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Regularity of Minimizers for a Class of Anisotropic Free Discontinuity Problems

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This paper contains existence and regularity results for solutions $u : \Omega \to \mathbb{R}^{n^N}$ of a class of free discontinuity problems i.e.: the energy to minimize consists of both a bulk and a surface part. The main feature of the class of problems considered here is that the energy density of the bulk part is supposed to be fully anisotropic with p-growth in the scalar case, $n = 1$. Similar results for the vectorial case $n > 1$ are obtained for radial energy densities, being anisotropic again with p-growth.

1. Introduction

Within the framework of the so called *free discontinuity problems*, a class of minimum problems characterized by a competition between volume and surface energies, functionals of the type

$$\mathcal{G}(K,u) = \int_{\Omega\setminus K} f(x,u(x),\nabla u(x))\,dx + \alpha \int_{\Omega\setminus K} |u(x)-g(x)|^q\,dx + \beta\mathcal{H}^{N-1}(K\cap\Omega)$$

have been suggested as models to describe phenomena in image segmentation, fracture mechanics and phase transitions (see [11], [25], [19]). Here, $\Omega \subset \mathbb{R}^N$ is a bounded open set, $K \subset \mathbb{R}^N$ is a closed set, $u \in [W^{1,p}(\Omega\setminus K)]^n$, $g \in [L^\infty(\Omega)]^n$, and $\alpha, \beta > 0$, $p > 1$, $q \geq 1$. For instance, in the Mumford–Shah functional typically Ω is a rectangle in the plane, the datum $g : \Omega \to [0,1]$ represents the grey level of a picture, α and β are scale and contrast parameters and $f(z) = |z|^2$. Then, one looks for a piecewise smooth approximation u of the given image g outside a set of contours K. On the other hand, in fracture mechanics $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ represents an elastic deformation, which is assumed to be smooth outside the crack site K.

In order to find minimizers for $\mathcal{G}$, as usual one may relax the functional by extending it to a larger class, where lower semicontinuity and compactness results may be found more

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easily. Following the ideas contained in the seminal paper [16] of De Giorgi, Carriero and Leaci, we may thus associate to $\mathcal{G}$ the functional

$$\mathcal{F}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx + \alpha \int_\Omega |u(x) - g(x)|^q\, dx + \beta \mathcal{H}^{N-1}(S_u \cap \Omega) \ ,$$

where u belongs to the space SBV of the special functions of bounded variation and S_u is the approximate discontinuity set of u (*cf.* the definitions given at the beginning of Section 2). Under standard convexity (or quasiconvexity) assumptions on f the compactness and lower semicontinuity theorems of Ambrosio (see [5], [6], [7] and [8]) guarantee the existence of minimizers of $\mathcal{F}$. Then, the existence of a "classical" minimizer of $\mathcal{G}$ is obtained through a regularity argument. Indeed, if one proves that for a SBV minimizer u of $\mathcal{F}$

$$\mathcal{H}^{N-1}((\overline{S}_u \setminus S_u) \cap \Omega) = 0 \ , \tag{1}$$

then it is easy to prove that the pair $(\overline{S}_u, u)$ minimizes $\mathcal{G}$.

The existence result for the Mumford-Shah functional obtained in [16] (see also [14] for a different approach) has been subsequently extended to various types of minimum problems involving convex bulk energies ([13], [20], [26]) and in some special case even to quasi-convex integrands ([1], [2]). However in all these papers f is assumed to be positively homogeneous with respect to ∇u or at least close at infinity to a positively homogeneous function.

In this paper we prove the existence of minimizers for $\mathcal{G}$ without assuming any type of homogeneity for f. Moreover we treat also the case in which f depends on all variables (x, u, z), thus covering a larger class of variational models (for the precise statement see Theorem 3.1). For instance, in the scalar case our result applies to an integrand of the type $f(z) = |z|^p + h(z)$, where h is any convex function satisfying the growth condition

$$0 \le h(z) \le L|z|^p \qquad \forall z \in \mathbb{R}^N \ .$$

Similarly (see Theorem 4.1), a vectorial case which is covered by our result is when the bulk energy is of the form

$$\int_{\Omega \setminus K} [f(|\nabla u|) + \phi(\nabla u)]\ dx \ ,$$

where as above $f(t)$ is a strictly convex function growing at infinity like $|t|^p$ and ϕ is a lower order quasiconvex function.

Our proof is essentially based on two results. The first one is a version of the decay estimate of the energy in small balls which is obtained through a typical Γ-convergence argument. The second one is an L^∞ gradient estimate for a $W^{1,p}$ minimizer u of a functional of the type $\int_\Omega f(\nabla u) dx$, whose main feature is that f is not assumed to be differentiable, hence there is no Euler equation satisfied by u. In the scalar case this estimate was proved in [20]. Here, this estimate is extended to the case $n > 1$, under the further assumption that the functional depends on the modulus of the gradient, but again without requiring any differentiability of f.

2. Preliminary Results

In the sequel Ω denotes a bounded open set of $\mathbb{R}^N$ and $B_R(x_0)$ is the ball $\{x \in \mathbb{R}^N : |x - x_0| < R\}$. We write simply B_R instead of $B_R(x_0)$ if $x_0 = 0$. Also, $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbb{R}^N$, ω_N is the measure of the unit ball and $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

If $u \in [L^1_{\rm loc}(\Omega)]^n$ we say that u has *approximate limit* at $x \in \Omega$ if there exists $\tilde{u}(x) \in \mathbb{R}^n$ such that

$$\lim_{\varrho \downarrow 0} \fint_{B_\varrho(x)} |u(y) - \tilde{u}(x)|\, dy = 0 \ .$$

Notice that this definition of approximate limit is different from the one contained in Federer's book (see [18, 2.9.12]) and commonly used in literature. However it is easily checked that the two definitions are equivalent if u is locally bounded; moreover if u is a BV function the two definitions coincide for $\mathcal{H}^{N-1}$-a.e. x (see [10, Section 3.6]). Denoting by S_u the *approximate discontinuity set* of u, *i.e.* the set of points in Ω where the approximate limit of u does not exist, we recall that if $u \in [BV_{\rm loc}(\Omega)]^n$, then S_u is H^{N-1}-countably rectifiable (see [18, Th. 4.5.9] or [10, Th. 3.78]). It is then well known that if $u \in [BV_{\rm loc}(\Omega)]^n$ the distributional derivative Du can be decomposed as $Du = \nabla u \mathcal{L}^N + D^s u$, where ∇u is the density of Du with respect to the Lebesgue N-dimensional measure $\mathcal{L}^N$ and $D^s u$ is the singular part of Du with respect to $\mathcal{L}^N$.

We recall that the space of *special functions of bounded variation* $[SBV(\Omega)]^n$ introduced in [15] consists of all functions in $[BV(\Omega)]^n$ such that $D^s u$ is supported in S_u, *i.e.* $|D^s u|(\Omega \setminus S_u) = 0$. Therefore if $u \in [SBV(\Omega)]^n$, then (see [10, Sec. 4.1])

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner S_u \ ,$$

where ν_u is the approximate normal to S_u and u^+ and u^- are the upper and lower traces of u on S_u (oriented by ν_u). For the study of the main properties of SBV functions we refer to [15], [5], [6], [7] and to [10, Chap. 4].

If $u : \Omega \to \mathbb{R}$ is a measurable function we denote by m any median of u, *i.e.* any number with the property that

$$|\{x \in \Omega : \ u(x) < t\}| \leq \frac{|\Omega|}{2} \ \ \forall t < m \ , \qquad |\{x \in \Omega : \ u(x) > t\}| \leq \frac{|\Omega|}{2} \ \ \forall t > m \ .$$

The next theorem can be found in [16, Th. 3.5] or also in [10, Prop. 7.5 and Remark 7.6].

