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# Convex Functions of Legendre Type in General Banach Spaces

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Convex functions of Legendre type are constructed on arbitrary open convex sets in Banach spaces that satisfy appropriate rotundity and smoothness conditions. A simple direct proof of the existence of "universal" barriers on arbitrary open convex sets in $\mathbb{R}^n$ is given.

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## 1. Introduction

In [2, 3] study was made of convex functions of Legendre type. Following [2], we will say a proper convex lower semicontinuous function $f$ is *essentially smooth*, if $\partial f$ is locally bounded and is a singleton at each point of $\mathrm{dom}(\partial f)$. The proper convex lower semicontinuous function $f$ is *essentially strictly convex* if $(\partial f)^{-1}$ is locally bounded, and $f$ is strictly convex on every line segment that lies entirely in $\mathrm{dom}(\partial f)$. The function $f$ is *Legendre* if it is both essentially smooth and essentially strictly convex.

Let us remark that essential strict convexity is both a weakening and strengthening of classical strict convexity. Indeed, the strict convexity is of $f$ is only required on convex subsets of $\mathrm{dom}(\partial f)$, whereas the requirement that $(\partial f)^{-1}$ is locally bounded is a form of coercivity. In [2, Theorem 5.11], it is shown that on finite dimensional spaces a function is essentially strictly convex if and only if $f$ is strictly convex on every subset of $\mathrm{dom}(\partial f)$. Therefore, the terminology we have adopted from [2] is consistent with the classical terminology of Rockafellar in finite dimensional spaces [9]. However, even on $\ell_2$, a strictly convex function need not be essentially strictly convex [2, Example 5.14]. Nevertheless, [2] shows that this is the appropriate infinite dimensional analog of finite dimensional essential strict convexity to develop the appropriate duality properties with essential smoothness, and on which to build the theory of Legendre functions in reflexive and more general Banach spaces.

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As in [8], a function $f$ is said to be *$\beta$-differentiable* at $x \in X$ and $f'(x) \in X^*$ is called its $\beta$-derivative, if for each $S$ in the bornology $\beta$, the limit

$$\lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t} = \langle f'(x), y \rangle$$

exists uniformly for $y \in S$. A *bornology* $\beta$ on a Banach space $X$, is a family of bounded symmetric sets that is closed under scalar multiplication, and directed upwards. We use the notation $\tau_\beta$ to denote the topology on $X^*$ of uniform convergence on $\beta$-sets. The following standard bornologies are of particular interest to us: $\beta = G$ the *Gateaux* bornology of finite symmetric sets; $\beta = W$ the *weak Hadamard* bornology of weakly compact symmetric sets, and $\beta = F$ the *Fréchet* bornology of bounded symmetric sets. Clearly $G$ is the smallest bornology, and so Gateaux differentiability is the weakest form of $\beta$-differentiability, whereas Fréchet differentiability is the strongest form of $\beta$-differentiability.

We will further say that an essentially smooth convex function $f$ is *essentially $\beta$-smooth* if $f$ is $\beta$-differentiable at points where $\partial f$ is nonempty. Analogously, we say that $f$ is a *$\beta$-Legendre* function if it is essentially $\beta$-smooth and essentially strictly convex. A function $f$ is said to be a *barrier function* for a convex set $C$ if $f$ is defined on int$C$ and $f(x_n) \to \infty$ whenever $d(x_n, C^c) \to 0$. As in [2], we will say a function $f$ is *supercoercive* if $\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = \infty$.

In [2] several examples of Legendre functions on convex sets in finite dimensional, or more generally reflexive, Banach spaces are given, whereas [3] shows the utility of having such functions in certain projection algorithms. The monograph [7] explores, among other things, certain constructions of essentially smooth convex barrier functions and demonstrates through several applications that their existence on convex sets has extremely useful consequences in optimization theory.

In this note, we study necessary and/or sufficient conditions under which Legendre or Legendre barrier functions can be constructed on convex sets with nonempty interior in general Banach spaces. In particular, the second section presents some elementary facts from convex analysis which will be useful in the third section where convex functions of Legendre type are constructed on convex sets with nonempty interior in several broad classes of Banach spaces. The methods here rely on geometric properties of Banach spaces and the ability to approximate convex functions with appropriately smooth convex functions. The final section presents a simple direct proof of the (log-)convexity of universal barriers in finite dimensional spaces considered by Nesterov and Nemirovskii in [7], as well as a strengthening and refinement thereof. In contrast to the infinite dimensional case, the finite dimensional examples are explicit and rely on measure theoretic and analytic arguments.

## 2. Elementary Facts from Convex Analysis

The first result lists a few useful properties of supercoercive functions, further information can be found in [2, §3].

**Fact 2.1.** *Suppose $f$ is a supercoercive convex function. Then:*

*(a) $\phi_n \in \partial f(x_n)$ and $\|x_n\| \to \infty$, imply $\|\phi_n\| \to \infty$;*

*(b) $(\partial f)^{-1}$ is bounded on bounded sets;*

*(c) if, additionally, $f$ is strictly convex, then $f$ is essentially strictly convex.*

**Proof.** Fix $x_0 \in \mathrm{dom} f$. Then the subgradient inequality implies that $\|\phi_n\| \geq |f(x_n) - f(x_0)|/\|x_n\|$; thus $\|\phi_n\| \to \infty$ by supercoercivity which proves (a). Now (b) is a consequence of (a), while (c) is a consequence of (b) and the definitions involved. □

The next result is a $\beta$-differentiability version of the Implicit Function Theorem for gauges. This result will be useful in showing that the existence of $\beta$-smooth norms is necessary in (many of) our constructions. A proof relying on elementary convexity arguments has been included for completeness.

