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Anisotropic Elliptic Equations in L^m*

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In this paper, we prove the existence of solutions to anisotropic nonlinear elliptic equations with right hand side term in $L^m(\Omega)$ and obtain the appropriate function space for the weak solutions. This paper gives a generalization of some results given in [1] and [3].

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1. Introduction

Let Ω be an open bounded set of $R^N (N \geq 2)$, $p_i > 1$, $(i = 1, 2, \cdots, N)$ and $a : \Omega \times R \times R^N \to R^N$ be a Carathéodory function. We assume that there exist two real positive constants α, β and a nonnegative function $h \in L^1(\Omega)$ such that for any $s \in R$, $\xi \in R^N$, $\eta \in R^N$ and for almost every $x \in \Omega$, every component $a_j(x, s, \xi)$ of a,

$$a(x, s, \xi)\xi \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{1}$$

$$|a_j(x, s, \xi)| \leq \beta(h(x) + |s|^{\overline{p}} + \sum_{i=1}^{N} |\xi_i|^{p_i})^{1-\frac{1}{p_j}}, \tag{2}$$

where $\overline{p}$ satisfies $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{p_i}$.

$$[a(x, s, \xi) - a(x, s, \eta)][\xi - \eta] > 0, \quad \xi \neq \eta. \tag{3}$$

The aim of this paper is to obtain a solution of the anisotropic elliptic equation

$$(P) \qquad \begin{cases} -\operatorname{div}(a(x, u, Du)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense of the distributions. When $f \in L^m(\Omega)$ with m satisfies

$$1 < m < \overline{m} = \frac{N\overline{p}}{N\overline{p} - N + \overline{p}}. \tag{4}$$

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We also assume

$$2-\frac{1}{N}<p_i<\frac{(N-1)\bar{p}}{N-\bar{p}},\quad \text{and } \overline{p}<N, i=1,2,\cdots,N. \tag{5}$$

Set

$$m^*=\frac{Nm}{N-m} \tag{6}$$

and

$$W_0^{1,(r_i)}(\Omega)=\{u\in W_0^{1,1}(\Omega)\mid D_iu\in L^{r_i}(\Omega)\},(r_i\ge 1,i=1,2,\cdots,N). \tag{7}$$

If a does not depend on x and s, namely $a(x,s,\xi)\equiv a(\xi)$, $a(\xi)$ is the vector field whose components are $|\xi_i|^{p_i-2}\xi_i$ $(i=1,2,\cdots,N;p_i>1)$. In [1], it has been proved that there exists a weak solution $u\in\bigcap_{i=1}^N W_0^{1,(r_i)}(\Omega)$ with $1\le r_i<\frac{p_i(\overline{p}-1)N}{\overline{p}(N-1)}$ when $f\in M_b(\Omega)$, and there exists a weak solution $u\in\bigcap_{i=1}^N W_0^{1,(q_i)}(\Omega)$ with $q_i=\frac{p_i(\overline{p}-1)N}{\overline{p}(N-1)}$ when $f\in L^1\log L^1(\Omega)$ too.

If $p_1=p_2=\cdots=p_N=p$, the existence results have been proved in [3] when $f\in M_b(\Omega)$,$f\in L^1\log L^1(\Omega)$ and $f\in L^m(\Omega)$ with $1<m<\frac{Np}{Np-N+p}$.

We consider the existence of weak solutions to problem (P) when $f\in L^m(\Omega)$ $(m>1)$ here. If $\overline{p}=N$, then $\overline{m}=1$, and if f is in $L^m(\Omega)$, then $m>\overline{m}=1$, and problem (P) is known to have a weak solution in $\bigcap_{i=1}^N W_0^{1,(p_i)}(\Omega)$ by [4] (since $f\in(\bigcap_{i=1}^N W_0^{1,(p_i)}(\Omega))'$).

Let us now assume that $\overline{p}<N$. Then $\overline{m}>1$ and if f is in $L^m(\Omega)$, $m\ge\overline{m}$, Problem (P) is known to have a weak solution in $\bigcap_{i=1}^N W_0^{1,(p_i)}(\Omega)$ by [4] (since $f\in(\bigcap_{i=1}^N W_0^{1,(p_i)}(\Omega))'$). The only case of interest is when f is in $L^m(\Omega)$ with $1<m<\overline{m}$, and we prove the following theorem.

