Journal of Convex Analysis
Volume 8 (2001), No. 2, 489–509

Monotonicity of the Integral Mean and Convex Functions

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Received August 31, 2000
Revised manuscript received February 12, 2001

A set A will be said convexly majorized by a set B if the integral mean of any convex function over A is not exceeding its mean over B. Sufficient conditions and necessary conditions are presented about this relation. Methods will be introduced which generate such sets A and B.

1. Introduction

Mean-value inequalities appear in a number of contexts in analysis, in particular, in potential theory, in complex analysis and in the theory of partial differential equations, principally as a tool to obtain estimates. See, for instance [1, 2, 4], and [6].

The aim of this paper is to find out for which pair of sets the mean of every convex function over one set is dominated by the mean of the same function over the other set. Although the context of convex functions is rather elementary, we hope that the detailed analysis of this situation will shed some light of more complex problems of this kind arising in other theories.

Let A be a bounded subset of $\mathbf{R}^n$ of positive Lebesgue measure and let $v : A \to \mathbf{R}$ be continuous. The integral mean of v over A, denoted by v_A, is defined as

$$v_A := \frac{1}{m(A)} \int_A v(x)dx, \tag{1}$$

where $m(A)$ denotes the Lebesgue measure of A.

If $A \subset \mathbf{R}^n$, then $\mathrm{co}(A)$ (resp. $\overline{\mathrm{co}}(A)$) denotes the convex hull (resp. the closed convex hull) of A. A new relation concerning integral means will be introduced first.

Definition 1.1. Let A and B be bounded subsets of $\mathbf{R}^n$ of positive Lebesgue measure. The set A is said to be convexly majorized by B if the inequality $v_A \le v_B$ holds for every real continuous convex function v defined on $\overline{\mathrm{co}}(A \cup B)$. This relation will be denoted by $A \prec B$.

*Work supported in part by the NSERC of Canada under grant A-8421.

ISSN 0944-6532 / $ 2.50

The set of functions, which are real, continuous and convex on the $\overline{\text{co}}(A \cup B)$ forms a convex cone, which will be denoted by $K_{A,B}$.

The main topic of this paper is to initiate the study of the relation $A \prec B$ and to present several results about it. A complete description will be given when A and B are compact sets in $\mathbf{R}$ and some sufficient and some necessary conditions will be given when A and B are bounded sets of positive Lebesgue measure in $\mathbf{R}^n$. Methods will be introduced which generate sets A and B satisfying the relation $A \prec B$. Some new inequalities will also be established.

Let A be a bounded subset of $\mathbf{R}^n$ of positive Lebesgue measure. The barycenter of A, denoted by $\overline{x}_A$, is defined by the formula

$$\overline{x}_A := \frac{1}{m(A)} \int_A x dx.$$

It might be worthwhile to notice that the notions reviewed above belong naturally to the affine geometry of the Euclidean space. Specifically, convex functions and sets are preserved by affinities (invertible affine transformations), the Lebesgue measure is preserved by affinities up to a multiplicative constant, while the mean, our main object, is the exact invariant of affinities. In this respect, one of the results of this paper (Theorem 6.11) is particularly striking. It is saying that if A and B are ellipsoids with common barycenter, then A is convexly majorized by B, if and only if $A \subset B$.

We observe that ellipsoids are naturally characterized in the affine geometry as sets whose boundaries are compact hypersurfaces such that the group of affine transformations preserving them acts on them transitively.

If A and B are bounded subsets of $\mathbf{R}^n$ of positive Lebesgue measure, μ_A will denote the measure, which is obtained by the restriction of the Lebesgue measure to A divided by $m(A)$. (Similarly for B.) Hence $\mu_A(\mathbf{R}^n) = \mu_B(\mathbf{R}^n) = 1$. Thus $A \prec B$ if and only if $\mu_A(v) \leq \mu_B(v)$ for all $v \in K_{A,B}$, where $\mu_A(v)$ denotes the integral of v on A with respect to μ_A. Therefore μ_B is a balayage of μ_A relative to $K_{A,B}$. The notion of balayage (see [9]) has been considered previously. Here, we consider the problem of balayage for a special family of measures, and obtain several results about them. Relations between balayage and the Hardy-Littlewood-Polya order studied in [3], and [7], while a detailed study of this later concept can be found in [8] and [11].

We are grateful to the Referee for pointing out some interesting relations, See, also our comments after the proof of Theorem 3.1,

2. Preliminaries and Notation

In this section some known results, which will be needed in the sequel, are discussed. A new inequality, which originally motivated this study, will also be presented. First, some of the notation used in this paper will be introduced.

The symmetric difference of two sets A and B which is the set $(A \setminus B) \cup (B \setminus A)$ will be denoted by $A \triangle B$. The dual space of $\mathbf{R}^n$ will be denoted by $(\mathbf{R}^n)^\star$. If $g \in (\mathbf{R}^n)^\star$, then $\text{Ker}(g)$ denotes the kernel of g. If A is a set in $\mathbf{R}^n$, then $\overline{A}$ denotes its closure, $\text{int}A$ its interior, and χ_A its characteristic function. If $f : \mathbf{R}^n \to \mathbf{R}^n$ then $\text{supp}\, f$ (resp. $\text{ess supp}\, f$) denotes the support (resp. the essential support) of f.

The following result can be found in [5].

Theorem 2.1. *A continuous function $f : (c,d) \to \mathbf{R}$ is convex on (c,d) if and only if the inequality*

$$f(a) \leq \frac{1}{2h} \int_{a-h}^{a+h} f(x)dx \tag{2}$$

holds for all $h > 0$ and for all $a \in (c,d)$ such that $c < a-h < a+h < d$.

The following equivalent reformulation of Theorem 2.1 will be useful in the sequel.

Corollary 2.2. *A continuous function $f : (c,d) \to \mathbf{R}$ is convex on (c,d) if and only if the inequality*

$$f(\frac{x_1 + x_2}{2}) \leq \int_0^1 f(x_1 + t(x_2 - x_1))dt$$

holds for all $c < x_1 < x_2 < d$.

It is easy to show the one-dimensional version of the following statement, which yields also the n-dimensional result.

Lemma 2.3. *If $w : \mathbf{U} \subset \mathbf{R}^n \to \mathbf{R}$ is a non-negative convex function on a convex neighborhood $\mathbf{U}$ of 0 with $w(0) = 0$, then for all scalars $t \geq 1$ the inequality $w(x) \leq w(tx)$ holds provided that both $x \in \mathbf{U}$ and $tx \in \mathbf{U}$.*

Next, a new criterion of convexity will be given. This result will be generalized considerably in this paper. A proof is provided only because of its simplicity.

Theorem 2.4. *A real continuous function v is convex on an interval (c,d) if and only if the inequality*

$$\int_{a-h}^{a+h} v(x)dx \leq \int_{a-2h}^{a-h} v(x)dx + \int_{a+h}^{a+2h} v(x)dx \tag{3}$$

or equivalently, the inequality

$$v_{[a-h,a+h]} = \frac{1}{2h} \int_{a-h}^{a+h} v(x)dx \leq \frac{1}{4h} \int_{a-2h}^{a+2h} v(x)dx = v_{[a-2h,a+2h]} \tag{4}$$

holds for all $a \in (c,d)$ and for all $h > 0$ such that $c < a-2h < a+2h < d$.