Proposition 2.1. *Let $B \subset \mathbb{R}^N$ be a ball, let $(u_h) \subset SBV(B)$ be a sequence such that for some $p > 1$*

$$\sup_{h \in \mathbb{N}} \int_B |\nabla u_h|^p \, dx < \infty \ , \qquad \lim_{h \to \infty} \mathcal{H}^{N-1}(S_{u_h}) = 0$$

and let m_h be medians of u_h in B. Then there exist a subsequence (u_{h_k}) and a function $u \in W^{1,p}(B)$ such that

$$u_{h_k}(x) - m_{h_k} \to u(x) \qquad\qquad \mathcal{L}^N\text{ -a.e. in } B \ .$$

Moreover there exist constants α_k, β_k such that, setting

$$\overline{u}_{h_k} = (u_{h_k} \vee \alpha_k) \wedge \beta_k \ ,$$

then

$$\overline{u}_{h_k} - m_{h_k} \to u \qquad\qquad \text{in } L^p(B)$$

and

$$|\{u_{h_k} \neq \overline{u}_{h_k}\}| \leq \gamma \left[\mathcal{H}^{N-1}(S_{u_{h_k}})\right]^{N/(N-1)} , \tag{2}$$

where γ is a suitable constant depending only on the dimension N.

The following result, due to Ambrosio [8] (see also [10, Th. 5.29]), provides the lower semicontinuity result needed to prove the existence of minimizers for the functional $\mathcal{F}$.

Theorem 2.2. *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nN} \to [0,\infty)$ be a Carathéodory function satisfying*

$$0 \leq f(x,u,\xi) \leq a(x) + \psi(|u|)(1+|\xi|^p) \qquad \forall (x,u,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{nN} \tag{3}$$

for some $p > 1$, $a \in L^1(\Omega)$ and some increasing function $\psi : [0,\infty) \to [0,\infty)$. If $\xi \mapsto f(x,u,\xi)$ is quasiconvex in $\mathbb{R}^{nN}$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and any $u \in \mathbb{R}^n$, then

$$\liminf_{h\to\infty} \int_\Omega f(x,u_h,\nabla u_h)\,dx \geq \int_\Omega f(x,u,\nabla u)\,dx$$

for any sequence $(u_h) \subset [SBV(\Omega)]^n$ converging in $[L^1(\Omega)]^n$ to $u \in [SBV(\Omega)]^n$ and satisfying

$$\sup_{h\in\mathbb{N}} \left[\int_\Omega |\nabla u_h|^p\,dx + \mathcal{H}^{N-1}(S_{u_h})\right] < \infty .$$

Whenever f is a Carathéodory integrand satisfying (3), $c > 0$ and $E \subset \Omega$ is a Borel set, we set

$$F(u,c,E) = \int_E f(x,u(x),\nabla u(x))\,dx + c\mathcal{H}^{N-1}(S_u \cap E) .$$

As in [9], the following definition is introduced to measure how far is a SBV function u from being a minimizer of the above functional F.

Definition 2.3 (Deviation from minimality). Let $u \in [SBV_{\text{loc}}(\Omega)]^n$ be such that $F(u,c,A) < \infty$ for all open sets $A \subset\subset \Omega$. We call *deviation from minimality* $\text{Dev}(u,c,\Omega)$ of u the smallest $\lambda \in [0,\infty]$ such that

$$F(u,c,A) \leq F(v,c,A) + \lambda$$

for all $v \in [SBV_{\text{loc}}(\Omega)]^n$ and all open $A \subset\subset \Omega$ satisfying $\{v \neq u\} \subset\subset A \subset\subset \Omega$.

If $\text{Dev}(u,c,\Omega) = 0$ we say that u is a *local minimizer* of $F(u,c,\Omega)$ in Ω.

Notice however that if $u \in [W^{1,p}_{\text{loc}}(\Omega)]^n$ we say that u is a local minimizer of the functional

$$F_0(u,\Omega) = \int_\Omega f(x,u(x),\nabla u(x))\,dx \tag{4}$$

if $F_0(u,A) \leq F_0(v,A)$ for all $v \in [W^{1,p}_{\text{loc}}(\Omega)]^n$ and all open $A \subset\subset \Omega$ such that $\{v \neq u\} \subset\subset A \subset\subset \Omega$.

The following lemma is an easy consequence of the definition of deviation from minimality. It is obtained comparing the energy of a function u in a ball $B_{\varrho'}$ with the energy of the function $v\chi_{B_\varrho} + u\chi_{B_{\varrho'}\setminus B_\varrho}$, where $\varrho < \varrho'$. In the estimate the area of the set $\{\tilde{u} \neq \tilde{v}\} \cap \partial B_\varrho$ appears since the comparison function is not approximately continuous in this set.

Lemma 2.4. *Let $u, v \in [SBV_{\mathrm{loc}}(B_R)]^n$, $\varrho < \varrho' < R$. If $\mathcal{H}^{N-1}(S_v \cap \partial B_\varrho) = 0$ and $F(u,c,B_{\varrho'}) < \infty$, $F(v,c,B_{\varrho'}) < \infty$, then*

$$F(u,c,B_\varrho) \leq F(v,c,B_\varrho) + c\mathcal{H}^{N-1}(\{\tilde{u} \neq \tilde{v}\} \cap \partial B_\varrho\}) + \mathrm{Dev}(u,c,B_{\varrho'}) \ ,$$

$$\mathrm{Dev}(v,c,B_\varrho) \leq F(v,c,B_\varrho) - F(u,c,B_\varrho) + \mathcal{H}^{N-1}(\{\tilde{u} \neq \tilde{v}\} \cap \partial B_\varrho\}) + \mathrm{Dev}(u,c,B_{\varrho'}) \ .$$

The proof of the lemma may be found in [16] or [10, Lemma 7.3].

The following result, which is also known as *biting lemma*, and of which there exist various versions and proofs in literature (see *e.g.* [3], [12], [24]) is used in the proof of Theorem 2.6.

Lemma 2.5. *Let $(u_h) \subset [L^1(\Omega)]^n$ be bounded. Then for all $\varepsilon > 0$ there exist a Borel set $A_\varepsilon \subset \Omega$ with $|A_\varepsilon| < \varepsilon$ and a subsequence (u_{h_k}) such that the sequence $\big(u_{h_k}\chi_{\Omega\setminus A_\varepsilon}\big)$ is equiintegrable.*

Let $f_h : B_R \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a sequence of Carathéodory functions such that for some $p > 1$

$$0 \leq f_h(x,u,z) \leq L(1+|z|^p) \qquad \forall (x,u,z) \in B_R \times \mathbb{R} \times \mathbb{R}^N \ . \tag{5}$$

If $v \in SBV_{\mathrm{loc}}(\Omega)$, $c > 0$ we set

$$F_h(v,c,B_\varrho) = \int_{B_\varrho} f_h(x,v,\nabla v)\,dx + c\mathcal{H}^{N-1}(S_v \cap B_\varrho)$$

and we write $\mathrm{Dev}_h(v,c,B_\varrho)$ for the deviation from minimality.

The next theorem (see also [16]) is essentially a sort of Γ-convergence result. Roughly speaking, it states that a sequence (u_h) of "almost minimizers" of the functionals F_h converges to a local minimizer of a functional of the type considered in (4) provided that the functions f_h converge strongly enough to f and the measures of the discontinuity sets S_{u_h} are infinitesimal.

