p-1}} \varphi \exp\{-\psi(u_h)\}\, dx + \\
&\qquad\quad - t^{p-1} \int_\Omega \frac{\varphi^{p-1}}{\varrho_h^{p-1}} \exp\{-\psi(u_h)\}\, dx + \frac{1}{\varrho_h^{p-1}} \langle \varphi_h, \varphi \exp\{-\psi(u_h)\} \rangle .
\end{aligned}$$

Note that, since by step II we have $z_h \to z$ in $W_0^{1,p}(\Omega)$, the term

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla\varphi \exp\{-\psi(u_h)\}\, dx$$

passes to the limit, yielding

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla\varphi \exp\{-\overline{\psi}\}\, dx.$$

By means of (29), we have

$$D_s L(x, u_h, \nabla u_h) - \psi'(u_h) \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \leqslant 0,$$

then passing to the limit as $h \to +\infty$, we obtain

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla\varphi \exp\{-\bar{\psi}\}\, dx - \alpha \int_\Omega z^{p-1} \varphi \exp\{-\bar{\psi}\}\, dx \geqslant 0,$$

for each $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ with $\varphi \geqslant 0$ which yields

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla\varphi\, dx \geqslant \alpha \int_\Omega z^{p-1} \varphi\, dx \tag{34}$$

for each $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$.

In a similar fashion, considering in (30) the admissible test functions

$$v = u_h - \left(\varphi \wedge \frac{z_h - \vartheta/\varrho_h}{\exp(\overline{\psi})} \right) \exp(\psi(u_h))$$

with $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ and $\varphi \geqslant 0$ and dividing by ϱ_h^{p-1}, recalling that $z_h \to z$ strongly, we get

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla \left[\varphi \wedge \frac{z}{\exp \overline{\psi}} \right] dx \leqslant \alpha \int_\Omega z^{p-1} \left[\varphi \wedge \frac{z}{\exp \overline{\psi}} \right] dx,$$

for each $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ with $\varphi \geqslant 0$. Actually this holds for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$. By substituting φ with $t\varphi$ with $t > 0$ we obtain

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla \left[\varphi \wedge \frac{z}{t \exp \overline{\psi}}\right] dx \leqslant \alpha \int_\Omega z^{p-1} \left[\varphi \wedge \frac{z}{t \exp \overline{\psi}}\right] dx.$$

Letting $t \to +\infty$, and taking into account (34), it results

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla\varphi \, dx = \alpha \int_\Omega z^{p-1}\varphi \, dx \tag{35}$$

for each $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$. Clearly (35) holds for any $\varphi \in W_0^{1,p}(\Omega)$, so that z is a positive eigenfunction related to α. This is a contradiction by [17, Remark 1, pp. 161]. □

Theorem 5.3. *Let $c \in \mathbb{R}$, $\alpha > \lambda_1$ and $t > 0$. Then f_t satisfies the $(PS)_c-$condition.*

Proof. Since $f_t(u) = \frac{f(tu)}{t^p}$, it is sufficient to combine Theorem 5.2, Theorem 4.5 and Proposition 4.3. □

6. Min–Max estimates

Let us first introduce the "asymptotic functional" $f_\infty : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f_\infty(u) = \begin{cases} \int_\Omega L_\infty(x, \nabla u) \, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx + \int_\Omega \phi_1^{p-1} u \, dx & \text{if } u \in K_\infty \\ +\infty & \text{if } u \notin K_\infty \end{cases}$$

where

$$K_\infty = \left\{ u \in W_0^{1,p}(\Omega) : \; u \geqslant 0 \text{ a. e. in } \Omega \right\}.$$

Proposition 6.1. *There exist $r > 0$, $\sigma > 0$ such that*

(a) for every $u \in W_0^{1,p}(\Omega)$ with $0 < \|u\|_{1,p} \leqslant r$ then $f_\infty(u) > 0$;

(b) for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{1,p} = r$ then $f_\infty(u) \geqslant \sigma > 0$.