Proof. It is clear that inequalities (3) and (4) are equivalent. Let v be convex on (c,d). Without loss of generality we can assume that v is positive on (c,d). Let $g : \mathbf{R} \to \mathbf{R}$ be the unique affine function which is determined by the conditions $g(a-h) = v(a-h)$ and $g(a+h) = v(a+h)$. Then

$$\begin{aligned} \int_{a-h}^{a+h} v(x)dx \leq \int_{a-h}^{a+h} g(x)dx &= \int_{a-2h}^{a-h} g(x)dx + \int_{a+h}^{a+2h} g(x)dx \\ &\leq \int_{a-2h}^{a-h} v(x)dx + \int_{a+h}^{a+2h} v(x)dx \end{aligned}$$

with equality if and only if v is affine on $(a-2h, a+2h)$. Conversely, if v is continuous on (c, d) and if v satisfies (4) there, then for each n positive integer the inequality

$$\frac{2^n}{2h}\int_{a-h/2^n}^{a+h/2^n} v(x)dx \le \frac{1}{2h}\int_{a-h}^{a+h} v(x)dx \tag{5}$$

holds. Now (5) shows that v satisfies (2), therefore v is convex on (c, d). □

It is easy to show that the following holds.

Corollary 2.5. *If v is a real differentiable convex function on an interval (c, d) and if $c < a-2h < a+2h < d$, then*

$$v_{[a-2h,a+2h]} - v_{[a-h,a+h]} \ge \frac{v'(a+h) - v'(a-h)}{8} h. \tag{6}$$

3. General Results

It is easy to see that the relation $\prec$ is transitive. Furthermore, if A and B are bounded sets in $\mathbf{R}^n$ of positive Lebesgue measure such that $m(A \triangle B) = 0$, then $A \prec B$ and $B \prec A$. The converse of this result holds also, and it will be shown later. Necessary conditions for the relation $A \prec B$ will be presented first.

Theorem 3.1. *Let A and B be bounded sets of positive measure in $\mathbf{R}^n$. If the relation $A \prec B$ holds then*

(i) A and B have the same barycenter, i. e. $\overline{x}_A = \overline{x}_B$.

(ii) If, in adddition, A and B are compact, then $m(A \setminus co(B)) = 0$.

(iii) If, in addition, A and B are convex and compact, then $A \subset B$.

Proof. Assume that the relation $A \prec B$ holds. Let $\ell : \mathbf{R}^n \to \mathbf{R}$ be an affine function. Since the inequality $v_A \le v_B$ holds for both $v = \ell$ and for $v = -\ell$, it follows that $\ell_A = \ell_B$. The special cases $\ell(x) = x_j$ for $j = 1, ..., n$ yield that the jth coordinate of the barycenter of A is the same as the jth coordinate of the barycenter of B for $j = 1, ..., n$. Hence the proof of (i) is complete.

Assume that (ii) does not hold. Then there is an $x^\star \in A$, such that $x^\star$ is a point of density (one) of A but $x^\star \notin \text{co}(B)$. Since $\{x^\star\}$ and $\text{co}(B)$ are disjoint convex and compact sets the separation theorem (Corollary 11.4.2. of [12]) indicates that there is an affine function ℓ and a real number α such that $\ell(x^\star) > \alpha$ and $\ell(x) < \alpha$ for every $x \in B$. Let $v(x) = \max(\ell(x), \alpha)$. Clearly v is a continuous convex function on $\mathbf{R}^n$ with

$$\int_B v dx = \alpha m(B).$$

Since $v(x^\star) > \alpha$ and $v(x) \ge \alpha$ for all $x \in A$, one obtains that

$$\int_A v dx > \alpha m(A).$$

Hence it follows that $v_A > \alpha = v_B$, which is a contradiction.

Assume next that (iii) is false. Then there is an $x^\star \in A$ such that $x^\star \notin B$. Notice that the distance between $x^\star$ and B is positive. Since the relative interior of A is non-empty (Theorem 6.2 of [12]) and since $m(A) > 0$, the relative interior of A is the same as the $\operatorname{int} A$. Thus there is an $x_1 \in \operatorname{int} A$. Then the points of the form $(1-\lambda)x_1 + \lambda x^\star \in \operatorname{int} A$ for all $0 \leq \lambda < 1$. (Theorem 6.1 of [12].) Hence there is an $x_2 \in \operatorname{int} A$ but $x_2 \notin B$. Obviously, x_2 is a point of density of A. It was shown in the proof of (ii), that this situation leads to a contradiction. □

Our Referee remarked that in the proof of Theorem 3.1, we use only the cone K consisting of functions of the form $v(x) = \max\{\ell(x), \alpha\}$ where ℓ is an affine function and α is a constant. The Referee was also asking whether balayage relative to K and $K_{A,B}$ are equivalent. We don't know the answer.

The next result presents sufficient conditions for the relation $A \prec B$.

Theorem 3.2. *Let A and B be bounded measurable sets in $\mathbf{R}^n$ with $0 < m(A) < m(B)$ and such that they have the same barycenter. If A and B are similar with respect to their common barycenter, then for every real function v, which is continuous and convex on $\overline{\mathrm{co}}(B)$ the inequality $v_A \leq v_B$ holds.*

Proof. It can be assumed without loss of generality that the common barycenter of A and B is at 0. Let v be a real function which is continuous and convex on $\overline{\mathrm{co}}(B)$. There is a not necessarily unique linear function $\ell : \mathbf{R}^n \to \mathbf{R}$ such that $v(x) - \ell(x) \geq 0$ for all $x \in \overline{\mathrm{co}}(B)$. Let $w(x) := v(x) - \ell(x)$. Then w is a non-negative, continuous and convex function on $\overline{\mathrm{co}}(B)$, with $w(0) = 0$. The linearity of ℓ together with $\overline{x}_A = \overline{x}_B = 0$ indicate that $\ell_A = \ell_B$. Therefore

$$v_B - v_A = \ell_B + w_B - \ell_A - w_A = w_B - w_A.$$

Thus it will suffice to prove that $w_B \geq w_A$. The similarity of A and B with respect to 0 and $0 < m(A) < m(B)$ imply that there is a scalar $c > 1$ so that $B = cA$. Thus $m(B) = c^n m(A)$. Let $F(x) = cx : \mathbf{R}^n \to \mathbf{R}^n$. Hence $F(A) = B$ and the Jacobian of F is c^n. Applying the change of variables $y = F(x)$ and the fact that $w(cx) \geq w(x)$ for all $x \in A$, which can be deduced from Lemma 2.3 one obtains that

$$\begin{aligned} w_B = \frac{1}{m(B)} \int_B w(y)dy &= \frac{1}{c^n m(A)} \int_A w(cx)c^n dx = \frac{1}{m(A)} \int_A w(cx)dx \\ &\geq \frac{1}{m(A)} \int_A w(x)dx = w_A. \end{aligned}$$

□

The next two results can be derived from Theorem 3.2.