Theorem 2.6. *Let (f_h) be a sequence of Carathéodory functions satisfying (5), such that $z \to f_h(x,u,z)$ is convex for all $(x,u) \in B_R \times \mathbb{R}$ and for any $h \in \mathbb{N}$. Let $(u_h) \subset SBV(B_R)$, m_h medians of u_h in B_R, $(c_h) \subset (0,\infty)$. Assume that*

(a) $$\sup_{h\in\mathbb{N}} \int_{B_R} |\nabla u_h|^p\,dx < \infty \ , \quad \lim_{h\to\infty} \mathcal{H}^{N-1}(S_{u_h}) = 0 \ ,$$

(b) $$\sup_{h\in\mathbb{N}} F_h(u_h,c_h,B_R) < \infty \ ,$$

(c) $$\lim_{h\to\infty} \mathrm{Dev}_h(u_h,c_h,B_R) = 0 \ ,$$

(d) $$\lim_{h\to\infty} u_h(x) = u(x) \in W^{1,p}(B_R) \quad \mathcal{L}^N \text{ -a.e. in } B_R \ , \qquad m_h \to m \in \mathbb{R} \ .$$

Then, if f_h converges to f uniformly on compact subsets of $B_R \times \mathbb{R} \times \mathbb{R}^N$, the function u is a local minimizer of the functional $v \to \int_{B_R} f(x, v, \nabla v)dx$ in $W^{1,p}(B_R)$ and

$$\lim_{h\to\infty} F_h(u_h, c_h, B_\varrho) = \int_{B_\varrho} f(x, v, \nabla v)\, dx \qquad \forall \varrho \in (0, R) \ .$$

Proof. Replacing $f_h(x, u, z)$ with $f_h(x, u + m_h, z)$, u_h with $u_h - m_h$ and u with $u - m$, since $f_h(x, u + m_h, z)$ converges uniformly on compact sets to $f(x, u + m, z)$, we may always assume that $m_h = m = 0$ for all h.

Let us first notice that from Proposition 2.1 and assumptions (d), (a) it easily follows that $\overline{u}_h$ converges to u in $L^p(B_R)$ and that $|\{u_h \neq \overline{u}_h\}| \to 0$. Since for any $h \in \mathbb{N}$ and for $\mathcal{L}^1$-a.e. $\varrho \in (0, R)$ we have that $\mathcal{H}^{N-1}(S_{\overline{u}_h} \cap \partial B_\varrho) = 0$, from Lemma 2.4 we deduce that

$$F_h(u_h, c_h, B_\varrho) \leq F_h(\overline{u}_h, c_h, B_\varrho) + c_h \mathcal{H}^{N-1}(\{\tilde{u}_h \neq \tilde{\overline{u}}_h\} \cap \partial B_\varrho) + \mathrm{Dev}_h(u_h, c_h, B_R)\,,$$

$$\begin{aligned}\mathrm{Dev}_h(\overline{u}_h, c_h, B_\varrho) \leq{}& F_h(\overline{u}_h, c_h, B_\varrho) - F_h(u_h, c_h, B_\varrho) \\ &+ c_h \mathcal{H}^{N-1}(\{\tilde{u}_h \neq \tilde{\overline{u}}_h\} \cap \partial B_\varrho) + \mathrm{Dev}_h(u_h, c_h, B_R)\end{aligned}$$

for $\mathcal{L}^1$-a.e. $\varrho \in (0, R)$. Moreover from (5) we have also that

$$F_h(\overline{u}_h, c_h, B_\varrho) \leq F_h(u_h, c_h, B_\varrho) + L|\{u_h \neq \overline{u}_h\} \cap B_\varrho| \qquad \forall \varrho \in (0, R) \ . \tag{6}$$

From the coarea formula and using (2) we get

$$\begin{aligned} a_h := c_h \int_0^R \mathcal{H}^{N-1}\left(\{\tilde{u}_h \neq \tilde{\overline{u}}_h\} \cap \partial B_\varrho\right)\, d\varrho &= c_h |\{u_h \neq \overline{u}_h\}| \\ &\leq c_h \gamma(N) \left[\mathcal{H}^{N-1}(S_{u_h})\right]^{N/(N-1)} \leq c_h^{-1/(N-1)} \gamma(N) M^{N/(N-1)} \ , \end{aligned}$$

where M is the supremum in (b).

We claim that $\lim_h a_h = 0$. This is clear from the last inequality above if $\lim_h c_h = \infty$. On the other hand if there exists a subsequence (c_{h_k}) such that $\lim_k c_{h_k} = c < \infty$ then from the inequalities above we still have $\lim_k a_{h_k} = 0$. Hence the claim follows.

Therefore there exists a subsequence $(\overline{u}_{h_k})$ such that

$$\lim_{k\to\infty} c_{h_k} \mathcal{H}^{N-1}\left(\{\tilde{u}_{h_k} \neq \tilde{\overline{u}}_{h_k}\} \cap \partial B_\varrho\right) = 0 \qquad \text{for } \mathcal{L}^1 \text{ -a.e. } \varrho \in (0, R) \ ;$$

moreover, using (a) and (b), we may extract a further (not relabelled) subsequence such that the measures

$$\mu_k = |\nabla \overline{u}_{h_k}|^p \mathcal{L}^N + c_{h_k} \mathcal{H}^{N-1} \llcorner S_{\overline{u}_{h_k}} \to \mu \qquad \text{weakly}^* \text{ in } B_R \ .$$

Let us now set for all $\varrho < R$

$$\alpha(\varrho) = \limsup_{k\to\infty} F_{h_k}(u_{h_k}, c_{h_k}, B_\varrho) \ .$$

From (6) and the two inequalities preceding (6) we may conclude that

$$\alpha(\varrho) = \limsup_{k\to\infty} F_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_\varrho) \qquad \text{for } \mathcal{L}^1 \text{-a.e. } \varrho \in (0,R) \tag{7}$$

and that for all $\varrho \in (0,R)$

$$\lim_{k\to\infty} \mathrm{Dev}_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_\varrho) = 0 \ .$$

We claim that for a.e. ϱ

$$\int_{B_\varrho} f(x,u,\nabla u)\,dx \le \liminf_{k\to\infty} F_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_\varrho) \ . \tag{8}$$

Let us assume with no loss of generality that $\lim_k F_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_\varrho)$ exists. Given $\varepsilon > 0$, by Lemma 2.5 there exist a Borel subset A_ε of B_R and a (not relabelled) subsequence such that the sequence $(|\nabla u_h|^p \chi_{B_R\setminus A_\varepsilon})$ is equiintegrable. For any $r \in \mathbb{N}$ we have

$$F_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_\varrho) \ge \int_{(B_\varrho\setminus A_\varepsilon)\cap M_{k,r}} f_{h_k}(x, \overline{u}_{h_k}, \nabla\overline{u}_{h_k})\,dx \ ,$$

where $M_{k,r} = \{x \in B_R : \ |\overline{u}_{h_k}(x)| + |\nabla\overline{u}_{h_k}(x)| \le r\}$; hence by the convergence of f_h to f we deduce that

$$\begin{aligned}\lim_{k\to\infty} F_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_\varrho) &\ge \liminf_{k\to\infty} \int_{(B_\varrho\setminus A_\varepsilon)\cap M_{k,r}} f(x, \overline{u}_{h_k}, \nabla\overline{u}_{h_k})\,dx \\ &\ge \liminf_{k\to\infty}\int_{B_\varrho\setminus A_\varepsilon} f(x, \overline{u}_{h_k}, \nabla\overline{u}_{h_k})\,dx - \limsup_{k\to\infty}\int_{(B_\varrho\setminus A_\varepsilon)\setminus M_{k,r}} L(1+|\nabla\overline{u}_{h_k}|^p)\,dx \ .\end{aligned}$$

Using Theorem 2.2 and letting $r \to \infty$, from the equiintegrability of the functions $|\overline{u}_{h_k}(x)|^p + |\nabla\overline{u}_{h_k}(x)|^p$ in $B_\varrho \setminus A_\varepsilon$ we have

$$\lim_{k\to\infty} F_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_\varrho) \ge \int_{B_\varrho\setminus A_\varepsilon} f(x,u,\nabla u)\,dx \ ,$$

from which (8) follows letting ε go to 0.