**Fact 2.2.** *(Implicit Function Theorem For Gauges) Suppose $f$ is an lsc convex function and that $C := \{x : f(x) \leq \alpha\}$ where $f(0) < \alpha$ and $0 \in intC$. If $f$ is $\beta$-differentiable at all $x \in bnd(C)$, then $\mu_C$, the gauge of $C$ is $\beta$-differentiable at all $x$ where $\mu_C(x) > 0$.*

**Proof.** First, fix $x_0 \in \mathrm{bnd}(C)$ and let $\phi = \dfrac{f'(x_0)}{\langle f'(x_0), x_0\rangle}$. Then $\phi(x_0) = 1$, and so it is the natural candidate for being the $\beta$-derivative $\mu_C'(x_0)$. This is precisely, what we will prove. In what follows, we let $H = \{h : \phi(h) = 0\}$.

We now show that $\phi \in \partial\mu_C(x_0)$. Observe that if $\phi(u) = 1$, then $u = x_0 + h$ for some $h \in H$, and so $\langle f'(x_0), h\rangle = 0$. Now $f(u) \geq f(x_0) + \langle f'(x_0), h\rangle$ by the subgradient inequality. Consequently $f(u) \geq \alpha$ and so $\mu_C(u) \geq 1 = \phi(u)$. Because $\mu_C$ is positively homogeneous, this shows $\mu_C(x) \geq \phi(x)$ for all $x \in X$. Therefore,

$$\phi(x) - \phi(x_0) = \phi(x) - 1 \leq \mu_C(x) - 1 = \mu_C(x) - \mu_C(x_0) \quad \text{for all } x \in X,$$

which shows that $\phi \in \partial\mu_C(x_0)$. Moreover, $\phi$ is the only subgradient of $\mu_C$ at $x_0$. Indeed, if $\psi \in \partial\mu_C(x_0)$, then the subdifferential inequality implies $\psi(x_0) = 1$. Also, if $h \in H$, then $\psi(h) = \psi(x_0) - \psi(x_0 - h) \leq \mu_C(x_0) - \mu_C(x_0 - h) \leq 0$ which implies $\psi(h) = 0$ for all $h \in H$. Consequently, the kernels of $\psi$ and $\phi$ are the same, and $\psi(x_0) = \phi(x_0) = 1$, and so $\phi = \psi$. Therefore $\partial\mu_C(x_0) = \phi$ and so $\phi$ is the Gateaux derivative of $\mu_C$ at $x_0$.

Because both $\phi$ and $\mu_C$ are positively homogeneous, it follows that $\phi$ is the Gateaux derivative of $\mu_C$ at $\lambda x_0$ for each $\lambda > 0$. Finally, the derivative mapping $x \to f'(x)$ is norm to $\tau_\beta$-continuous on $\mathrm{bnd}(C)$ (see Fact 2.3(b)), and because $\mu_C'(x) = \dfrac{f'(x)}{\langle f'(x), x\rangle}$ for all $x$ in the boundary of $C$, it follows that $x \to \mu_C'(x)$ is norm to $\tau_\beta$ continuous at all $x$ where $\mu_C(x) \neq 0$. According to Fact 2.3, $\mu_C$ is $\beta$-differentiable at all $x$ where $\mu_C(x) \neq 0$. □

The equivalence of (a) and (d) in the next result is a $\beta$-differentiability version of Šmulyan's criterion for convex functions, while (b) is the well-known continuity characterization of $\beta$-differentiability that was used in the proof of Fact 2.2. We record Fact 2.3 and its proof here for completeness.

**Fact 2.3.** *Suppose the lsc convex function $f$ is continuous at $x_0$. Then, the following are equivalent.*

*(a) $f$ is $\beta$-smooth at $x_0$.*

*(b)* $\phi_n \to_{\tau_\beta} \phi$ *whenever* $\phi_n \in \partial f(x_n)$, $\phi \in \partial f(x_0)$ *and* $x_n \to x_0$.

*(c)* $\phi_n \to_{\tau_\beta} \phi$ *whenever* $\phi_n \in \partial_{\epsilon_n} f(x_n)$, $\phi \in \partial f(x_0)$, $x_n \to x_0$ *and* $\epsilon_n \downarrow 0$.

*(d)* $\phi_n \to_{\tau_\beta} \phi$ *whenever* $\phi_n \in \partial_{\epsilon_n} f(x_0)$, $\phi \in \partial f(x_0)$ *and* $\epsilon_n \downarrow 0$.

**Proof.** (a) $\Rightarrow$ (d): Suppose that (d) does not hold, then there exist $\epsilon_n \downarrow 0$, $\phi_n \in \partial_{\epsilon_n} f(x_0)$, $\phi \in \partial f(x_0)$ and $\eta > 0$ such that

$$\sup_W |\phi_n - \phi| > \eta \quad \text{for all } n \text{ and some } \beta\text{-set } W.$$

Now choose $h_n \in W$ such that $(\phi_n - \phi)(h_n) \geq \eta$ and let $t_n = 2\epsilon_n/\eta$. Then

$$\begin{aligned} f(x_0 + t_n h_n) - f(x_0) - \phi(t_n h_n) &\geq \phi_n(t_n h_n) - \phi(t_n h_n) - \epsilon_n \\ &\geq \frac{2\epsilon_n}{\eta}\eta - \epsilon_n = \epsilon_n \end{aligned}$$

Therefore $f$ is not $\beta$-differentiable at $x_0$.