Corollary 3.3. *Let A and B be bounded measurable sets in $\mathbf{R}^n$ with $0 < m(A) < m(B)$. If A and B are symmetric and similar with respect to some $x_0 \in \mathbf{R}^n$, with a similarity factor $c > 1$, then $v_A \leq v_B$ for every real continuous convex function v on $\overline{\mathrm{co}}(B)$.*

Corollary 3.4. *If A and B are two concentric balls in $\mathbf{R}^n$ with $A \subset B$, then the relation $A \prec B$ holds.*

An n-dimensional extension of Theorem 2.1 will be presented next.

Theorem 3.5. *A continuous function $v : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if the inequality*

$$v(x_0) \leq \frac{1}{m(B)} \int_B v(x)dx = v_B \tag{7}$$

holds for every $x_0 \in \mathbf{R}^n$ and for all B which are convex and compact sets of positive measure, and which are symmetric with respect to x_0.

Proof. Assume first that v is a convex function on $\mathbf{R}^n$. Let $x_0 \in \mathbf{R}^n$ and let $B \subset \mathbf{R}^n$ be a convex and compact set of positive measure which is symmetric with respect to x_0. Notice that x_0 is an interior point of B. Consider the set

$$A_\epsilon := \{y = x_0 + \epsilon(x - x_0) : x \in B\}$$

where $\epsilon > 0$ is sufficiently small so that $A_\epsilon \subset B$. Now, Theorem 3.2 implies that $A_\epsilon \prec B$, or equivalently that $v_{A_\epsilon} \leq v_B$ for every sufficiently small positive number ϵ. Since

$$v(x_0) = \lim_{\epsilon \to 0^+} \frac{1}{m(A_\epsilon)} \int_{A_\epsilon} v(x)dx = \lim_{\epsilon \to 0^+} v_{A_\epsilon}$$

it follows that v satisfies (7). To prove the converse, consider distinct points x^1 and x^2 in $\mathbf{R}^n$ and let $x_0 = (x^1 + x^2)/2$. Fix an inner product in $\mathbf{R}^n$ and let S_δ be the ball of radius δ, centered at 0, lying in the $(n-1)-$dimensional hyperplane orthogonal to the vector $x^2 - x^1$. Let $B_\delta := [x^1, x^2] + S_\delta$. The assumptions indicate that (7) holds with $B = B_\delta$. One can conclude by Fubini theorem using the continuity of v that

$$\lim_{\delta \to 0+} v_{B_\delta} = \int_0^1 v(x^1 + t(x^2 - x^1))dt,$$

i.e.

$$v(x_0) \leq \int_0^1 v(x^1 + t(x^2 - x^1))dt. \tag{8}$$

Clearly (8) remains valid if x^1 is replaced by $x^3 = x_0 - h(x^2 - x^1)$ and x^2 is replaced by $x^4 = x_0 + h(x^2 - x^1)$ where h is an arbitrary positive number. One can deduce from Corollary 2.2 that v is convex on the line which is going through the points x^1 and x^2. Furthermore, since these points were chosen arbitrarily, the proof is complete. □

Theorem 3.2 admits the following converse.

Theorem 3.6. *If v is a real continuous function on $\mathbf{R}^n$ with the property that the inequality $v_A \leq v_B$ holds for every pairs of compact sets A and B which are similar and symmetric with respect to some $x_0 \in \mathbf{R}^n$ with a similarity factor $c > 1$, then v is convex.*

Proof. It was shown in the proof of the previous theorem that if the function v has the indicated properties then v satisfies (7) for each $x_0 \in \mathbf{R}^n$ and for every compact convex set B of positive Lebesgue measure, which is symmetric with respect to x_0. Then the previous result indicates that v is convex on $\mathbf{R}^n$. □

The next few results describe some additional properties of the relation $\prec$.

Lemma 3.7. *Let A and B be bounded subsets of $\mathbf{R}^n$ with $A \subset B$ and $m(A) > 0, m(B \setminus A) > 0$. Then*
(a) the relation $A \prec B$ holds if and only if $A \prec B \setminus A$;
(b) the relation $A \prec B$ holds if and only if $B \prec B \setminus A$.

Proof. It can be deduced from (1) and the additivity of the Lebesgue measure that

$$m(B)v_B = \int_B v dx = \int_A v dx + \int_{B\setminus A} v dx = m(A)v_A + m(B \setminus A)v_{B\setminus A}.$$

Dividing the previous identity by $m(B)$, one obtains that

$$v_B = \frac{m(A)}{m(B)} v_A + \frac{m(B \setminus A)}{m(B)} v_{B\setminus A}.$$

Because $m(B \setminus A) = m(B) - m(A)$, it follows that

$$v_B - v_A = \frac{m(B \setminus A)}{m(B)} (v_{B\setminus A} - v_A). \tag{9}$$

Therefore $A \prec B$, if and only if $A \prec B \setminus A$, which proves part (a).

Observe that

$$v_{B\setminus A} - v_B = v_{B\setminus A} - \frac{m(A)}{m(B)} v_A - \frac{m(B \setminus A)}{m(B)} v_{B\setminus A} = \frac{m(A)}{m(B)} (v_{B\setminus A} - v_A).$$

Now, it can be concluded with the aid of (9), that

$$v_{B\setminus A} - v_B = \frac{m(A)}{m(B \setminus A)} (v_B - v_A),$$

which proves part (b) of the assertion. □

Let $0 < a < b$. Theorem 3.2 indicates that $[-a, a] \prec [-b, b]$. Then part (b) of Lemma 3.7 yields the following.

Corollary 3.8. *If $0 < a < b$ then $[-b, b] \prec [-b, -a] \cup [a, b]$.*

The following result can be obtained with the aid of Theorem 3.2 and the fundamental theorem of calculus.

Corollary 3.9. *If $f : (c, d) \to \mathbf{R}$ and if f' is convex on (c, d) then for all fixed $x \in (c, d)$*

$$\frac{f(x + h) - f(x - h)}{2h}$$

is nondecreasing as a function of h for $0 < h < \min(d - x, x - c)$.

Corollary 3.8 indicates that Theorem 3.2 is not valid for a general pair of sets $A \subset B$ which are only symmetric but not similar with respect to their common barycenter. If, in addition, A and B are both convex then the conclusions of Theorem 3.2 are true in $\mathbf{R}$, but as the next example will show they are not remain true already in $\mathbf{R}^2$.

Example 3.10. Let $A = [-1,1] \times [-1,1]$ and let B be the parallelogram bounded by the lines $x = -1, x = 1, y = 2 + x$ and $y = -2 + x$. Then there is a convex function $v(x,y)$ with $v_A > v_B$ even though $A \subset B$ and A and B are convex and compact sets which are symmetric with respect to their common barycenter $\overline{x}_A = \overline{x}_B = 0$.

Proof. The square A is a proper subset of B. The area of A is 4, while the area of B is 8. Let $P_1 = (-1,1)$ and $P_2 = (1,3)$ be two of the vertices of the parallelogram B. Select the point $Q_1 = (-1/2, 3/2)$, and notice that Q_1 is lying on the line segment $\overline{P_1P_2}$. Let $Q_2 = (-1,0)$ and $P = (-2/3, 1)$ and notice that P is the point of intersection of the line segment $\overline{Q_1Q_2}$ and the line $y = 1$. The area of the triangle $\triangle(Q_2P_1P)$ is $1/6$, while the area of $\triangle(Q_2P_1Q_1)$ is $1/4$.