Let $v \in W^{1,p}(B_R)$ be a function such that $\{v \ne u\} \subset\subset B_R$ and let $\varrho < \varrho' < R$ so that the inequality in (7) holds for ϱ, $\mu(\partial B_\varrho) = \mu(\partial B_{\varrho'}) = 0$ and $\{v \ne u\} \subset\subset B_\varrho$. From the inequality

$$F_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_{\varrho'}) \le F_{h_k}(\eta v + (1-\eta)\overline{u}_{h_k}, c_{h_k}, B_{\varrho'}) + \mathrm{Dev}_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_{\varrho'}) \ ,$$

where $\eta \in C^1_0(B_{\varrho'})$, $0 \le \eta \le 1$, $\eta = 1$ on B_ϱ and $|\nabla\eta| \le 2/(\varrho' - \varrho)$, and using the controls from above on f_h, we get

$$\begin{aligned}F_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_\varrho) \le F_{h_k}(v, c_{h_k}, B_\varrho) + c\int_{B_{\varrho'}\setminus B_\varrho}\left[1+|\nabla\overline{u}_{h_k}|^p + |\nabla v|^p + \frac{|\overline{u}_{h_k} - v|^p}{(\varrho'-\varrho)^p}\right]dx \\ + c_{h_k}\mathcal{H}^{N-1}(S_{\overline{u}_{h_k}} \cap B_{\varrho'} \setminus B_\varrho) + \mathrm{Dev}_{h_k}(\overline{u}_{h_k}, c_{h_k}, B_{\varrho'})\end{aligned}$$

for a suitable constant $c \geq 1$ depending only on L and p. Letting k go to ∞ and recalling that $v = u$ outside B_ϱ, we obtain easily

$$\alpha(\varrho) \leq \int_{B_\varrho} f(x, v, \nabla v)\, dx + c \int_{B_{\varrho'} \setminus B_\varrho} (1 + |\nabla v|^p)\, dx + \gamma \mu(B_{\varrho'} \setminus B_\varrho) \ .$$

Therefore, letting $\varrho' \downarrow \varrho$ we get that for $\mathcal{L}^1$-a.e. ϱ and any $v \in W^{1,p}(B_R)$ such that $\{v \neq u\} \subset\subset B_\varrho$

$$\limsup_{k\to\infty} F_{h_k}(u_{h_k}, c_{h_k}, B_\varrho) \leq \int_{B_\varrho} f(x, v, \nabla v) \ , dx \ .$$

Since this inequality holds in particular for $v = u$, from (8) it follows that for $\mathcal{L}^1$-a.e. ϱ there exists

$$\lim_{k\to\infty} F_{h_k}(u_{h_k}, c_{h_k}, B_\varrho) = \int_{B_\varrho} f(x, u, \nabla u)\, dx$$

and that u has the claimed minimizing property. This concludes the proof. □

3. Existence of minimizers

In this section we prove that SBV minimizers of the anisotropic functional

$$\mathcal{F}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx + \alpha \int_\Omega |u - g|^q\, dx + \beta \mathcal{H}^{N-1}(S_u \cap \Omega)$$

satisfy the regularity property (1) (see Theorem 3.5). As we said in the introduction, from this result we then get the existence of a minimizing pair (K, u), where $K \subset \mathbb{R}^N$ is a closed set and $u \in W^{1,p}(\Omega \setminus K)$, for the functional

$$\mathcal{G}(K, u) = \int_{\Omega\setminus K} f(x, u(x), \nabla u(x))dx + \alpha \int_{\Omega\setminus K} |u - g|^q dx + \beta \mathcal{H}^{N-1}(K \cap \Omega)\,, \qquad (9)$$

where $\alpha, \beta > 0$, $q \geq 1$, $g \in L^\infty(\Omega)$.

We assume that the function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ can be represented as $f = (\mu^2 + |z|^2)^{p/2} + h(x, u, z) + \phi(x, u, z)$. We assume that h is a continuous function, convex in z and such that for all (x, u, z)

$$0 \leq h(x, u, z) \leq L(\mu^2 + |z|^2)^{p/2} \qquad (\mathrm{H}_1)$$

with $L \geq 0$, $0 \leq \mu \leq 1$, $p > 1$, and that for all $(x, u, z), (y, v, z)$

$$|h(x, u, z) - h(y, v, z)| \leq \sigma(|x - y| + |u - v|)(\mu^2 + |z|^2)^{p/2} \qquad (\mathrm{H}_2)$$

where $\sigma : [0, \infty) \to [0, \infty)$ is a continuous, bounded and increasing function such that $\sigma(0) = 0$.

Moreover ϕ is a Carathéodory function, convex in z, such that for all (x, u, z)

$$|\phi(x, u, z)| \leq L(1 + |z|^r) \qquad \text{for some } r < p\ . \qquad (\mathrm{H}_3)$$

The proof of the existence result follows the lines of the one originally given in [16] for the case $f = |z|^2$ and relies upon the decay estimate (10) below. Notice that when $c = 0$ and u is a $W^{1,p}$ local minimizer of an integral of the type $\int_\Omega f(\nabla v)dx$, (10) implies that u is locally Lipschitz continuous. Indeed this property is stated in the following result which has been proved in [20].

Theorem 3.1. *If $h : \mathbb{R}^N \to [0, \infty)$ is a convex function satisfying (H_1) and $f = (\mu^2 + |z|^2)^{p/2} + h(z)$, then any local minimizer $u \in W^{1,p}(\Omega)$ of the functional*

$$v \mapsto \int_\Omega f(\nabla v)\, dx$$

is locally Lipschitz continuous in Ω and

$$\sup_{x \in B_{R/2}(x_0)} |\nabla u(x)|^p \leq C_0 \fint_{B_R(x_0)} (\mu + |\nabla u(x)|^p)\, dx$$

for all balls $B_R(x_0) \subset \Omega$, with C_0 depending only on N, p, L.

It is interesting to note that a function f such that $0 \leq f(z) \leq C(\mu^2 + |z|^2)^{p/2}$ can be split as $(\mu^2 + |z|^2)^{p/2}$ plus a convex function h satisfying (H_1) if and only if (see [21]) for all $z \in \mathbb{R}^N$, $\varphi \in C^1_0(Q)$

$$\int_Q f(z + \nabla\varphi(y))\, dy \geq \int_Q \left[f(z) + \nu(\mu^2 + |z|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}} |\nabla\varphi|^2 \right] dy \ ,$$

where Q is the unit cube and $\nu > 0$ is a suitable constant.

Lemma 3.2. *Let $f : \mathbb{R}^{nN} \to [0, \infty)$ be a quasiconvex function such that for some $p \geq 1$*

$$0 \leq f(\xi) \leq L(1 + |\xi|^p) \qquad \forall \xi \in \mathbb{R}^{nN}$$

and let $(t_h) \subset (0, \infty)$ be a sequence such that $\lim_h t_h = \infty$. Then, there exists a subsequence (t_{h_k}) such that

$$\frac{f(t_{h_k}\xi)}{t^p_{h_k}}$$

converge to a quasiconvex function f_∞ uniformly on compact sets of $\mathbb{R}^{nN}$.

Proof. Setting $f_h(\xi) = f(t_h\xi)/t^p_h$ the result immediately follows noticing that in any ball B_R the functions f_h are bounded and equicontinuous. □

The next result provides a crucial estimate of the decay of F in small balls by a blow-up argument.

Lemma 3.3 (Decay estimate). *Let f be a function verifying (H_1), (H_2) and (H_3). For any $M, c > 0$, $\Omega' \subset\subset \Omega$ and $\tau \in (0, 1)$ there exist $\varepsilon(M, c, \tau, \Omega')$, $\vartheta(M, c, \tau, \Omega')$ such that if $u \in SBV(\Omega)$, $\nabla u \in [L^p(\Omega)]^N$, $\|u\|_{L^\infty(\Omega)} \leq M$ and $B_\varrho(x) \subset\subset \Omega$, with $x \in \Omega'$, $\varrho < \varepsilon^2$, and if*

$$F(u, c, B_\varrho(x)) \leq \varepsilon\varrho^{N-1} \ , \qquad \mathrm{Dev}(u, c, B_\varrho(x)) \leq \vartheta F(u, c, B_\varrho(x)) \ ,$$

then

$$F(u, c, B_{\tau\varrho}(x)) \leq C_1 \tau^N F(u, c, B_\varrho(x)) \ , \tag{10}$$

where C_1 depends only on N, p, L.