(d) $\Rightarrow$ (c): This follows because $f$ is Lipschitz on a certain neighborhood of $x_0$, say $B_r(x_0)$, with some Lipschitz constant $M$. Consequently, if $x_n \in B_r(x_0)$ and $\phi_n \in \partial_{\epsilon_n} f(x_n)$, then $\phi_n \in \partial_{\epsilon_n + M\|x_n - x_0\|} f(x_0)$.

(c) $\Rightarrow$ (b) is obvious. To prove (b) $\Rightarrow$ (a) we suppose $f$ is not $\beta$-differentiable at $x_0$. Then there exist $t_n \downarrow 0$ and $h_n \in W$ where $W$ is a $\beta$-set and $\epsilon > 0$ such that

$$f(x_0 + t_n h_n) - f(x_0) - \phi(t_n h_n) \geq \epsilon t_n \quad \text{where} \quad \phi \in \partial f(x_0).$$

Let $\phi_n \in \partial f(x_0 + t_n h_n)$. Now,

$$\phi_n(t_n x_n) \geq f(x_0 + t_n h_n) - f(x_0) \geq \phi(t_n h_n) + \epsilon t_n$$

and so $\phi_n \not\to_{\tau_\beta} \phi$. $\square$

## 3. Essentially Smooth Convex Functions on General Convex Sets

This section explores constructions of essentially smooth functions and of functions of Legendre type in various classes of Banach spaces. The first result gives us conditions on a Banach space under which we can quite generally construct Legendre functions.

**Theorem 3.1.** *Suppose $X$ is a Banach space such that distance functions to closed convex sets are $\beta$-differentiable on their complements. Let $C$ be a closed convex subset of $X$ having nonempty interior. Then there is a convex function $f$ that is continuous on $C$ and $\beta$-differentiable on int$(C)$ such that:*

*(a)* $\|f'(x_n)\| \to \infty$ *as* $x_n \in C$ *and* $d(x_n, C^c) \to 0$*;*

*(b)* $\partial f(x) = \emptyset$ *if* $x \in$ *bnd*$(C)$

**Proof.** Without loss of generality, we may assume $0 \in \text{int}(C)$. Let $C_n = \{\lambda x : 0 \leq \lambda \leq 1 - 2^{-n}, x \in C\}$. Fix $\epsilon > 0$ such that $B_\epsilon \subset C$. First, we observe that

$$d(C_n, C^c) \geq \frac{\epsilon}{2^n} \quad \text{for all } n. \tag{3.1}$$

Indeed, if $x \in C_n$, then $x = (1-2^{-n})z$ for some $z \in C$. Now suppose $\|h\| \leq \epsilon$, then $h \in C$. Therefore, $x + 2^{-n}h = (1-2^{-n})z + 2^{-n}h \in C$. Thus, $x + B_{\epsilon 2^{-n}} \subset C$ which proves (3.1).

The hypothesis implies that $d^2(\cdot, D)$ is $\beta$-differentiable whenever $D$ is a convex set, and in particular, $\|\cdot\|^2$ is $\beta$-differentiable. The desired function is then defined by

$$f(x) = \|x\|^2 + \sum_{n=1}^{\infty} \frac{n2^{n+2}}{\epsilon^2} d(x, C_n)^2.$$

(Note if for some $\delta > 0$, we have $d(x, C_n) > \delta$ for all $n$, e.g. if $x \in \mathrm{int}(C^c)$ then this sum diverges.)

To see that $f$ is $\beta$-differentiable on $\mathrm{int}(C)$, fix $x_0 \in \mathrm{int}(C)$. Now for some $N$, $x_0 \in \mathrm{int}(C_N)$ and $d(x, C_n) = 0$ for all $x \in C_N$ and for all $n \geq N$. Consequently, on $\mathrm{int}(C_N)$, $f$ is a finite sum of $\beta$-differentiable functions. Therefore $f$ is $\beta$-differentiable on $\mathrm{int}(C)$.

Consider $D_N = \{x \in C : \|x\| \leq N\}$. For $x \in D_N$, we have $d(x, C_n) \leq N2^{-n}$ for all $n$. Therefore,

$$\sum_{n=1}^{\infty} \left|\frac{n2^{n+2}}{\epsilon^2} d(x, C_n)^2\right| \leq \sum_{n=1}^{\infty} \frac{4Nn^2}{2^n \epsilon^2} \quad \text{for all } x \in D_N.$$

By the Weierstrass M-test, $f$ is continuous on $D_n$ for each $n$; and, in particular, $f$ is defined on $\mathrm{bnd}(C)$.