Theorem 1.1. *Assume that* (1)–(3) *and* (5). *Let* $1<m<\overline{m}=\frac{N\overline{p}}{N\overline{p}-N+\overline{p}}$ *and* f *be in* $L^m(\Omega)$. *Then problem* (P) *exists a weak solution* $u\in\bigcap_{i=1}^N W_0^{1,(q_i)}(\Omega)$, *with* $q_i=\frac{p_i(\overline{p}-1)m^*}{\overline{p}}$.

Remark 1.2. The Theorem extends the results of Proposition 1 in [2] and Theorem 3 in [3]. Furthermore it can be even as a regularity theorem regarding the solution u obtained in Theorem 1 in [1].

2. Proof of Theorem 1.1

In order to prove the Theorem 1.1, we need the following nonisotropic Sobolev inequality (cf. [1, 5]).

Lemma 2.1. *If* $u\in\bigcap_{i=1}^N W_0^{1,(r_i)}(\Omega)$, $r_i\ge 1(i=1,2,\cdots,N)$, *then*

$$\|u\|_{L^s(\Omega)}\le C_1(\prod_{i=1}^N\|D_iu\|_{L^{r_i}(\Omega)})^{\frac{1}{N}}, \tag{8}$$

where $s=\overline{r}^*=\frac{N\overline{r}}{N-\overline{r}}$ *if* $\overline{r}<N$, $\overline{r}$ *satisfies* $\frac{1}{\overline{r}}=\frac{1}{N}\sum_{i=1}^N\frac{1}{r_i}$, C_1 *is a positive contant depending only on* N *and* $r_i,(i=1,2,\cdots,N)$; *if* $\overline{r}\ge N$, *then* (8) *is satisfied for every* $s\in[1,+\infty)$ *and* C_1 *depends also on* s *and* $\operatorname{meas}\Omega$.

By the density property, we may choose a sequence $\{f_k\} \subset C_0^\infty(\Omega)$,

$$f_k \longrightarrow f \ \text{ strongly in } L^m(\Omega), \text{as } k \longrightarrow \infty, \tag{9}$$

such that

$$\|f_k\|_{L^m(\Omega)} \leq \|f\|_{L^m(\Omega)}, \ k = 1, 2, \cdots . \tag{10}$$

We consider the following approximation problem:

$$(P_k) \qquad \begin{cases} -\operatorname{div}(a(x, u_k, Du_k)) = f_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

In the following, we will give a generalization of Estimate 3 in [3].

Lemma 2.2. *Assume* (1)–(3), (9)–(10) *and* (5). *Let* $1 < m < \overline{m}$, *then for any given* $k \geq 1$, *there exists a weak solution* $u_k \in \bigcap_{i=1}^N W_0^{1,(p_i)}(\Omega)$ *to problem* (P_k), *moreover, we have*

$$\|D_i u_k\|_{L^{q_i}(\Omega)} \leq C_2, \ q_i = \frac{p_i(\overline{p} - 1)m^*}{\overline{p}}, i = 1, 2, \cdots, N \tag{11}$$

and

$$\|u_k\|_{L^{\overline{q}^*}(\Omega)} \leq C_2, \tag{12}$$

where $\overline{q}^* = \frac{N\overline{q}}{N-\overline{q}}$, $\overline{q} = \frac{N}{\sum_{i=1}^N \frac{1}{q_i}}$, C_2 *is a positive constant independent of* k.

Proof. For any given $k \geq 1$, by [4], it is easy to prove that problem (P_k) admits a weak solution $u_k \in \bigcap_{i=1}^N W_0^{1,(p_i)}(\Omega)$ such that

$$\int_\Omega a(x, u_k, Du_k) Dv dx = \int_\Omega f_k v dx, \forall v \in \bigcap_{i=1}^N W_0^{1,(p_i)}(\Omega). \tag{13}$$

To prove Lemma 2.2, we use a choice of a test functions as in [6]. For $0 < s < 1$, define ϕ as

$$\phi(y) = \int_0^y (1 + |t|)^{-s} dt, \quad \forall y \in R. \tag{14}$$

It is easy to see that $\phi(u_k) \in \bigcap_{i=1}^N W_0^{1,(p_i)}(\Omega)$, taking $v = \phi(u_k)$ in (13), we obtain

$$\int_\Omega a(x, u_k, Du_k)\phi' Du_k dx = \int_\Omega f_k \phi(u_k) dx. \tag{15}$$