Proof. To prove the assertion it is enough to assume $\tau \in (0,1/2)$ (otherwise just take $C_1 = 2^N$). Given such a τ we argue by contradiction assuming that there exist two sequences (ε_h), (ϑ_h) with $\lim_h \varepsilon_h = \lim_h \vartheta_h = 0$, SBV functions u_h, with $\|u_h\|_\infty \leq M$, and balls $B_{\varrho_h}(x_h)$, with $x_h \in \Omega'$, $\varrho_h \leq \varepsilon_h^2$, such that

$$F(u_h, c, B_{\varrho_h}(x_h)) \leq \varepsilon_h \varrho_h^{N-1} \ , \quad \mathrm{Dev}(u_h, c, B_{\varrho_h}(x_h)) \leq \vartheta_h F(u_h, c, B_{\varrho_h}(x_h)) \ ,$$

$$F(u_h, c, B_{\tau\varrho_h}(x_h)) > C_1 \tau^N F(u_h, c, B_{\varrho_h}(x_h)) \ ,$$

where $C_1 = (L+1)C_0 + 1$ and C_0 is the constant given in Theorem 3.1. Setting for all $h \in \mathbb{N}$, $y \in B_1$

$$v_h(y) = \varrho_h^{(1-p)/p} \gamma_h^{1/p} u_h(x_h + \varrho_h y) \ ,$$

with $\gamma_h = \varepsilon_h^{-1}$, we denote by m_h any median of v_h in B_1 and define $w_h = v_h - m_h$,

$$f_h(y,u,z) = \gamma_h \varrho_h f(x_h + \varrho_h y, \varrho_h (\gamma_h \varrho_h)^{-1/p}(m_h + u), (\gamma_h \varrho_h)^{-1/p} z) \ .$$

With these notations we have

$$F_h(w_h, c\gamma_h, B_1) = \rho_h^{1-N} \varepsilon_h^{-1} F(u_h, c, B_{\varrho_h}(x_h)) \leq 1 \ , \quad \mathrm{Dev}_h(w_h, c\gamma_h, B_1) \leq \vartheta_h \ ,$$

$$F_h(w_h, c\gamma_h, B_\tau) > C_1 \tau^N F_h(w_h, c\gamma_h, B_1) \ . \tag{11}$$

Since the sequence $(|\nabla w_h|)$ is bounded in $L^p(B_1)$ and 0 is a median for all w_h, Proposition 2.1 implies that, passing if necessary to a subsequence, we may assume that $w_h(x) \to w(x)$ $\mathcal{L}^N$-a.e. in B_1, with $w \in W^{1,p}(B_1)$. Extracting eventually a further subsequence, since the sequence (u_h) is bounded in $L^\infty(\Omega)$, we may also assume that $\varrho_h(\gamma_h\varrho_h)^{-1/p} m_h \to m \in \mathbb{R}$ and that $x_h \to x_0$ in Ω. Finally since $\varrho_h \leq \varepsilon_h^2$ we have also that $\gamma_h \varrho_h \to 0$, and that $\varrho_h(\gamma_h\varrho_h)^{-1/p} \to 0$. Notice that given $R > 0$ if $y \in B_1$, $|u| + |z| < R$, by (H_2), (H_3) we have

$$|f_h(y,u,z) - \gamma_h \varrho_h \tilde{f}(x_0, m, (\gamma_h\varrho_h)^{-1/p} z)| \leq \omega_{h,R} \ ,$$

where $\tilde{f}(x_0,m,z) = |z|^p + h(x_0,m,z)$ and $\omega_{h,R}$ is infinitesimal as $h \to \infty$. Using Lemma 3.2 we may therefore conclude that up to another subsequence

$$f_h(y,u,z) \to \tilde{f}_\infty(z)$$

uniformly on compact subsets of $B_1 \times \mathbb{R} \times \mathbb{R}^N$, where $\tilde{f}_\infty(z) = |z|^p + h_\infty(x_0,m,z)$. Moreover it can be easily checked that $h_\infty(x_0,m,z)$ verifies the assumption (H_1) (with $\mu = 0$). Thus, Theorem 2.6 allows us to conclude that w is a local minimizer of the functional $v \mapsto \int_{B_1} \tilde{f}_\infty(\nabla v)\, dy$ and that

$$\lim_{h\to\infty} F_h(w_h, c\gamma_h, B_\varrho) = \int_{B_\varrho} \tilde{f}_\infty(\nabla w)\, dy \qquad \forall \varrho \in (0,1) \ .$$

Hence by Theorem 3.1 w is locally Lipschitz continuous in B_1 and

$$\sup_{y\in B_{1/2}} |\nabla w(y)|^p \leq C_0 \fint_{B_1} |\nabla w|^p\, dy \leq C_0 \fint_{B_1} \tilde{f}_\infty(\nabla w)\, dy.$$

Therefore

$$\lim_{h\to\infty} F_h(w_h, c\gamma_h, B_\tau) \le (L+1)C_0\tau^N \int_{B_1} \tilde{f}_\infty(\nabla w)\,dy \le$$
$$\le (L+1)C_0\tau^N \limsup_{h\to\infty} F_h(w_h, c\gamma_h, B_1)$$

and the contradiction with (11) proves the assertion. □

Let us now recall the definition of quasi minimizer (see [9] or [10, Def. 7.17]).

Definition 3.4. A function $u \in SBV_{\rm loc}(\Omega)$ is a quasi minimizer of the functional $F(v, c, \Omega)$ if there exists $\omega \ge 0$ such that for all $B_\varrho(x) \subset\subset \Omega$

$$\mathrm{Dev}(u, c, B_\varrho(x)) \le \omega\varrho^N \quad . \tag{12}$$

We denote by $\mathcal{M}_\omega(\Omega)$ the class of quasi minimizers satisfying (12).

Notice that in the vectorial case $n > 1$ Definition 2.3 of deviation from minimality and the definition above still make sense. Henceforth we shall denote by $[\mathcal{M}_\omega(\Omega)]^n$ the class of the quasi minimizers $u \in [SBV_{\rm loc}(\Omega)]^n$ satisfying (12).

The next result, based on the decay lemma, can now be proved exactly as Theorem 7.21 in [10].

Theorem 3.5 (Density lower bound). *Let f be a convex function satisfying* (H$_1$), (H$_2$), (H$_3$). *Let $M > 0$ and $\Omega' \subset\subset \Omega$. There exist positive ϑ_0 and ϱ_0 such that if $u \in \mathcal{M}_\omega(\Omega)$, $\|u\|_{L^\infty(\Omega)} \le M$, then*

$$\mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) > \vartheta_0\varrho^{N-1}$$

for all $B_\varrho(x) \subset\subset \Omega$ with $x \in S_u \cap \Omega'$ and $\varrho < \varrho_0$. Moreover

$$\mathcal{H}^{N-1}((\overline{S}_u \setminus S_u) \cap \Omega) = 0 \quad .$$

Let us now assume that the function f satisfies also the the following conditions

$$f(x, u, 0) = \min_{z\in\mathbb{R}^N} f(x, u, z)\,, \qquad f(x, u, 0) \le f(x, v, 0) \ \text{ if } |u| \le |v|\,, \tag{13}$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$ and any $u, v \in \mathbb{R}$. Notice that then $\mathcal{F}(u_M) \le \mathcal{F}(u)$, where $u_M = (u \vee -M) \wedge M$ and $M = \|g\|_\infty$. Therefore in order to minimize $\mathcal{F}$ we may restrict to those $SBV(\Omega)$ functions u such that $\|u\|_\infty \le M$. From this observation we may thus easily get the following result.

Theorem 3.6. *Let f satisfy the assumptions of Theorem* 3.5 *and* (13). *Then, there exists a minimizer $u \in SBV(\Omega) \cap L^\infty(\Omega)$ of $\mathcal{F}$. Moreover the pair $(\overline{S}_u, u)$ is a minimizer of the functional $\mathcal{G}$ defined in* (9).