To prove (a), suppose $x_k \in C$ and $d(x_k, C^c) \to 0$. Now $f$ is supercoercive, and so if $\|x_k\| \to \infty$, then $\|f'(x_k)\| \to \infty$ (Fact 2.1(a)). Therefore, by standard subsequence arguments, we may assume that $\{x_k\}$ is bounded, say $\|x_k\| \leq M$ for all $k$. Choose $n_k \to \infty$ such that $d(x_k, C^c) < \epsilon 2^{-(n_k+1)}$ and so (3.1) implies that $d(x_k, C_{n_k}) \geq \epsilon 2^{-(n_k+1)}$. Now choose $1 - 2^{-n_k} \leq \lambda_k \leq 1$ such that $\lambda_k x_k \in \mathrm{bnd}(C_{n_k})$. Because $0 \in C_n$ for each $n$, it follows that $d(x_k, C_n) \geq d(\lambda_k x_k, C_n)$ for each $n$. Therefore

$$\frac{f(x_k) - f(\lambda_k x_k)}{\|x_k - \lambda_k x_k\|} \geq \frac{n_k 2^{n_k+2}}{\epsilon^2} \frac{d^2(x_k, C_{n_k})}{(1-\lambda_k)M} \geq \frac{n_k}{M}.$$

Therefore $\|f'(x_k)\| \geq \frac{n_k}{M}$. Notice that (b) follows from the same argument since $d(\bar{x}, C^c) = 0$ for $\bar{x} \in \mathrm{bnd}(C)$. $\square$

**Remark 3.2.** If the dual norm on $X^*$ is LUR [resp. SC], then the above theorem applies with Fréchet [resp. Gateaux] differentiability. On $L_1(\Omega)$ where $\Omega$ is a $\sigma$-finite measure space, the above theorem applies with weak Hadamard differentiability.

**Proof.** If the dual norm on $X^*$ is LUR [resp. SC], then distance functions to closed convex sets are Fréchet differentiable [resp. Gateaux differentiable] on their complements [1] (see also [5, Corollary VIII.3.16]). For the case of $L_1(\Omega)$ where $\Omega$ is a $\sigma$-finite measure space, [4, Theorem 2.4] shows that there is an equivalent norm $|||\cdot|||$ on $L_1(\Omega)$ such that its dual norm $|||\cdot|||^*$ on $L_\infty(\Omega)$ satisfies $x_n \to_{\tau_W} x$ in the Mackey topology of uniform convergence on weakly compact subset of $L_1(\Omega)$ whenever $|||x^*|||^* = 1$, $|||x_n^*|||^* \leq 1$ and $|||x^* + x_n^*|||^* \to 2$ (this dual norm is *Mackey-LUR*). Because the dual norm is Mackey-LUR, it can be shown as in [5, Corollary VIII.3.16] that distance functions to convex sets on $(L_1(\Omega), |||\cdot|||)$ will be weak-Hadamard differentiable on their complements. $\square$

A characterization of open sets admitting certain essentially $\beta$-smooth convex barrier functions is presented in the next result. Let us recall that a function $f : X \to \mathbb{R}$ is said to be *coercive* if $\lim_{\|x\|\to\infty} f(x) = \infty$.

**Theorem 3.3.** *Let $X$ be a Banach space, and $C$ be an open convex set containing $0$. Then the following are equivalent.*

*(a) There is a $\beta$-differentiable convex function $f$ whose domain is $C$ such that $f(x_n) \to \infty$ and $\|f'(x_n)\| \to \infty$ if $d(x_n, C^c) \to 0$, or if $\|x_n\| \to \infty$.*

*(b) There is a coercive convex barrier function $f$ that is $\beta$-differentiable on $C$.*

*(c) There are continuous gauges $\{\mu_n\}$ such that $\mu_n$ is $\beta$-differentiable when $\mu_n(x) \neq 0$ and $\mu_n \downarrow \mu_C$ pointwise where $\mu_C$ is the gauge of $C$, and $X$ admits a $\beta$-differentiable norm.*

*(d) $X$ admits a $\beta$-differentiable norm and there is a sequence of $\beta$-differentiable convex functions $\{f_n\}$ that are bounded on bounded sets and $f_n \downarrow \mu_C$ pointwise where $\mu_C$ is the gauge of $C$.*

(a) $\Rightarrow$ (b): This portion is obvious.

(b) $\Rightarrow$ (c): Let $C_n = \{x : f(x) \leq n\}$ for all $n$ such that $f(0) < n$. Then $C_n \subset \operatorname{int}(C)$ and Fact 2.2 implies that $\mu_{C_n}$ is $\beta$-differentiable at all $x$ where $\mu_{C_n}(x) \neq 0$. It is not difficult to check that $\mu_{C_n} \downarrow \mu_C$ pointwise. Because $C_n$ is bounded with 0 in its interior, it follows that $|||\cdot|||$ defined by $|||x||| = \mu_{C_n}(x) + \mu_{C_n}(-x)$ is an equivalent $\beta$-differentiable norm on $X$.

(c) $\Rightarrow$ (d): Let $h_n : \mathbb{R} \to \mathbb{R}$ be nondecreasing $C^\infty$-smooth convex functions such that $h_n(t) = 2/n$ if $t \leq 1/n$ and $h_n(t) \downarrow t$ for all $t \geq 0$. Because $h_n$ is constant on a neighborhood of 0 and $\mu_C$ is differentiable at $x$ where $\mu_C(x) > 0$, it follows that $f_n = h_n \circ \mu_C$ is $\beta$-differentiable everywhere, and that $\{f_n\} \downarrow \mu_C$ pointwise.