Noting (1) and (14), (15) yields

$$\sum_{i=1}^N \int_\Omega \frac{|D_i u_k|^{p_i}}{(1 + |u_k|)^s} dx \leq \frac{1}{\alpha(1 - s)} \int_\Omega |f_k|(1 + |u_k|)^{1-s} dx. \tag{16}$$

For any $q_i < p_i$ and $1 \le i \le N$, Hölder's inequality and (16) imply that

$$\begin{aligned}\int_\Omega |D_i u_k|^{q_i} dx &\le \Big(\int_\Omega \frac{|D_i u_k|^{p_i}}{(1+|u_k|)^s} dx\Big)^{\frac{q_i}{p_i}} \Big(\int_\Omega (1+|u_k|)^{\frac{sq_i}{p_i-q_i}} dx\Big)^{1-\frac{q_i}{p_i}} \\ &\le [\alpha(1-s)]^{-\frac{q_i}{p_i}} \Big(\int_\Omega |f_k|(1+|u_k|)^{1-s} dx\Big)^{\frac{q_i}{p_i}} \Big(\int_\Omega (1+|u_k|)^{\frac{sq_i}{p_i-q_i}} dx\Big)^{1-\frac{q_i}{p_i}}. \qquad (17)\end{aligned}$$

If

$$\overline{q}^* = \frac{sq_i}{p_i - q_i}, \qquad (18)$$

(10), Hölder's inequality and (17) yield

$$\begin{aligned}&\int_\Omega |D_i u_k|^{q_i} dx \\ &\le [\alpha(1-s)]^{-\frac{q_i}{p_i}} \|f_k\|_{L^m(\Omega)}^{\frac{q_i}{p_i}} \Big(\int_\Omega (1+|u_k|)^{(1-s)m'} dx\Big)^{\frac{q_i}{m'p_i}} \Big(\int_\Omega (1+|u_k|)^{\overline{q}^*} dx\Big)^{1-\frac{q_i}{p_i}} \\ &\le [\alpha(1-s)]^{-\frac{q_i}{p_i}} \|f\|_{L^m(\Omega)}^{\frac{q_i}{p_i}} \Big(\int_\Omega (1+|u_k|)^{(1-s)m'} dx\Big)^{\frac{q_i}{m'p_i}} \Big(\int_\Omega (1+|u_k|)^{\overline{q}^*} dx\Big)^{1-\frac{q_i}{p_i}} \qquad (19) \\ &= C_3 \Big(\int_\Omega (1+|u_k|)^{(1-s)m'} dx\Big)^{\frac{q_i}{m'p_i}} \Big(\int_\Omega (1+|u_k|)^{\overline{q}^*} dx\Big)^{1-\frac{q_i}{p_i}},\end{aligned}$$

where $C_3 = [\alpha(1-s)]^{-\frac{q_i}{p_i}} \|f\|_{L^m(\Omega)}^{\frac{q_i}{p_i}}$, $m' = \frac{m}{m-1}$.
If

$$m'(1-s) = \overline{q}^*, \qquad (20)$$

we get

$$\int_\Omega |D_i u_k|^{q_i} dx \le C_4 + C_5 \Big(\int_\Omega |u_k|^{\overline{q}^*} dx\Big)^{1-\frac{q_i}{p_i}+\frac{q_i}{m'p_i}} \qquad (21)$$

where C_4 and C_5 are two positive constant independent of k.
By (18) and (20), we obtain

$$\overline{q} = (\overline{p}-1)m^*, q_i = \frac{p_i}{\overline{p}}(\overline{p}-1)m^*. \ i = 1, 2, \cdots, N. \qquad (22)$$

Taking $r_i = q_i$, $s = \overline{q}^*$ in Lemma 2.1, we have

$$\Big(\int_\Omega |u_k|^{\overline{q}^*} dx\Big) \le C_1^{\overline{q}^*} \Big(\prod_{j=1}^N \|D_j u_k\|_{L^{q_j}(\Omega)}\Big)^{\frac{\overline{q}^*}{N}} \qquad (23)$$

where C_1 is a positive constant depending only on N and $q_i (i = 1, 2, \cdots, N)$, but independent of k. Putting (23) into (21), we get for any i, with $1 \le i \le N$

$$\int_\Omega |D_i u_k|^{q_i} dx \le C_4 + C_5 C_1^{\overline{q}^*(1-\frac{q_i}{p_i}+\frac{q_i}{m'p_i})} \Big(\prod_{j=1}^N \|D_j u_k\|_{L^{q_j}(\Omega)}\Big)^{\frac{\overline{q}^*}{N}(1-\frac{q_i}{mp_i})}. \qquad (24)$$