Let v be a convex function such that: $v(P_1) = v(-1,1) = 1, v(Q_1) = v(Q_2) = 0, v$ is affine on the triangle $\triangle(Q_2P_1Q_1)$ and v is 0 on the remainder of the parallelogram B. The volume of the tetrahedron Q_2P_1PS, where $S = (-1,1,1) \in \mathbf{R}^3$, is $\iint_A v dxdy$ and it is one third of the area of $\triangle(Q_2P_1P)$. Thus, it is $1/18$. Similarly, $\iint_B v dx$ is the volume of the tetrahedron $Q_2P_1Q_1S$, which is $1/12$. Therefore

$$v_A = \frac{\iint_A v dxdy}{4} = \frac{1}{72}, \quad \text{while} \quad v_B = \frac{\iint_B v dxdx}{8} = \frac{1}{96}.$$

$\square$

Lemma 3.11. *Let A and B be bounded subsets of $\mathbf{R}^n$ of positive Lebesgue measure. If both of the relations $A \prec B$ and $B \prec A$ hold then $m(A \triangle B) = 0$.*

Proof. Choose $s > 0$ so that the open ball $B_s(0) = \{x \in \mathbf{R}^n : |x| < s\}$ contains $A \cup B$. It is well-known that any $\mathcal{C}^2$ smooth function ϕ which is defined on a neighborhood of $\overline{B_s(0)}$ can be written on $\overline{B_s(0)}$ as $\phi = u - v$, where u and v are convex on $\overline{B_s(0)}$. Thus, if $A \prec B$ and $B \prec A$, then $u_A = u_B$ for every function u which is convex on $\overline{B_s(0)}$. Therefore, $\phi_A = \phi_B$ for every $\mathcal{C}^2$ smooth function on $\mathbf{R}^n$. Now, if $m(A \setminus B)$ were positive, (the case $m(B \setminus A) > 0$ can be treated similarly) there would exist an $x_0 \in A \setminus B$ so that x_0 is a point of density for A. It is easy to see that for each $r > 0$ there exists a $\mathcal{C}^2$ smooth $\phi_r : \mathbf{R}^n \to \mathbf{R}$ with the following properties:

(i) $\operatorname{supp} \phi_r \subset \overline{B_r(x_0)}$,

(ii) $\phi_r(x_0) = 1$ and $0 \leq \phi_r(x) \leq 1$ for all $x \in \mathbf{R}^n$,

(iii) $\phi_r(x) \geq 1/2$ for all $x \in \mathbf{R}^n$ such that $|x - x_0| \leq r/2$.

Then, it follows that

$$(\phi_r)_B \leq \frac{m(B_r(x_0) \cap B)}{m(B_r(x_0))} \frac{m(B_r(x_0))}{m(B)} \quad \text{while} \quad (\phi_r)_A \geq \frac{t_{r/2} m(B_{r/2}(x_0))}{2m(A)},$$

where for $r > 0$

$$t_r = \frac{m(A \cap \overline{B_r(x_0)})}{m(\overline{B_r(x_0)})}.$$

Since $t_r \to 1$ and $m(B_r(x_0) \cap B)/m(B_r(x_0)) \to 0$ as $r \to 0$ one obtains that

$$\frac{(\phi_r)_B}{(\phi_r)_A} \to 0 \quad \text{as} \quad r \to 0,$$

which is a contradiction. □

The affine invariance of the integral mean and the preservation of the relation $A \prec B$ under a non-singular affine mapping will be shown next.

Theorem 3.12. *Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a nonsingular affine map. Let A and B be bounded sets in $\mathbf{R}^n$ of positive Lebesgue measure. Then*

(i) If v is a continuous function on $T(A)$ then $v_{T(A)} = (v \circ T)_A$,

(ii) If $A \prec B$ then $T(A) \prec T(B)$.

Proof. Since T is a nonsingular affine map its Jacobian, denoted by $\mathrm{Jac}(T)$, satisfies the relation

$$\frac{|\mathrm{Jac}(T)|}{m(T(A))} = \frac{1}{m(A)}. \tag{10}$$

With the aid of (10) and the definition of the barycenter one can conclude that

$$\begin{aligned} v_{T(A)} &= \frac{1}{m(T(A))} \int_{T(A)} v(y)dy = \frac{1}{m(T(A))} \int_A v(T(x))dT(x) \\ &= \frac{1}{m(T(A)} \int_A (v \circ T)(x)|\mathrm{Jac}(T)|dx = \frac{1}{m(A)} \int_A (v \circ T)(x)dx = (v \circ T)_A. \end{aligned}$$

To prove (ii) consider a convex function v on $\mathbf{R}^n$ and assume that $A \prec B$. One can deduce by (i) that $v_{T(A)} = (v \circ T)_A \le (v \circ T)_B = v_{T(B)}$. □

4. The one-dimensional case

In this section a complete characterization of the relation $A \prec B$ will be given in $\mathbf{R}$ for compact sets.

Let A be a compact subset of $\mathbf{R}$ of positive Lebesgue measure, and let $x_0 = \overline{x}_A$ and let $[a, b]$ be the smallest closed interval containing A, i.e. $[a, b] = \mathrm{co}(A)$. Hence, $a < x_0 < b$. With the aid of the functions

$$\phi(x) = \int_{x_0}^x \chi_A(s)ds, \quad a \le x \le b \quad \text{and} \quad \Phi(x) = \int_{x_0}^x \phi(s)ds, \quad a \le x \le b.$$

the designator function of $A, \Psi_A(x)$ is defined as follows:

$$\Psi_A(x) = \begin{cases} \frac{1}{m(A)}[\Phi(x) - \Phi(b) - (x-b)\phi(b)], & \text{for } x_0 \le x \le b, \\ \frac{1}{m(A)}[\Phi(x) - \Phi(a) - (x-a)\phi(a)], & \text{for } a \le x < x_0. \end{cases}$$

Notice that ϕ is a non-decreasing and absolutely continuous function on $[a, b]$, in addition, Φ is non-negative and convex on $[a, b]$. Furthermore, $\Phi(x_0) = 0, \Phi'(x_0) = 0$, and $\Phi''(x)$ exists a. e. and equals to $\chi_A(x)$ a. e. in $[a, b]$. It is easy to see that Φ is the only convex function with these properties on $[a, b]$. Next a characterization of Ψ_A is given.

Theorem 4.1. *Let A be a compact subset of* $\mathbf{R}$ *of positive Lebesgue measure with barycenter at $\overline{x}_A = x_0$ and with $co(A) = [a,b]$. Then the designator function $\Psi_A : [a,b] \to \mathbf{R}$ has the following properties.*

(i) Ψ_A *is continuous on* $[a,b]$;

(ii) Ψ_A *is non-negative on* $[a,b]$ *with* $\Psi_A(a) = \Psi_A(b) = 0$;

(iii) $(\Psi_A)'_+(a) = (\Psi_A)'_-(b) = 0$;

(iv) Ψ_A *is convex on* $[a,x_0]$ *and on* $[x_0,b]$;

(v) Ψ'_A *exists and continuous for all* $x \in [a,b], x \neq x_0$ *and* Ψ'_A *has one-sided derivatives at* x_0 *with a jump discontinuity at* x_0.