(d) $\Rightarrow$ (a): Given $f_n \downarrow \mu_C$ pointwise, it follows that $g_n \downarrow \mu_C$ pointwise where $g_n = (1+\frac{1}{n})f_n$. Now let $h_n$ be a $C^\infty$-smooth nondecreasing convex function on $\mathbb{R}$ such that $h_n(t) = 0$ if $t \leq 1$, and $h_n(t) \geq n$ if $t \geq 1 + \frac{1}{2n}$. Let $\|\cdot\|$ be an equivalent $\beta$-differentiable norm on $X$ and define $f$ by

$$f(x) = \|x\|^2 + \sum_{n=1}^{\infty} h_n(g_n(x)).$$

Then $f$ is convex, because $h_n$ is convex and nondecreasing and $g_n$ is convex. Also, if $x_0 \in C$, then $\mu_C(x_0) < 1$. Therefore, $g_n(x_0) \leq g_N(x_0) < \alpha < 1$ for all $n \geq N$ and some $\alpha$. Let $O = \{x : g_N(x) < \alpha\}$ then $O$ is an open neighborhood of $x_0$ and

$$f(x) = \|x\|^2 + \sum_{n=1}^{N-1} h_n(g_n(x)) \quad \text{for all } x \in O.$$

Therefore $f$ is $\beta$-differentiable on $O$.

Now suppose $d(x_k, C^c) \to 0$. Because $f$ is supercoercive, as in the proof of Theorem 3.1, we may assume that $\{x_k\}$ is bounded. Let $F_n = \{x : g_n(x) \geq 1 + \frac{1}{2n}\}$. Since $g_n$ is Lipschitz on bounded sets, and $g_n \geq 1 + \frac{1}{n}$ on $C^c$, it follows that $x_k \in F_n$ for all $k \geq N$. Therefore, $f(x_k) \geq h_n(g_n(x_k)) \geq n$ for all $k \geq N$. Thus $f(x_k) \to \infty$ and, by the subgradient inequality, $\|f'(x_k)\| \to \infty$ since $\{x_k\}$ is bounded. $\square$

**Corollary 3.4.** *Suppose $X$ is a Banach space that admits an equivalent norm whose dual is LUR [resp. strictly convex]. Let $C$ be an open convex set in $X$. Then there is a convex function $f$ that is $C^1$-smooth on $C$ [resp. continuous and Gateaux differentiable on $C$] such that:*

*(a) $f(x_n) \to \infty$ and $\|f'(x_n)\| \to \infty$ as $x_n \in C$ and $d(x_n, C^c) \to 0$;*

*(b) if $x_n \in C$ and $\|x_n\| \to \infty$, then $f(x_n) \to \infty$ and $\|f'(x_n)\| \to \infty$.*

**Proof.** If $X^*$ admits a dual LUR norm, then every Lipschitz convex function can be approximated uniformly by $C^1$-smooth convex functions. This follows because the infimal convolution $f \,\square\, \|\cdot\|^2$ of a convex function $f$ and $\|\cdot\|$ whose dual is LUR, will be Fréchet differentiable (this follows from seminal work of Asplund and Rockafellar [1]), and a standard computation shows that the convergence of $f \,\square\, n\|\cdot\|^2$ will be uniform to $f$ when $f$ is Lipschitz. It follows that we can apply Theorem 3.3(d). An analogous result holds for dual strictly convex norms. $\square$

Let us note that our constructions can be made log-convex by considering $e^{f(x)}$. The following theorem relates the existence of various essentially smooth or functions of Legendre type to the existence of certain norms on the Banach space.

**Theorem 3.5.**
*(a) Suppose $X$ admits an essentially $\beta$-smooth lsc convex function on a bounded open convex set $C$ with $0 \in C$. Then $X$ admits a $\beta$-differentiable norm.*

*(b) Suppose $X$ admits a strictly convex lower semicontinuous function that is continuous at one point. Then $X$ admits a strictly convex norm.*

*(c) Suppose $X$ admits an essentially $\beta$-smooth lsc convex function $f$ on a bounded convex set $C$ with $0 \in int(C)$ that additionally satisfies $\|f'(x_n)\| \to \infty$ if $d(x_n, bnd(C)) \to 0$. Then $X$ admits a $\beta$-differentiable norm.*

*(d) If $X$ admits a Legendre function on a bounded open convex set, then $X$ admits an equivalent Gateaux differentiable norm that is strictly convex.*

*(e) If $X$ admits an equivalent strictly convex norm, and if every open convex set is the domain of an essentially $\beta$-smooth function, then every open convex set is the domain of some $\beta$-Legendre function.*

*(f) Suppose $X$ admits an equivalent strictly convex norm whose dual is strictly convex (resp. LUR), then every open convex set is the domain of some Legendre function (resp. Fréchet-Legendre function).*

**Proof.** (a) Let $h(x) = f(x)+f(-x)$ on $B := C\cap(-C)$. Notice that $h$ is $\beta$-differentiable on $\text{int}(B)$. It follows that $h$ is $\beta$-differentiable on the open convex set $B$, and also $h(x) = +\infty$ for $x \notin B$. Choose $f(0) < \alpha < \infty$. Because $f$ is continuous at 0 the set $D = \{x : f(x) \leq \alpha\}$ has nonempty interior and $D \subset B$. The bounded convex set $D$ is also symmetric, and so Fact 2.2, implies that the norm, defined as the gauge of $D$, is an equivalent $\beta$-differentiable norm on $X$.

(b) Suppose $f$ is strictly convex, and continuous at $x_0$. By replacing $f$ with $f - \phi$ and translating as necessary, we may assume $x_0 = 0$ and $f(0) = 0$ is the minimum of $f$. Also, replacing $f$ with $f + \|\cdot\|^2$ gives us a function that is both strictly convex and coercive (the sum of a convex function and a strictly convex function is strictly convex). Now since $f$ is continuous at 0, so is $h$ where $h$ is as in (a). Because $h$ is continuous at 0 and coercive, $B = \{x : h(x) \leq 1\}$ is a bounded convex set, with nonempty interior. Because $h$ is strictly convex and symmetric, the gauge of $B$ is a strictly convex equivalent norm on $X$.