Proof. The existence of a minimizer u of $\mathcal{F}$ follows immediately from the comments made above, using the closure and compactness results due to Ambrosio (see [5], [7] and also [10, Theorems 4.7 and 4.8]) together with Theorem 2.2.

If u is such a minimizer, since $\mathcal{F}(u) \leq \mathcal{F}(0)$, we have that $|\nabla u| \in L^p(\Omega)$, hence $u \in W^{1,p}(\Omega \setminus \overline{S}_u)$. Also, a simple comparison argument shows that $u \in \mathcal{M}_\omega(\Omega)$ for $c = \beta$ and for some ω depending on L, p, N, $\|g\|_\infty$. Therefore Theorem 3.5 applies, and $\mathcal{H}^{N-1}((\overline{S}_u \setminus S_u) \cap \Omega) = 0$. Let (K, v) be any competing pair such that $\mathcal{G}(K, v) < \infty$. Arguing as before we may assume without loss of generality that v is bounded in Ω, hence (see [16] or [10, Prop. 4.4]) $v \in SBV(\Omega)$ and $\mathcal{H}^{N-1}(S_v \setminus K) = 0$, thus by the minimality of u we get

$$\mathcal{G}(\overline{S}_u, u) = \mathcal{F}(u) \leq \mathcal{F}(v) \leq \mathcal{G}(K, v) \ ,$$

which proves the assertion. □

4. The Vectorial Case

In this section we prove the existence of minimizers for the functional $\mathcal{G}$ in the case $n > 1$. As the reader can easily check, the arguments used in the proof of the existence result in the scalar case can be carried over with no substantial change also in the vectorial case. The only difficulty now is that we must have the right counterpart of the local Lipschitz estimate provided by Theorem 3.1. On the other hand, it is well known that in general we cannot expect global regularity of solutions to non linear elliptic systems or of minimizers of vectorial functionals unless we have some special structure assumption. A significant case in which one can get global regularity is when the coefficients of the system or the functional depend only on the modulus of the gradient (see [27], [23], [4]). The following theorem extends the regularity results obtained in those papers.

Theorem 4.1. *Let $h : [0, \infty) \to [0, \infty)$ be an increasing convex function such that for all $t \in \mathbb{R}$*

$$0 \leq h(t) \leq L(\mu^2 + t^2)^{p/2} \ ,$$

where $p > 1$, $L \geq 0$, $0 \leq \mu \leq 1$. If $u \in [W^{1,p}(\Omega)]^n$ is a local minimizer of the functional

$$v \mapsto \int_\Omega f(|\nabla v|)\, dx \ ,$$

where $f(t) = (\mu^2 + t^2)^{p/2} + h(t)$, then u is locally Lipschitz continuous in Ω and

$$\sup_{x \in B_{R/2}(x_0)} |\nabla u(x)|^p \leq C_0 \fint_{B_R(x_0)} (1 + |\nabla u(x)|^p)\, dx$$

for all balls $B_R(x_0) \subset \Omega$, with C_0 depending only on N, m, p, L.

Following [20] the proof of this result is obtained in two steps. We first give a precise sup estimate for the gradient of a $W^{1,p}$ minimizer in the case when f is smooth and satisfies the usual ellipticity assumptions. Then this estimate is carried out to the general case by means of an approximation argument.

Throughout this section we use Einstein's convention for repeated indices.

Lemma 4.2. *Let $G : \mathbb{R}^{nN} \to [0, +\infty[$ be a C^2 function such that*

$$0 \leq G(\xi) \leq L(\mu^2 + |\xi|^2)^{p/2} \ ,$$

$$|\nabla^2 G(\xi)| \leq \Lambda(\mu^2 + |\xi|^2)^{(p-2)/2} \ , \qquad G_{\xi^i_\alpha \xi^j_\beta}(\xi)\eta^i_\alpha \eta^j_\beta \geq \nu(\mu^2 + |\xi|^2)^{(p-2)/2}|\eta|^2 \ ,$$

for every $\xi, \eta \in \mathbb{R}^{nN}$, where $L, \Lambda, \nu > 0$, $p > 1$, $0 \leq \mu \leq 1$. Let us assume that $G(\xi) = g(|\xi|^2)$ with $g \in C^2(\mathbb{R})$. Then there exists a constant $C_0 = C_0(N, n, p, L, \nu)$, independent of Λ and μ, such that if $u \in [W^{1,p}(\Omega)]^n$ is a local minimizer of the functional

$$v \mapsto \int_\Omega G(\nabla v)\,dx \ ,$$

then for every $B_R(x_0) \subset \Omega$

$$\sup_{B_{R/2}(x_0)} (\mu^2 + |\nabla u|^2)^{p/2} \leq C_0 \fint_{B_R(x_0)} (\mu^2 + |\nabla u|^2)^{p/2} \ . \tag{14}$$

Proof. From the regularity results proved in the papers quoted above we already know that $u \in [W^{2,2}_{\text{loc}}(\Omega)]^n \cap [W^{1,\infty}_{\text{loc}}(\Omega)]^n$, thus the proof reduces to show that the estimate (14) holds with a constant independent on Λ and μ. Let us consider the Euler–Lagrange system satisfied by u

$$\int_\Omega G_{\xi^i_\alpha}(\nabla u)\nabla_\alpha \varphi^i\,dx = 0 \qquad \forall \varphi \in [C^1_0(\Omega)]^n \ .$$

Let us fix $s \in \{1, \dots, N\}$, $\eta \in C^1_0(\Omega)$, $\psi \in [C^2(\Omega)]^n$; choosing $\varphi = \eta^2 \nabla_s \psi$ we get

$$\int_\Omega G_{\xi^i_\alpha}(\nabla u)\nabla_\alpha(\nabla_s \psi^i)\eta^2\,dx = -2\int_\Omega G_{\xi^i_\alpha}(\nabla u)(\nabla_s\psi^i)\eta\nabla_\alpha\eta\,dx \ .$$

We integrate by parts the integral on the left hand side, thus getting

$$\begin{aligned}&\int_\Omega G_{\xi^i_\alpha\xi^j_\beta}(\nabla u)\nabla_s(\nabla_\beta u^j)(\nabla_\alpha\psi^i)\eta^2\,dx \\ &\qquad = 2\int_\Omega G_{\xi^i_\alpha}(\nabla u)(\nabla_s\psi^i)\eta\nabla_\alpha\eta\,dx - 2\int_\Omega G_{\xi^i_\alpha}(\nabla u)(\nabla_\alpha\psi^i)\eta\nabla_s\eta\,dx\end{aligned}$$

Setting $\psi = V^\gamma \nabla_s u$, where $\gamma > 0$ and $V = \mu^2 + |\nabla u|^2$, inserting ψ (which is an admissible test function) into the above equation and summing up, we obtain

$$\begin{aligned}&\int_\Omega G_{\xi^i_\alpha\xi^j_\beta}(\nabla u)\nabla_s(\nabla_\beta u^j)\left[\nabla_\alpha(\nabla_s u^i)V^\gamma + \gamma(\nabla_s u^i)V^{\gamma-1}\nabla_\alpha(|\nabla u|^2)\right]\eta^2\,dx \\ &\quad = 2\int_\Omega G_{\xi^i_\alpha}(\nabla u)\eta\Big\{\nabla_\alpha\eta\left[V^\gamma\nabla_{ss}u^i + \gamma V^{\gamma-1}(\nabla_s u^i)\nabla_s(|\nabla u|^2)\right] \\ &\qquad\qquad -\nabla_s\eta\left[V^\gamma\nabla_\alpha(\nabla_s u^i) + \gamma V^{\gamma-1}(\nabla_s u^i)\nabla_\alpha(|\nabla u|^2)\right]\Big\}\,dx \ .\end{aligned} \tag{15}$$