Proof. To justify (i) one needs to show only the continuity of Ψ_A at x_0. Furthermore, since $\Phi(x_0) = 0$, only the equality

$$\lim_{x\to x_0+} \Psi_A(x) = \tfrac{1}{m(A)}[-\Phi(b) - (x_0 - b)\phi(b)] = \tag{11}$$
$$\Psi_A(x_0) = \lim_{x\to x_0-0} \Psi_A(x) = \tfrac{1}{m(A)}[-\Phi(a) - (x_0 - a)\phi(a)].$$

has to be verified. It will be assumed that $x_0 = 0$, since this situation can be always achieved by translating A to the left by x_0. The definition of ϕ indicates that $\phi(b) = m(A^+)$ and $\phi(a) = -m(A^-)$ where $A^+ = A \cap [0,b]$ and $A^- = A \cap [a,0]$. It can be derived by Fubini theorem that

$$\Phi(b) = \int_0^b \int_0^x \chi_A(s)dsdx = \int_0^b (b-s)\chi_A(s)ds.$$

Thus

$$-\Phi(b) + b\phi(b) = \int_0^b s\chi_A(s)ds. \tag{12}$$

By means of a similar argument it can be shown that

$$\Phi(a) = \int_0^a \int_0^x \chi_A(s)dsdx = \int_a^0 s\chi_A(s)ds + a\phi(a).$$

Therefore,

$$-\Phi(a) + a\phi(a) = -\int_a^0 s\chi_A(s)ds. \tag{13}$$

Since the barycenter of A is at 0, we have that $\int_a^b s\chi_A(s)ds = 0$, or equivalently that $\int_0^b s\chi_A(s)ds = -\int_a^0 s\chi_A(s)ds$. Now (11) follows from (12) and (13).

Notice that (11) and (12) show that $\Psi_A(x_0) = \Psi_A(0) > 0$. Clearly $\Psi_A(a) = \Psi_A(b) = 0$ and $(\Psi_A)'$ is non-negative on $[a,x_0]$ and non-positive on $[x_0,b]$. Thus the proof of (ii) is complete. The verification of each of (iii), (iv) and (v) is straightforward. □

The following result will be useful in the sequel.

Lemma 4.2. *Let A be a compact subset of* $\mathbf{R}$ *of positive Lebesgue measure with barycenter at $x_0 = \overline{x}_A$. Let $co(A) = [a,b]$ and let $\Psi_A(x)$ be the designator function of A. Let w be a real $\mathcal{C}^2$ function on $[a,b]$ such that $w(x_0) = 0$. Then*

$$\int_a^b w''(x)\Psi_A(x)dx = \frac{1}{m(A)}\int_A w(x)dx = w_A.$$

Proof. Since Ψ'_A has a jump discontinuity at x_0, we apply the method of integration by parts, separately twice on $[a, x_0]$ and twice on $[x_0, b]$. Note that the method and the formula of integration by parts are valid also when both of the factors are absolutely continuous (Theorem 6.3.a. of [1]). Thus, we obtain

$$\begin{aligned}\textstyle\int_a^{x_0} w''\Psi_A(x)dx = \Psi_A(x_0)w'(x_0) - \Psi_A(a)w'(a) - \\ (\Psi_A)'_-(x_0)w(x_0) + (\Psi_A)'_+(a)w(a) + \textstyle\int_a^{x_0} w\Psi''_A dx,\end{aligned} \tag{14}$$

and

$$\begin{aligned}\textstyle\int_{x_0}^{b} w''\Psi_A(x)dx = \Psi_A(b)w'(b) - \Psi_A(x_0)w'(x_0) - \\ (\Psi_A)'_-(b)w(b) + (\Psi_A)'_+(x_0)w(x_0) + \textstyle\int_{x_0}^{b} w\Psi''_A dx.\end{aligned} \tag{15}$$

It can be derived by adding (14) and (15) and by (ii) and (iii) of Theorem 4.1 that

$$\int_a^b w''\Psi_A dx = w(x_0)((\Psi_A)'_+(x_0) - (\Psi_A)'_-(x_0)) + \int_a^b w\Psi''_A dx.$$

Since

$$\Psi''_A = \frac{1}{m(A)}\chi_A(x) \quad \text{for a. e.} \quad x \in [a, b],$$

one can conclude that

$$\int_a^b w''\Psi_A dx = w(x_0)((\Psi_A)'_+(x_0) - (\Psi_A)'_-(x_0)) + w_A. \tag{16}$$

Because $w(x_0) = 0$, the proof is complete. □

Corollary 4.3. *Let A be a compact subset of* $\mathbf{R}$ *of positive Lebesgue measure with barycenter at $x_0 = \overline{x}_A$ and with $co(A) = [a, b]$. Let Ψ_A be the designator function of A, then*

$$(\Psi_A)'(x_0 + 0) - (\Psi_A)'(x_0 - 0) = -1,$$

or in the distributional sense $\Psi''_A = -\delta_{x_0} + \chi_A/m(A)$.

Proof. The substitution of $w(x) \equiv 1$ into (16) yields

$$0 = (\Psi_A)'(x_0 + 0) - (\Psi_A)'(x_0 - 0) + \frac{1}{m(A)}\int_A 1dx,$$

from which the conclusion follows. □

Remark 4.4. Let A be a compact subset of $\mathbf{R}$ such that for any open interval I we have that $m(I \cap A) > 0$, whenever $I \cap A \neq \emptyset$. The designator function Ψ_A determines the set A uniquely since

$$A = \overline{\text{ess supp}\, \Psi''_A \setminus \{\overline{x}_A\}},$$

and $\overline{x}_A$ is the unique maximum point of Ψ_A.

We are now able to show the main result of this section.

Theorem 4.5. *Let A and B be compact subsets of* $\mathbf{R}$ *of positive Lebesgue measure with the same barycenter. In addition, A is such that $m(I \cap A) > 0$, for any open interval I for which $I \cap A \neq \emptyset$. Then $A \prec B$, if and only if $\Psi_A(x) \leq \Psi_B(x)$ for all $x \in co(A)$.*

Proof. Let x_0 be the common barycenter of A and B and let $[c,d] = co(B)$. Assume first that $\Psi_A(x) \leq \Psi_B(x)$ for all $x \in co(A) := [a,b]$. This condition implies that $[a,b] \subset [c,d]$. It needs to be shown that if v is a continuous convex function on $[c,d]$ then $v_A \leq v_B$. By means of a standard argument one can see that it is enough to show this inequality for $\mathcal{C}^2$ smooth convex functions. Assume, therefore that v is a real $\mathcal{C}^2$ convex function on $[c,d]$. Observe that v can be written as $v(x) = w(x) + \ell(x)$, where $\ell(x) = v(x_0) + v'(x_0)(x - x_0)$. Since x_0 is the common barycenter of A and B we have that $\ell_A = \ell_B$. Clearly, w is a $\mathcal{C}^2$ smooth convex function on $[c,d]$ with $w(x_0) = 0$ and $w'(x_0) = 0$. Therefore w'' is non-negative on $[c,d]$. Lemma 4.2 and the non-negativity of w'' on $[c,d]$ imply that

$$v_A = \ell_A + w_A = \ell_B + \int_a^b w''(x)\Psi_A(x)dx \leq \ell_B + \int_c^d w''(x)\Psi_B(x)dx = v_B.$$