(c) As in (b), we may assume $f(0) = 0$ is the minimum of $f$. Now let $h$ and $B$ be as above; we next observe that $\inf\{h(x) : x \in \mathrm{bnd}(B)\} > 0$. Indeed, suppose $x_n \in \mathrm{bnd}(B)$ and $h(x_n) \to 0$. Notice that $x_n$ or $-x_n$ is in the boundary of $C$. Without loss, assume that $x_n \in \mathrm{bnd}(C)$. We know that $f(x_n) \to 0$ (since 0 is the minimum of $f$ and $h$). Thus, the Bronsted-Rockafellar theorem implies there is a sequence $y_n \in C$ with $\|x_n - y_n\| \to 0$ and $\phi_n \in \partial f(y_n)$ while $\|\phi_n\| \to 0$. This violates the condition $\|f'(y_n)\| \to \infty$ as $d(y_n, \mathrm{bnd}(B)) \to 0$. Hence there is an $\alpha$ such that $0 < \alpha < \inf\{h(x) : x \in \mathrm{bnd}(B)\}$, and we may apply Fact 2.2 as in (a).

(d) Construct $h$ and $B$ as in (a). Because $h$ is strictly convex on $B \supset D$, the norm constructed in (a) is strictly convex.

(e) Let $B$ be the unit ball with respect to an equivalent strictly convex norm $\|\cdot\|$, and let $C$ be the interior of $B$. Because $C$ is bounded and open, there is an essentially $\beta$-smooth convex function whose domain is $C$ that satisfies Theorem 3.3(a). According to Theorem 3.3(c), there are $\beta$-differentiable gauges $\mu_n$ decreasing pointwise to $\|\cdot\|$. By letting $\|\cdot\|_n = \frac{1}{2}[\mu_n(x) + \mu_n(-x)]$ we get $\beta$-differentiable norms $\|\cdot\|_n \downarrow \|\cdot\|$. Now $\|\cdot\|_1 \leq K\|\cdot\|$ and so the norms $\|\cdot\|_n$ are equi-Lipschitz. Following [6], we define $|||\cdot|||$ by

$$|||x||| = \sqrt{\sum_{n=1}^{\infty} \frac{1}{2^n}\|x\|^2}.$$

This norm is $\beta$-differentiable because of the uniform convergence of the sum of derivatives. Moreover, it is strictly convex, because if $|||x||| = |||y||| = 1$ and $2|||x|||^2 + 2|||y|||^2 - |||x+y|||^2 = 0$ we must have $2\|x\|_n^2 + 2\|y\|_n^2 - \|x+y\|_n^2 = 0$ for all $n$. Because $\|\cdot\|_n \to \|\cdot\|$ pointwise, we have $2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 0$ and so $x = y$ by the strict convexity of $\|\cdot\|$. Then adding $|||\cdot|||^2$ to any essentially $\beta$-smooth convex function, produces a supercoercive essentially $\beta$-smooth strictly convex function whose domain is $C$. Hence Fact 2.1(c) implies this function is $\beta$-Legendre.

(f) This follows from (e) and Corollary 3.4. □

The previous result shows that many spaces do not have functions of Legendre type, while many others have an abundance of such functions. We make a brief list of some such spaces in the following:

**Example 3.6.**
(a) The spaces $\ell_\infty/c_0$ and $\ell_\infty(\Gamma)$ where $\Gamma$ is uncountable, admit no essentially strictly convex, and hence no Legendre functions.

(b) If $X$ is a WCG space, then every open convex subset of $X$ is the domain of a Legendre function.

(c) If $X^*$ is WCG, or if $X$ is a WCG Asplund space, then every open convex subset of $X$ is the domain of a Fréchet-Legendre function.

**Proof.** If the spaces in (a) were to have such functions, then they would have strictly convex norms by Theorem 3.5(b). It is well-known (see [5, Chapter II]) that these spaces do not have equivalent strictly convex norms. In [5, Chapter VII] it is shown that WCG spaces have strictly convex norms whose duals are strictly convex, and so (b) follows from Theorem 3.5(f). Similarly, [5, Chapter VII] shows that spaces as in (c) have LUR norms whose duals are also LUR, and so (c) follows from Theorem 3.5(f). □

If $f$ is essentially smooth with domain $C$, then $\|f'(x_n)\| \to \infty$ if $x_n \to \bar{x}$ where $\bar{x} \in \mathrm{bnd}(C)$ [2, Theorem 5.6(v)]. However, this does not ensure that $\|f'(x_n)\| \to \infty$ when $d(x_n, \mathrm{bnd}(C)) \to 0$ (as was required in Theorem 3.5(c)), even for bounded sets as is shown in the following example.

**Example 3.7.** There is an essentially smooth convex function $f$ whose domain is a closed convex set $C$ such that $\|f'(x_n)\| \not\to \infty$ as $d(x_n, \mathrm{bnd}(C)) \to 0$.