Since $G_{\xi^i_\alpha\xi^j_\beta}(\xi) = 4g''(|\xi|^2)\xi^i_\alpha\xi^j_\beta + 2g'(|\xi|^2)\delta^{ij}\delta_{\alpha\beta}$, we easily get from Lemma 4.3 below that

$$\begin{aligned}&G_{\xi^i_\alpha\xi^j_\beta}(\nabla u)\nabla_s(\nabla_\beta u^j)\nabla_s u^i\nabla_\alpha(|\nabla u|^2) \\ &\quad = 2g''(|\nabla u|^2)\nabla_\alpha u^i\nabla_s u^i\nabla_s(|\nabla u|^2)\nabla_\alpha(|\nabla u|^2) + g'(|\nabla u|^2)|\nabla(|\nabla u|^2)|^2 \\ &\quad \geq \frac{\nu}{2}(\mu^2 + |\nabla u|^2)^{(p-2)/2}|\nabla(|\nabla u|^2)|^2 \ ,\end{aligned}$$

whereas the ellipticity assumption on G implies that

$$G_{\xi^i_\alpha \xi^j_\beta}(\nabla u)\nabla_s(\nabla_\beta u^j)\nabla_\alpha(\nabla_s u^i) \geq \nu(\mu^2 + |\nabla u|^2)^{(p-2)/2}|\nabla^2 u|^2 \ .$$

Using these two estimates to control the left hand side of (15) we then obtain

$$\nu \int_\Omega V^{\gamma+(p-2)/2}\eta^2|\nabla^2 u|^2\,dx + \frac{\gamma\nu}{2}\int_\Omega V^{\gamma-1+(p-2)/2}\eta^2|\nabla(|\nabla u|^2)|^2\,dx$$
$$\leq C\int_\Omega V^{\gamma+(p-1)/2}\eta|\nabla\eta||\nabla^2 u|dx + C\gamma\int_\Omega V^{\gamma-1+p/2}\eta|\nabla\eta||\nabla(|\nabla u|^2)|dx$$

for some C depending only on N, n, p, L. Therefore using Young's inequality we immediately get

$$\frac{\nu}{2}\int_\Omega V^{\gamma+(p-2)/2}\eta^2|\nabla^2 u|^2\,dx + \frac{\gamma\nu}{4}\int_\Omega V^{\gamma-1+(p-2)/2}\eta^2|\nabla(|\nabla u|^2)|^2\,dx$$
$$\leq \frac{C'(1+\gamma)}{\nu}\int_\Omega V^{\gamma+p/2}|\nabla\eta|^2\,dx$$

from which we may easily deduce that for all $\gamma \geq 0$

$$\int_\Omega V^{\gamma-1+(p-2)/2}\eta^2|\nabla(|\nabla u|^2)|^2\,dx \leq \frac{C''}{\nu^2}\int_\Omega V^{\gamma+p/2}|\nabla\eta|^2 \ ,$$

where C'' does not depend on γ. Setting $\beta = \frac{p}{4} + \frac{\gamma}{2}$ the inequality above becomes

$$\int_\Omega |\nabla(V^\beta\eta)|^2\,dx \leq \tilde{C}\beta^2\int_\Omega V^{2\beta}|\nabla\eta|^2\,dx \ ,$$

which, assuming with no loss of generality that $B_1 \subset \Omega$ and $\eta \in C^1_0(B_1)$ immediately implies by Sobolev–Poincaré inequality

$$\|V^\beta\eta\|_{L^{2\chi}(B_1)} \leq \sqrt{\tilde{C}}\beta\|V^\beta|\nabla\eta|\|_{L^2(B_1)} \ ,$$

with $\chi = \frac{N}{N-2}$ if $N \geq 3$ or any $\chi > 1$ if $N = 2$. Applying this inequality for any $i \in \mathbb{N}$ with $\beta_i = \frac{p}{4}\chi^{i-1}$ and choosing $\eta \in C^1_0(B_{r_i})$, where $r_i = \frac{1}{2} + \frac{1}{2^i}$, with $\eta = 1$ on $B_{r_{i+1}}$, $0 \leq \eta \leq 1$, $|\nabla\eta| \leq 2^{i+2}$, we get

$$\|V\|_{L^{2\beta_{i+1}}(B_{r_{i+1}})} \leq (4\sqrt{\tilde{C}}\beta_i 2^i)^{1/\beta_i}\|V\|_{L^{2\beta_i}(B_{r_i})} \qquad \forall i \in \mathbb{N} \ .$$

Iterating this estimate, and recalling that $B_{1/2} \subset B_{r_i}$ for every $i \in \mathbb{N}$ and that $2\beta_1 = p/2$ we easily get

$$\|V\|_{L^{2\beta_{i+1}}(B_{1/2})} \leq C_0\|V\|_{L^{p/2}(B_1)} \ ,$$

where $C_0 = \Pi_{i=1}^\infty(\tilde{C}\beta_i 2^i)^{1/\beta_i}$. Hence the result follows when $R = 1$. The general case $R > 0$ can be recovered by a scaling argument. □

In the sequel we shall use the following result whose simple proof is omitted.

Lemma 4.3. *If $G : \mathbb{R}^{nN} \to [0, \infty)$ satisfies the assumptions of Lemma 4.2, then for every $\xi \in \mathbb{R}^{nN}$, $\lambda \in \mathbb{R}^N$, we get*

$$4g''(|\xi|^2)\xi^i_\alpha \xi^i_\beta \lambda_\alpha \lambda_\beta + 2g'(|\xi|^2)|\lambda|^2 \geq \nu(\mu^2 + |\xi|^2)^{(p-2)/2}|\lambda|^2 \ . \tag{16}$$

We are now in position to prove the claimed regularity result.

Proof of Theorem 4.1. Extending h to $\mathbb{R}$ by even reflection we may assume that also f is a convex, even function defined on all $\mathbb{R}$. From [20, Lemma 2.4] we obtain that there exists a sequence (f_h) of functions of class $C^2(\mathbb{R})$ converging to f on compact sets of $\mathbb{R}$ and such that

$$C^{-1}\left(\mu^2 + \frac{1}{h^2} + t^2\right)^{p/2} \leq f_h(t) \leq C(L+1)\left(\mu^2 + \frac{1}{h^2} + t^2\right)^{p/2} \ ,$$

$$C^{-1}\nu\left(\mu^2 + \frac{1}{h^2} + t^2\right)^{(p-2)/2} \leq f''_h(t) \leq \Lambda_h\left(\mu^2 + \frac{1}{h^2} + t^2\right)^{(p-2)/2}$$

for some constant C not depending on μ, h, L and for $\Lambda_h > 0$. Since f is an even function it can be easily checked that the construction made in [20, Lemma 2.4] yields that also the approximating functions f_h are even, hence $f'_h(0) = 0$.

Setting for every $h \in \mathbb{N}$ and for every $t > -1/h^2$ $g_h(t) = f_h(\sqrt{t + 1/h^2})$, the functions g_h are of class $C^2([0, \infty))$. Moreover the functions $G_h(\xi) = g_h(|\xi|^2)$ converge uniformly to $f(|\xi|)$ on compact sets of $\mathbb{R}^{nN}$. We claim that the G_h verify the assumptions of Lemma 4.2 with μ^2 replaced by $\mu^2 + 1/h^2$ and ν replaced by $C'\nu$ for some constant C' independent on h. The verification of the first two assumptions is straightforward, hence we just limit ourselves to check that the ellipticity condition is satisfied. Infact if $t \geq 0$ for any h we have

$$f'_h(t) = \int_0^t f''_h(s)\,ds \geq C^{-1}\nu \int_0^t \left(\mu^2 + \frac{1}{h^2} + s^2\right)^{\frac{p-2}{2}} ds \geq C_1 \nu t \left(\mu^2 + \frac{1}{h^2} + t^2\right)^{\frac{p-2}{2}} .$$

Let us fix $\xi, \eta \in \mathbb{R}^{nN}$ and set $t_0 = \sqrt{|\xi|^2 + 1/h^2}$. From this inequality we get

$$\begin{aligned} G_{h,\xi^i_\alpha \xi^j_\beta}(\xi)\eta^i_\alpha \eta^j_\beta &= 4g''_h(|\xi|^2)|\langle \xi, \eta\rangle|^2 + 2g'_h(|\xi|^2)|\eta|^2 \\ &= \frac{f''_h(t_0)}{t_0^2}|\langle \xi, \eta\rangle|^2 + \frac{f'_h(t_0)}{t_0}\left[|\eta|^2 - \frac{|\langle \xi, \eta\rangle|^2}{t_0^2}\right] \geq C'\nu|\eta|^2\left(\mu^2 + \frac{1}{h^2} + t_0^2\right)^{\frac{p-2}{2}} , \end{aligned}$$

with $C' = \min\{C_1, C^{-1}\}$. Hence the claim is proved.