To prove the converse, consider an arbitrary nonnegative continuous function $f : [c,d] \to \mathbf{R}$. It is well-known that there is a $\mathcal{C}^2$ smooth convex function $w : [c,d] \to \mathbf{R}$, which is a solution of the initial value problem $w''(x) = f(x)$ for $x \in [c,d]$ with $w(x_0) = 0 = w'(x_0)$. Then Lemma 4.2 indicates $w_A = \int_A f(x)\Psi_A(x)dx$ and $w_B = \int_B f(x)\Psi_B(x)dx$. Hence for every such an f the inequality

$$\int_a^b f(x)\Psi_A(x)dx \leq \int_c^d f(x)\Psi_B(x)dx. \tag{17}$$

holds. It can be deduced from Theorem 3.1 that $[a,b] \subset [c,d]$. Recall that Ψ_A is continuous on $[a,b]$. Since for every $\epsilon > 0$ and for every $x_1 \in [a,b]$ there exists a non-negative and continuous function $f_\epsilon : \mathbf{R} \to \mathbf{R}$ with $f_\epsilon(x_1) > 0$ and $\operatorname{supp} f_\epsilon \subset (x_1 - \epsilon, x_1 + \epsilon)$, the assumption $\Psi_A(x_1) > \Psi_B(x_1)$ for some $x_1 \in [a,b]$ contradicts (17). □

5. Designator function in $\mathbf{R}^n$

Let A be a compact subset of $\mathbf{R}^n$ of positive Lebesgue measure whose barycenter is at $\overline{x}_A = 0$. Consider $g \in (\mathbf{R}^n)^\star$ with $g \neq 0$. In this section we are assuming that an $(n-1)$-dimensional Lebesgue measure has been selected on $\operatorname{Ker}(g)$ (there will not be relation of those chosen for different $g's$) and the $(n-1)$-dimensional Lebesgue measure on the affine hyperplane $\{x \in \mathbf{R}^n : g(x) = t\}$, denoted by $\{g = t\}$ will be obtained by a parallel translation of the Lebesgue measure from $\operatorname{Ker}(g)$. We shall denote by $\alpha^g(t)$ the $(n-1)$-dimensional volume of the section $A \cap \{g = t\}$. (Of course, this depends on the choice of the $(n-1)$-dimensional Lebesgue measure on $\operatorname{Ker}(g)$.) Clearly, $\alpha^g(t)$ is a non-negative L^∞ function. Let $[a,b] = [a^g, b^g]$ be the smallest closed interval containing the essential support of α^g. Let

$$|\alpha^g| = \int_{-\infty}^{\infty} \alpha^g(t)dt,$$

and let $\Phi_\alpha : \mathbf{R} \to \mathbf{R}$ be the unique non-negative $\mathcal{C}^1$ convex function such that $\Phi'_\alpha(t)$ is absolutely continuous on $\mathbf{R}$ and such that $\Phi''_\alpha(t) = \alpha^g(t)$ a.e. and $\Phi_\alpha(0) = \Phi'_\alpha(0) = 0$.

Consider

$$\Psi_\alpha(t) = \begin{cases} \frac{1}{|\alpha^g|}[\Phi_\alpha(t) - \Phi_\alpha(b^g) - (t - b^g)(\Phi_\alpha)'(b^g)] & \text{for } 0 \le t \le b^g, \\ \frac{1}{|\alpha^g|}[\Phi_\alpha(t) - \Phi_\alpha(a^g) - (t - a^g)(\Phi_\alpha)'(a^g)] & \text{for } a^g \le t < 0 \\ 0, & \text{for } t \notin [a^g, b^g]. \end{cases} \tag{18}$$

The notation $\Psi_A(g, t) = \Psi_\alpha(t)$ will be also used, to display the dependence from A and g. A formula for $\Psi_A(g, t)$ will be derived which is independent from the choice of the Lebesgue measure on $\mathrm{Ker}(g)$. Some characterizations of Ψ_α will be given first.

Theorem 5.1. *Let A be a compact and essential subset of $\mathbf{R}^n$ (i. e. any proper compact subset of A has smaller Lebesgue measure) with barycenter at $\overline{x}_A = 0$. Fix $g \neq 0$, $g \in (\mathbf{R}^n)^\star$. Then the map $\Psi_\alpha : \mathbf{R} \to \mathbf{R}$ has the following properties:*
(i) Ψ_α is continuous and non-negative with $\Psi_\alpha(a^g) = \Psi_\alpha(b^g) = 0$ and

$$\Psi_\alpha(0) = \frac{1}{|\alpha^g|} \int_0^{b^g} t\alpha^g(t)dt = -\frac{1}{|\alpha^g|} \int_{a^g}^0 t\alpha^g(t)dt.$$

(ii) supp $\Psi_\alpha = [a^g, b^g]$;
(iii) $\Psi'_\alpha(a^g) = \Psi'_\alpha(b^g) = 0$;
(iv) $\Psi_\alpha(t)$ is convex on $[a^g, 0]$ and on $[0, b^g]$;
(v) $\Psi'_\alpha(t)$ exists and continuous for all $t \neq 0$ and it has one-sided derivatives at $t = 0$, satisfying

$$\Psi'_\alpha(0+) - \Psi'_\alpha(0-) = -1;$$

(vi) For every real $\mathcal{C}^2$ smooth function w on $[a^g, b^g]$ the identity

$$\int_{a^g}^{b^g} w''(t)\Psi_\alpha(t)dt = w(0)[\Psi'_\alpha(0+) - \Psi'_\alpha(0-)] + \frac{1}{|\alpha^g|} \int_{a^g}^{b^g} w(t)\alpha^g(t)dt$$

holds.

Proof. All of the statements with the exception of (ii) can be derived basically the same way as they were done in the proofs of Theorem 4.1 and Lemma 4.2, and the definition of Ψ_α implies that $\mathrm{supp}\Psi_\alpha \subset [a^g, b^g]$. We will show that $\Psi_\alpha(t)$ is positive on the interval (a^g, b^g), which yields the proof of (ii). Since the convex hull of the essential support of $\alpha^g(t)$ equals to $[a^g, b^g]$, there are sequences $\{t_n\}$ and $\{s_n\}$ with $t_n \nearrow b^g$, $s_n \searrow a^g$ and such that s_n and t_n are density points of $\mathrm{supp}\, \alpha^g$. Therefore in the neighborhood of any of these points $\Phi_\alpha(t)$ cannot be equal to an affine function. Since $\Psi_\alpha(0) > 0, \Psi_\alpha(b^g) = 0$, and Ψ_α is nonincreasing on $[0, b^g)$ it follows that $\Psi_\alpha(t)$ is positive on $[0, b^g)$. One can show the statement with a similar argument when $t \in (a^g, 0]$. □

Next, a new formula will be derived for $\Psi_A(g, t)$.