**Proof.** Let $X = c_0$ with its usual norm and let $C$ be the closed unit ball of $c_0$. Let $h_n : [-1, 1] \to \mathbb{R}$ be continuous, even and convex such that $h$ is $C^1$-smooth on $(-1, 1)$, and $(h_n)'_-(1) = +\infty$, $h_n(1) = 1$ and $h_n(t) = 0$ for $|t| \leq 1 - \frac{1}{2n}$. Now extend $h_n$ to an lsc convex function on $\mathbb{R}$ by defining $h(t) = +\infty$ if $|t| > 1$. Define $f$ on $c_0$ by $f(x) = \sum_{n=1}^{\infty} h_n(x_n)$ where $x = (x_n)_{n=1}^{\infty}$. Then $f$ is $C^1$-smooth and convex on $\mathrm{int}C$ because it is a locally finite sum of such functions there. Since $(h_n)'_+(1) = \infty$, it follows that $\partial f(x) = \emptyset$ if $\|x\| = 1$, and clearly $f(x) = \infty$ if $\|x\| > 1$. Therefore, $f$ is essentially Fréchet smooth. However, if we consider $v_n = (1 - \frac{1}{n})e_n$, we have $f(v_n) = 0$ and $f'(v_n) = 0$ for each $n$ while $d(v_n, \mathrm{bnd}(C)) \to 0$. □

Contrasting Theorem 3.5 (c) with the previous example leads naturally to the following question.

**Question 3.8.** If $X$ admits an essentially $\beta$-smooth convex function on a bounded convex set with 0 in its interior, does $X$ admit a $\beta$-differentiable norm? Relatedly, if $X$ admits an essentially $\beta$-smooth convex function on a bounded convex set with 0 in its interior, does $X$ admit a $\beta$-differentiable convex function on a bounded convex set with 0 in its interior such that $\|f'(x_n)\| \to \infty$ whenever $d(x_n, \mathrm{bnd}(C)) \to 0$?

## 4. The finite dimensional case

This section begins with a direct proof of the log-convexity of the *universal* barrier for an arbitrary open convex set in a finite dimensional Banach space considered by Nesterov and Nermirovskii in [7]. Then we explore some refinements of this result.

**Theorem 4.1.** *Let $A$ be a nonempty open convex set in $\mathbb{R}^N$. Define, for $x \in A$,*

$$F(x) = \lambda_N((A - x)^o),$$

*where $\lambda_N$ is $N$-dimensional Lebesque measure and $(A - x)^0$ is the polar set. Then $F$ is an essentially Fréchet smooth,* log-*convex, barrier function for $A$.*

**Proof.** Without loss of generality, we assume $A$ is line free (else we can add the square of the norm on the lineality subspace). In this case the universal self-concordant barrier $b_A$ is a multiple of $\log F$. (As described in [7], self-concordance is central to the behavior of interior point methods. Because barrier properties (without considering concordance) are applicable more generally, it seems useful to exhibit the strengthened convexity and barrier properties directly as we do below.)

Now observe that $F$ is finite on $A$, because $B_\epsilon \subset A - x$ for some $\epsilon > 0$. Moreover, $F(x) = +\infty$ for $x \in \mathrm{bnd} A$. Indeed, this follows by translation invariance of the measure since $(A - x)^o$ contains a ray and has non-empty interior because $A$ is line free. (So we really need Haar measure which effectively limits this proof to finite dimensions.) Therefore, $F$ is a barrier function for $A$.

Now, by the spherical change of variable theorem

$$F(x) = \frac{1}{N} \int_{S_X} \frac{1}{(\delta_A^*(u) - \langle u, x\rangle)^N}\, du \tag{4.1}$$

where $du$ is surface measure on the sphere. Because the integrand is $C^N$ on $A$ and the gradient is locally bounded in $A$ it follows that that $F$ is essentially smooth. It remains to verify the log-convexity of $F$. For this we will use (4.1), indeed:

$$\begin{aligned}
F\left(\frac{x+y}{2}\right) &= \frac{1}{N} \int_{S_X} \frac{1}{(\delta_A^*(u) - \langle u, \frac{x+y}{2}\rangle)^N} du \\
&= \frac{1}{N} \int_{S_X} \frac{1}{\left(\frac{\delta_A^*(u) - \langle u,x\rangle}{2} + \frac{\delta_A^*(u) - \langle u,y\rangle}{2}\right)^N} du \\
&\leq \int_{S_X} \frac{1}{(\delta_A^*(u) - \langle u, x\rangle)^{N/2} (\delta_A^* - \langle u, y\rangle)^{N/2}}\, du && (4.2) \\
&\leq \sqrt{\int_{S_X} \frac{1}{(\delta_A^* - \langle u, x\rangle)^N}\, du \int_{S_X} \frac{1}{(\delta_A^*(u) - \langle u, x\rangle)^N}\, du} && (4.3) \\
&= \sqrt{F(x)F(y)}
\end{aligned}$$

where we have used the Arithmetic-Geometric Mean inequality in (4.2) and Cauchy-Schwartz in (4.3). Taking logs of both sides of the preceding inequality completes the proof. □

We now refine the above example to produce an essentially smooth convex function whose domain is a closed convex cone with nonempty interior. We restrict ourselves to the case of cones here because the most important applications have been in abstract linear programming over cones which are built up from products of cones of positive definite matrices, orthants and other simple cones. Moreover, the technical details for the case of general closed convex sets appeared to be more involved (and we already have this more generally by the less explicit methods of the previous section). Before proceeding, we will need the following lemma.