Therefore, there exist two positive constants C_6 and C_7 independent of k, such that

$$\|D_i u_k\|_{L^{q_i}(\Omega)} \le C_6 + C_7 \Big(\prod_{j=1}^{N} \|D_j u_k\|_{L^{q_j}(\Omega)}\Big)^{\frac{\overline{q}^*}{N}(\frac{1}{q_i} - \frac{1}{mp_i})}, \ i = 1, 2, \cdots, N. \tag{25}$$

Let

$$d = \prod_{j=1}^{N} \|D_j u_k\|_{L^{q_j}(\Omega)}. \tag{26}$$

By (25), we get

$$d \le C_8 + C_9 d^{\frac{\overline{q}^*}{N}\sum_{i=1}^{N}(\frac{1}{q_i} - \frac{1}{mp_i})} = C_8 + C_9 d^{\overline{q}^*(\frac{1}{\overline{q}} - \frac{1}{m\overline{p}})} \tag{27}$$

where C_8 and C_9 are two positive constants independent of k. By (22) and the conditions satisfied by m and $\overline{p}$, we have

$$\overline{q}^*\Big(\frac{1}{\overline{q}} - \frac{1}{m\overline{p}}\Big) < 1. \tag{28}$$

By (28) and (27), there exists a positive constant C_{10} independent of k, such that

$$d \le C_{10}. \tag{29}$$

Thus (11) follows from (29) and (25). Lemma 2.1 (taking $r_i = q_i$) and (11) yield (12), and by (5), we have $q_i > 1$ and $\frac{q_i}{p_i - 1} > 1$. This finishes the proof of Lemma 2.2. □

Proof of Theorem 1.1. Using Lemma 2.1 and Lemma 2.2, Theorem 1.1 can follow as in [3]. In fact, by (11) and (12), there exists a subsequence of $\{u_k\}$(still denoted by $\{u_k\}$) such that

$$D_i u_k \longrightarrow D_i u \quad \text{weakly in } L^{q_i}(\Omega), i = 1, 2, \cdots, N, \tag{30}$$

$$u_k \longrightarrow u \quad \text{strongly in } L^{\overline{q}}(\Omega), \tag{31}$$

$$u_k \longrightarrow u \quad \text{a. e. in } \Omega. \tag{32}$$

Using the same method as [3], we can prove

$$D_i u_k \longrightarrow D_i u \ \text{ a. e. in } \Omega, i = 1, 2, \cdots, N. \tag{33}$$

Since a is a Carathéodory function in $\Omega \times R \times R^N$, by (32) and (33), we get

$$a_i(x, u_k(x), Du_k(x)) \longrightarrow a_i(x, u(x), Du(x)), \quad \text{a. e. in } \Omega. \tag{34}$$

By (2), (11) and (12), there exists a positive constant C_{11} independent of k, such that

$$\|a_i(\cdot, u_k, Du_k)\|_{L^{\frac{p_i(\overline{p}-1)m^*}{(p_i-1)\overline{p}}}(\Omega)} \le C_{11}. \tag{35}$$

By (34) and (35), we obtain

$$a_i(\cdot, u_k, Du_k) \longrightarrow a_i(\cdot, u, Du) \ \text{ weakly in } L^{\frac{p_i(\overline{p}-1)m^*}{(p_i-1)\overline{p}}}(\Omega). \tag{36}$$

By (36) and (9), let $k \to \infty$ in (13), we get

$$\int_\Omega a(x, u, Du) Dv dx = \int_\Omega f v dx, \quad \forall v \in C_0^\infty(\Omega). \tag{37}$$

Therefore u is a weak solution to problem (P) and $u \in \bigcap_{i=1}^{N} W_0^{1,(q_i)}(\Omega)$ with $q_i = \frac{p_i(\overline{p}-1)}{\overline{p}} m^*$. Thus Theorem 1.1 is proved. □

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