Let now u be a local minimizer of the functional $\int_\Omega f(|\nabla v|)dx$. Let us fix a ball $B_R(x_0) \subset\subset \Omega$, and denote by u_h the solution of the minimum problem

$$\min\left\{\int_{B_R(x_0)} G_h(\nabla v)\,dx : \ v \in u + [W^{1,p}_0(B_R(x_0))]^n\right\} \ .$$

Applying Lemma 4.2 to u_h it easily follows that up to a subsequence $u_h \to u_\infty$ locally weakly* in $[W^{1,\infty}(B_R(x_0))]^n$ and that

$$\int_{B_R(x_0)} f(|\nabla u_\infty|)\,dx \leq \int_{B_R(x_0)} f(|\nabla u|)\,dx \ .$$

Hence, by the strict convexity of the function $\xi \to f(|\xi|)$ one gets that $u_\infty = u$. the conclusion then follows applying again Lemma 4.2 to u_h and letting $h \to \infty$ (for the details see the proof of Theorem 2.2 in [20]). □

Let us now consider the functional

$$\mathcal{F}(u)=\int_\Omega [f(x,|u(x)|,|\nabla u(x)|)+\phi(x,u(x),\nabla u(x))]dx + \alpha\int_\Omega |u-g|^q dx + \beta\mathcal{H}^{N-1}(S_u \cap \Omega)$$

where $\alpha, \beta > 0$, $q \geq 1$, $g \in [L^\infty(\Omega)]^n$. Here $f(x,s,t) = (\mu^2+t^2)^{p/2} + h(x,s,t)$, where $h : \Omega \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function, satisfying the following assumptions:

h is convex with respect to t, non decreasing in s and t, and for all (x,s,t)

$$0 \leq h(x,s,t) \leq L(\mu^2+t^2)^{p/2} \tag{H$'_1$}$$

with $L \geq 0$, $0 \leq \mu \leq 1$, $p > 1$;for all $(x,s,t), (y,s',t)$

$$|h(x,s,t) - h(y,s',t)| \leq \sigma(|x-y| + |s-s'|)(\mu^2+t^2)^{p/2} \tag{H$'_2$}$$

where $\sigma : [0,\infty) \to [0,\infty)$ is a continuous, bounded and increasing function such that $\sigma(0) = 0$;

$\phi : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nN} \to [0,\infty)$ is a Carathéodory function, quasiconvex in ξ, such that for all (x,u,ξ)

$$0 \leq \phi(x,u,\xi) \leq L(1+|\xi|^r) \qquad \text{for some } r < p \ . \tag{H$'_3$}$$

We can now state the vectorial counterpart of Theorem 3.5.

Theorem 4.4. *Let f, ϕ be functions satisfying* (H$'_1$), (H$'_2$), (H$'_3$). *Let $M > 0$ and $\Omega' \subset\subset \Omega$. Given $\omega \geq 0$ there exist ϑ_0 and ϱ_0 such that if $u \in [\mathcal{M}_\omega(\Omega)]^n$, $\|u\|_{L^\infty(\Omega)} \leq M$, then*

$$\mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) > \vartheta_0 \varrho^{N-1}$$

for all $B_\varrho(x) \subset\subset \Omega$ with $x \in S_u \cap \Omega'$ and $\varrho < \varrho_0$. Moreover

$$\mathcal{H}^{N-1}((\overline{S}_u \setminus S_u) \cap \Omega) = 0 \ .$$

Proof. As in the scalar case the result is obtained once we have a decay estimate of the type (10), which can be proved with the argument used in the proof of Lemma 3.3. The reader may easily check that this argument immediately extends to our situation provided that we use Theorem 4.1 instead of Theorem 3.1 and that for all $\varrho < R$

$$\lim_{h\to\infty} F_h(w_h, c\gamma_h, B_\varrho) = \int_{B_\varrho} \tilde{f}_\infty(|\nabla w|)\, dy \ ,$$

where the functions w_h and the functionals F_h and are defined as in the proof of Lemma 3.3 and $\tilde{f}_\infty(t) = |t|^p + h_\infty(x_0,|m|,t)$.

To show that this equality holds let us follow the proof of Theorem 2.6. The main difference is that now we have to use Proposition 2.1 to each component w_h^i. Let us still denote by $\overline{w}_h$ the functions whose components are $(w_h^i \vee \alpha_h^i) \wedge \beta_h^i$. All the various steps in the proof remain unchanged exept (6), which is based on the fact that for a scalar function $\nabla\overline{u} = 0$ in the set where $\{\overline{u} \neq u\}$, a fact which is not true anymore in the vectorial case. Notice that

$$F_h(\overline{w}_h, c_h, B_\varrho) \le F_h(w_h, c_h, B_\varrho) + \int_{B_\varrho \cap \{\overline{w}_h \neq w_h\}} [f_h(x, \overline{w}_h, \nabla\overline{w}_h) - f_h(x, w_h, \nabla w_h)] dx,$$

where the functions f_h are defined as in the proof of Lemma 3.3. However, using the fact that f is increasing with respect to $|u|$ and $|\nabla u|$, it is easy to check that the last integral in the equality can be controlled from above by

$$\gamma_h \varrho_h L \int_{B_\varrho \cap \{\overline{w}_h \neq w_h\}} \left(1 + (\gamma_h \varrho_h)^{-r/p} |\nabla w_h|^r\right) \, dx$$

a quantity which is infinitesimal. Then, the rest of the proof goes exactly as in the scalar case. □

Let $\xi = (\xi^i_\alpha)$ be a matrix in $\mathbb{R}^{nN}$. We denote by ξ° any matrix in $\mathbb{R}^{nN}$ obtained from ξ by replacing one row with all zeros and leaving the other rows unchanged. We assume that for all $(x, u, \xi), (x, v, \xi)$

$$\phi(x, u, \xi^\circ) \le \phi(x, v, \xi) \qquad \text{if } |u| \le |v| \ . \tag{17}$$

Notice that this condition does not imply necessarily that ϕ depends only on $|u|$. For instance if $n = N$ the function $\phi = a(x, u)|\det(\xi)|$ satisfies (17) whenever $a(x, u) \ge 0$. Also, it is easy to check that, under this assumption on ϕ, to minimize $\mathcal{F}$, we may restrict as in the scalar case to functions u which are bounded. Hence the following existence result holds.

Theorem 4.5. *Let f, ϕ satisfy the assumptions of Theorem 4.4 and (17). Then, there exists a minimizer $u \in [SBV(\Omega)]^n \cap [L^\infty(\Omega)]^n$ of $\mathcal{F}$. Moreover the pair $(\overline{S}_u, u)$ is a minimizer of the functional*

$$\mathcal{G}(K, v) = \int_{\Omega \setminus K} [f(x, |v(x)|, |\nabla v(x)|) + \phi(x, v(x), \nabla v(x))] \, dx$$
$$+ \alpha \int_{\Omega \setminus K} |v(x) - g(x)|^q \, dx + \beta \mathcal{H}^{N-1}(K \cap \Omega) \ ,$$

where $K \subset \mathbb{R}^N$ is closed and $v \in [W^{1,p}(\Omega \setminus K)]^n$.

Remark 4.6. We remark that if ϕ is an increasing function of $|u|$ and of the modulus of all the minors of ξ, following the same argument used in the proof of Theorem 4.1 of [22], it is easy to prove that the minimizers of $\mathcal{F}$ are bounded and hence that Theorem 4.5 holds too.

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