Proof. Let us consider the weakly closed set

$$K^* = \left\{ u \in K_\infty : \int_\Omega L_\infty(x, \nabla u) \, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx \leqslant \frac{1}{2} \int_\Omega L_\infty(x, \nabla u) \, dx \right\}.$$

In $K_\infty \setminus K^*$ the statements are evident. On the other hand, it is easy to see that

$$\inf \left\{ \int_\Omega v\phi_1^{p-1} \, dx : \; v \in K^*, \; \|v\|_{1,p} = 1 \right\} = \varepsilon > 0$$

arguing by contradiction. Therefore for each $u \in K^*$ we have

$$f_\infty(u) = \int_\Omega L_\infty(x, \nabla u) \, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx + \int_\Omega \phi_1^{p-1} u \, dx \geqslant c\|u\|_{1,p}^p + \varepsilon \|u\|_{1,p}$$

where $c \in \mathbb{R}$ is a suitable constant. Thus the statements follow. □

Proposition 6.2. *Let $r > 0$ be as in the Proposition* 6.1. *Then there exist $\bar{t} > 0$, $\sigma' > 0$ such that for every $t \geqslant \bar{t}$ and for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{1,p} = r$, then $f_t(u) \geqslant \sigma'$.*

Proof. By contradiction, we can find two sequences $(t_h) \subseteq \mathbb{R}$ and $(u_h) \subseteq W_0^{1,p}(\Omega)$ such that $t_h \geqslant h$ for each $h \in \mathbb{N}$, $\|u_h\|_{1,p} = r$ and $f_{t_h}(u_h) < \frac{1}{h}$. Up to a subsequence, (u_h) weakly converges in $W_0^{1,p}(\Omega)$ to some $u \in K_\infty$. Using (b) of [13, Theorem 5], it follows that

$$f_\infty(u) \leqslant \liminf_h f_{t_h}(u_h) \leqslant 0.$$

By (a) of Proposition 6.1, we have $u = 0$. On the other hand, since

$$\limsup_h f_{t_h}(u_h) \leqslant 0 = f_\infty(u),$$

using (c) of [13, Theorem 5] we deduce that (u_h) strongly converges to u in $W_0^{1,p}(\Omega)$, namely $\|u\|_{1,p} = r$. This is impossible. □

Proposition 6.3. *Let $\sigma', \bar{t}$ as in Proposition* 6.2. *Then there exists $\tilde{t} \geqslant \bar{t}$ such that for every $t \geqslant \tilde{t}$ there exist $v_t, w_t \in W_0^{1,p}(\Omega)$ such that $\|v_t\|_{1,p} < r$, $\|w_t\|_{1,p} > r$, $f_t(v_t) \leqslant \frac{\sigma'}{2}$ and $f_t(w_t) \leqslant \frac{\sigma'}{2}$. Moreover we have*

$$\sup \{f_t((1-s)v_t + sw_t) : \ 0 \leqslant s \leqslant 1\} < +\infty.$$

Proof. We argue by contradiction. We set $\tilde{t} = \bar{t} + h$ and suppose that there exists (t_h) such that $t_h \geqslant h + \bar{t}$ and such that for every v_{t_h}, w_{t_h} in $W_0^{1,p}(\Omega)$ with $\|v_{t_h}\|_{1,p} < r$, $\|w_{t_h}\|_{1,p} > r$ it results $f_{t_h}(v_{t_h}) > \frac{\sigma'}{2}$ and $f_{t_h}(w_{t_h}) > \frac{\sigma'}{2}$. It is easy to prove that there exists a sequence (u_h) in K_{t_h} which strongly converges to 0 in $W_0^{1,p}(\Omega)$ and therefore $\|u_h\|_{1,p} < r$ and $f_{t_h}(u_{t_h}) \leqslant \frac{\sigma'}{2}$ eventually as $h \to +\infty$. This contradicts our assumptions. In a similar way one can prove the statement for w_t, while the last statement is straightforward. □

7. Proof of the main result

Proof of Theorem 2.1. By combining Theorem 5.3, Propositions 6.2 and 6.3 we can apply Theorem 3.4 and deduce the assertion. □

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