Lemma 5.2. *Let A be a compact subset of $\mathbf{R}^n$ of positive Lebesgue measure with barycenter at zero. If $g \in (\mathbf{R}^n)^\star \setminus \{0\}$ then*

$$\Psi_A(g, t) = \begin{cases} \frac{1}{m(A)} \int_{A \cap \{g \ge t\}} (g - t)dx, & \text{if } t \ge 0, \\ \frac{1}{m(A)} \int_{A \cap \{g \le t\}} (t - g)dx, & \text{if } t < 0. \end{cases} \tag{19}$$

Proof. Fix $g \in (\mathbf{R}^n)^\star, g \neq 0$. Choose Lebesgue measures ℓ_n on $E = \mathbf{R}^n$, ℓ_{n-1} on $\mathrm{Ker}(g)$ and the standard one on $\mathbf{R}$. Then one can see by Fubini theorem that there is a positive constant c so that for every integrable and compactly supported function f on E the relation

$$\int_E f(x)d\ell_n(x) = c\int_{-\infty}^{\infty} ds(\int_{\{g=s\}} f d\ell_{n-1}) \tag{20}$$

holds. Subtituting $f(x) = \chi_A(x)$ into (20) yields

$$\ell_n(A) = c\int_{-\infty}^{\infty} ds\ell_{n-1}(A\cap\{g=s\}) = c\int_{-\infty}^{\infty}\alpha^g(s)ds = c|\alpha^g| = m(A).$$

Next, substituting $f(x) = (g(x)-t)\chi_{A\cap\{g\geq t\}}$, where t is a fixed non-negative number, into (20) gives

$$\int_{A\cap\{g\geq t\}}(g(x)-t)dx = \int_E f(x)d\ell_n(x) =$$
$$c\int_{-\infty}^{\infty} ds\int_{A\cap\{g=s\}}(g-t)\chi_{\{g\geq t\}}d\ell_{n-1} = c\int_t^{b^g}(s-t)\alpha^g(s)ds.$$

Let $w(t) := c\int_t^{b^g}(s-t)\alpha^g(s)ds$. Clearly $w(t) = 0$ for $t \geq b^g$. Furthermore, $w'(t) = c\int_{b^g}^t \alpha^g(s)ds$ and $w''(t) = c\alpha^g(t)$ for a. e. t. Thus $w'(t)$ is an absolutely continuous function and

$$\frac{d^2}{dt^2}[\frac{1}{m(A)}w(t)] = \frac{1}{|\alpha^g|}\alpha^g(t) = (\Psi_A)''(t), \qquad \text{for a. e.} \quad t\in[0,b^g]$$

and

$$\frac{1}{m(A)}w(b^g) = 0 = \Psi_A(b^g) \quad \text{and} \quad \frac{w'(b^g)}{m(A)} = 0 = (\Psi_A)'(b^g).$$

Hence

$$\frac{1}{m(A)}\int_{A\cap\{g\geq t\}}(g-t)dx = \frac{1}{m(A)}w(t) = \Psi_A(g,t),\ t\geq 0.$$

Notice that the identity for $t \leq 0$ can be obtained similarly. Hence the proof is complete. □

Remark 5.3. Observe that the two parts of the formula (19) are consistent for $t=0$, i. e.

$$\frac{1}{m(A)}\int_{A\cap\{g\geq 0\}} gdx = \frac{1}{m(A)}\int_{A\cap\{g\leq 0\}}(-g)dx.$$

Indeed, $\int_A gdx = 0$, because g is a linear functional and the barycenter of A is at 0.

The following alternative formula can be derived for $\Psi_A(g,t)$ when $t < 0$.

$$\begin{aligned}\Psi_A(g,t) &= \frac{1}{m(A)}\int_A (t-g(x))dx - \frac{1}{m(A)}\int_{A\cap\{g>t\}}(t-g(x))dx \\ &= t + \frac{1}{m(A)}\int_{A\cap\{g\geq t\}}(g(x)-t)dx.\end{aligned}$$

Thus, (19) can be written as:

$$\Psi_A(g,t) = \begin{cases} \frac{1}{m(A)} \int_A \max(g(x)-t,0)dx, & \text{for } t \geq 0, \\ t + \frac{1}{m(A)} \int_A \max(g(x)-t,0)dx, & \text{for } t < 0. \end{cases}$$

Define $\Psi_A(0,t) = 0$. The proofs of parts (i) and (iii) of the following result are obvious.

Theorem 5.4. *Let A be a compact subset of $\mathbf{R}^n$ of positive Lebesgue measure with barycenter at zero. Then the mapping $\Psi_A(g,t) : (\mathbf{R}^n)^\star \times \mathbf{R} \to \mathbf{R}$ is continuous. Furthermore,*
(i) $\Psi_A(g,t) = \min(t,0) + \frac{1}{m(A)} \int_A \max(g(x)-t,0)dx$;
(ii) for $c \in \mathbf{R}$, $\Psi_A(cg,ct) = |c|\Psi_A(g,t)$;
(iii) Ψ_A is a convex function on the set $\{(g,t) : t \geq 0\}$ and on the set $\{(g,t) : t < 0\}$.

Proof. The proof of (ii) can be derived easily from (19) when $c \geq 0$. Clearly, it suffices to show only the case $c = -1$. Let $t \geq 0$. Then with the aid of (19) we have that

$$\Psi_A(-g,-t) = \frac{1}{m(A)} \int_{\{-g \leq -t\} \cap A} (-t + g(x))dx =$$
$$\frac{1}{m(A)} \int_{\{g \geq t\} \cap A} (g(x) - t)dx = \Psi_A(g,t).$$

Since the case $t < 0$ can be treated essentially in the same way, it follows that the proof is complete. □

The function $\tilde{\Psi}_A(g) := \Psi_A(g,1)$ will be called the reduced designator function. It follows that $\tilde{\Psi}_A : (\mathbf{R}^n)^\star \to \mathbf{R}$ is a nonnegative convex function on $(\mathbf{R}^n)^\star$, with linear growth at ∞. Furthermore

$$\{g : \tilde{\Psi}_A(g) = 0\} = \{g : g(A) \subset (-\infty, 1]\} = (co(A))^\circ$$

is the polar set of the convex hull of A. Thus $\tilde{\Psi}_A$ determines uniquely the compact set A, if A is essential and convex.

Theorem 5.5. *If A and B are compact subsets of $\mathbf{R}^n$ with common barycenter at zero and if $A \prec B$, then $\tilde{\Psi}_A \leq \tilde{\Psi}_B$ on $(\mathbf{R}^n)^\star$.*

Proof. If $g \in (\mathbf{R}^n)^\star$, then $\max(g(x)-1,0)$ is a convex function on $\mathbf{R}^n$, Therefore,

$$\tilde{\Psi}_A(g) = \frac{1}{m(A)} \int_A \max(g(x)-1,0)dx =$$
$$[\max(g(x)-1,0)]_A \leq [\max(g(x)-1,0)]_B = \tilde{\Psi}_B(g).$$

□

Assume that A and B are bounded measurable sets in $\mathbf{R}^n$ with the same barycenter. It is an open question whether the inequality $\tilde{\Psi}_A \leq \tilde{\Psi}_B$ on $(\mathbf{R}^n)^\star$ is sufficient for the relation $A \prec B$ to be hold when $n \geq 2$.