**Lemma 4.2.** *Let $g(x,u) \geq 0$ be concave in $x$, let $\phi$ be convex and decreasing on $\mathbb{R}^+$ and consider*

$$G(x) = \phi^{-1}\left(\int \phi(g(x,u))\,\mu(du)\right)$$

*for a probability measure $\mu$. Assume additionally the mean $H_\phi$ defined by $H(a,b) = \phi\left(\frac{\phi^{-1}(a)+\phi^{-1}(b)}{2}\right)$ is concave. Then $G$ is concave.*

**Proof.** Using the fact that $g(x,u)$ is concave in $x$ and $\phi$ is decreasing in (4.4) below, and then Jensen's inequality with the concavity of $H_\phi$ in (4.5) below, we obtain:

$$\begin{aligned}
\phi\left(G\left(\frac{x+y}{2}\right)\right) &= \int \phi\left(g\left(\frac{x+y}{2}\right), u\right)\,\mu(du) \\
&\leq \int \phi\left(\frac{g(x,u)+g(y,u)}{2}\right)\,\mu(du) && (4.4)\\
&= H_\phi(\phi(g(x,u)), \phi(g(y,u)))\,\mu(du) \\
&\leq H_\phi\left(\int (\phi(g(x,u)\mu(du), \int \phi(g(y,u)\mu(du)\right) && (4.5)\\
&= H_\phi(\phi(G(x), \phi(G(y)) \\
&= \phi\left(\frac{\phi^{-1}(\phi(G(x)) + \phi^{-1}(\phi(G(y))}{2}\right) \\
&= \phi\left(\frac{G(x)+G(y)}{2}\right)
\end{aligned}$$

Because $\phi$ is decreasing, the previous inequality implies

$$G\left(\frac{x+y}{2}\right) \geq \frac{G(x)+G(y)}{2},$$

and so $G$ is concave. □

The next fact follows from properties of the Hessian, whose tedious computations are omitted, but are easily checked in a computer algebra system.

**Fact 4.3.** *Let $\phi(t) = t^\alpha$ with $\alpha < 0$. Then $H_\phi(a,b) = \phi\left(\dfrac{\phi^{-1}(a)+\phi^{-1}(b)}{2}\right)$ is concave.*

We now have the tools in hand to prove

**Theorem 4.4.** *Given $F(x) = \lambda_N((A-x)^o)$ as above, and let $A$ be an open convex cone, we define $G(x) = -(F(x)^{-p})$ where $0 < p < 1/N$ is fixed. Then $G$ is convex, essentially smooth, vanishes on bnd$(A)$ and has domain equal to the closure of $A$.*

**Proof.** Let $\phi(t) = t^{-N}$; then $\phi$ is convex and decreasing on $\mathbb{R}^+$. Moreover, Fact 4.3 ensures that $H_\phi$ is concave. Consequently,

$$\widetilde{G}(x) = F(x)^{-\frac{1}{N}} = \phi^{-1}\left(\int \phi(\delta_A^*(u) - \langle u, x\rangle)du\right)$$

is concave by Lemma 4.2 because $g(x,u) = \delta_A^*(u) - \langle u, x\rangle$ is concave in $x$. Because $t^\alpha$ is concave and increasing for $0 < \alpha < 1$, it follows that $\widetilde{G}^\alpha$ is concave. Because $G = -\widetilde{G}^\alpha$, we know that $G$ is convex. Also, $G$ is smooth on $A$ because $F$ is smooth and does not vanish there. Moreover, $G$ vanishes on $\mathrm{bnd}(A)$ because $F$ is infinite there.

Therefore, it remains to show that $G$ is essentially smooth, where we, of course, have defined $G(x) = +\infty$ for $x \notin \overline{A}$. To do this, we will check that $\|\nabla G(x_n)\| \to \infty$ as $x_n \to \bar{x}$ where $\bar{x} \in \mathrm{bnd}(A)$ (see [2]).

Now, we have (normalized):

$$\|\nabla F(x)\| \geq \int_{h\in A\cap S_X} \langle h, u\rangle \int_{u\in A^+\cap S_X} \langle u, x\rangle^{-N-1}\, \mu(du)\mu(dh).$$

where $S_X$ is the unit sphere and $A^+$ is the positive polar cone. Interchanging the order of integration, we write

$$\|\nabla F(x)\| \geq \int_{u\in A^+\cap S_X} \eta(u)\langle u, x\rangle^{-N-1}\, \mu(du),$$

where

$$\eta(u) := \int_{h\in A\cap S_X} \langle h, u\rangle, \mu(dh)\,.$$

It suffices to observe, by continuity, that $\inf\{\eta(u) : u \in A^+\cap S_X\} > 0$ (since $A^+$ is pointed) and so for some constant $K > 0$

$$K\|\nabla F(x)\| \geq \int_{u\in A^+\cap S_X} \langle u, x\rangle^{-N-1}\, \mu(du) \geq \left[\int_{u\in A^+\cap S_X} \langle u, x\rangle^{-N}\, \mu(du)\right]^{1+\frac{1}{N}},$$

on applying Hölder's inequality. Thus,

$$K^N\|\nabla F(x)\|^N \geq F(x)^{N+1}.$$

A direct computation of $\nabla G(x)$ shows that $\|\nabla G(x_n)\| \to \infty$ as $x_n \to \bar{x} \in \mathrm{bnd}(A)$. $\square$

Of course, the constructions in Section 3 guarantee the existence of essentially smooth convex functions on $A$ and $\overline{A}$, but they are not as explicit as those in this section. Moreover, the essentially smooth convex functions constructed in Theorem 3.1 defined on closed convex sets do not vanish (nor are they constant) on the boundaries of those sets. While it is easy to construct essentially smooth functions that vanish on their boundaries when the domain is the ball of a smooth norm, we are not aware of such constructions on general convex sets with nonempty interior in smooth Banach spaces.

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