The next result shows the linear invariance of $\tilde{\Psi}_A$.

Theorem 5.6. *Let A be a compact subset of $\mathbf{R}^n$ with barycenter at zero. If $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear and invertible then*

$$\tilde{\Psi}_{T(A)}(g) = \tilde{\Psi}_A(T^\star(g)),$$

where $T^\star$ denotes the adjoint of T.

Proof. The barycenter of $T(A)$ is also at zero. One obtains from (i) of Theorem 5.4 when $t = 1$ that

$$\tilde{\Psi}_{T(A)}(g) = [\max(g(x) - 1, 0)]_{T(A)}. \tag{21}$$

One can deduce from Theorem 3.12 that the right side of (21) can be written as

$$[\max(g(x) - 1, 0) \circ T]_A = [\max(T^\star(g) - 1, 0]_A = \tilde{\Psi}_A(T^\star(g)).$$

□

6. Two Constructions and Related Results

Two methods will be described in this section, which generate sets A and B satisfying the relation $A \prec B$ in $\mathbf{R}^n$ when $n \geq 2$. The first method will be prepared by an inequality, which can be derived from the classical Chebysev's inequality as it is given in [10]. Here, a direct proof will be given.

Lemma 6.1. *Let b be a fixed positive number and let f be a nonnegative even function defined on $[-b, b]$ which is nondecreasing on $[0, b]$. Then for any real function w which is convex on $[-b, b]$ the inequality*

$$\frac{1}{2b}\int_{-b}^{b} w(x)dx \int_{-b}^{b} f(x)dx \leq \int_{-b}^{b} w(x)f(x)dx \tag{22}$$

holds.

Proof. If (22) holds for a function f with every function w which is convex on $[-b, b]$, then we will say that (22) holds for f. It is easy to see that if (22) holds for a finite set of functions $f_1, f_2, ..., f_m$ and if $c_1, ..., c_m$ is a sequence of non-negative numbers, then it holds also for any f of the form

$$f(x) = \sum_{j=1}^{m} c_j f_j(x).$$

Let $0 < a < b$. Corollary 3.8 states that the relation $[-b, b] \prec [-b, -a] \cup [a, b]$ holds. Equivalently, it states that for any real function w which is convex on $[-b, b]$, the inequality

$$\frac{1}{2b}\int_{-b}^{b} w(x)dx \leq \frac{1}{2(b-a)}\int_{[-b,-a]\cup[a,b]} w(x)dx \tag{23}$$

holds. Let $f_a(x) := \chi_{[-b,-a]\cup[a,b]}$ with $0 < a < b$. Since (23) can be rewritten as

$$\frac{1}{2b}\int_{-b}^{b} w(x)dx \int_{-b}^{b} f_a(x)dx \leq \int_{-b}^{b} w(x)f_a(x)dx,$$

it follows that (22) holds for f_a when $0 < a < b$. Thus (22) holds for any f of the form

$$\begin{aligned} f(x) = \textstyle\sum_{j=1}^{m} c_j f_{a_j}, \quad &\text{where} \quad 0 < a_1 < ... < a_m < b \quad \text{and} \\ c_j > 0 \quad \text{for} \quad j = 1, ..., m \quad &\text{with} \quad \textstyle\sum_{j=1}^{m} c_j = 1. \end{aligned} \tag{24}$$

Since any function f which satisfies the assumptions of this lemma is the uniform limit of functions of the form (24), the proof of (22) is complete. □

Theorem 6.2. *Let $K \subset \mathbf{R}^n$ be a compact set of positive Lebesgue measure with barycenter at 0. Let b be a fixed positive number. Let $f : [-b, b] \to \mathbf{R}$ be a continuous and even function on $[-b, b]$ with $f(0) \geq 1$ and such that f is nondecreasing on $[0, b]$. Let*

$$A = [-b, b] \times K \subset \mathbf{R} \times \mathbf{R}^n \quad \textit{and} \quad B = \bigcup_{|x| \leq b} \{x\} \times (f(x)K),$$

where for a scalar t the set $tK = \{ty : y \in K\}$. Then for every real function v which is convex on a convex neighborhood of B, the inequality $v_A \leq v_B$ hold.

Proof. Let $v(x, y) = v(x, y_1, y_2, ..., y_n)$ be a convex function on a convex neighborhood of B. Let

$$w(x) := \int_K v(x, y)dy, \quad x \in [-b, b].$$

The convexity of v indicates that w is a convex function on $[-b, b]$. Note that the sets K and $f(x)K$ are similar with respect to their common barycenter $\overline{x}_K = 0$ in $\mathbf{R}^n$. Therefore, one can deduce from Theorem 3.2 that for every fixed $x \in [-b, b]$ the inequality

$$\frac{1}{m(f(x)K)} \int_{f(x)K} v(x, y)dy \geq \frac{1}{m(K)} \int_K v(x, y)dy = \frac{w(x)}{m(K)} \tag{25}$$

holds. Since $m(f(x)K) = (f(x))^n m(K)$, inequality (25) can be rewritten as

$$\int_{f(x)K} v(x, y)dy \geq (f(x))^n w(x).$$

Now with the aid of the previous inequality and Lemma 6.1 one obtains that

$$\begin{aligned} \int_B v(x, y)dxdy &= \int_{-b}^{b} dx \int_{f(x)K} v(x, y)dy \geq \int_{-b}^{b} (f(x))^n w(x)dx \\ &\geq \frac{1}{2b} \int_{-b}^{b} (f(x))^n dx \int_{-b}^{b} w(x)dx. \end{aligned} \tag{26}$$

An application of Fubini Theorem yields that

$$m(B) = m(K) \int_{-b}^{b} (f(x))^n dx$$

It can be derived from (26) and from $m(A) = 2bm(K)$ that

$$\begin{aligned} v_B &= \frac{1}{m(K) \int_{-b}^{b} (f(x))^n dx} \int_B v(x, y)dxdy \\ &\geq \frac{1}{m(K) \int_{-b}^{b} (f(x))^n dx} \frac{1}{2b} \int_{-b}^{b} (f(x))^n dx \int_{-b}^{b} w(x)dx \\ &= \frac{1}{2bm(K)} \int_{-b}^{b} dx \int_K v(x, y)dy = \frac{1}{m(A)} \int_A v dxdy = v_A. \end{aligned}$$

Hence the proof is complete. □

Applying repeatedly Theorem 6.2 with a constant function $f \geq 1$, where f can be different at each step and using the transitivity of the relation $A \prec B$ we obtain the proof of the following result.

Corollary 6.3. *If $0 < a_j < b_j$ for $j = 1, ..., n$ and $A = [-a_1, a_1] \times ... \times [-a_n, a_n]$ and $B = [-b_1, b_1] \times ... \times [-b_n, b_n]$, then $A \prec B$.*

More generally one can show the following.

Theorem 6.4. *Let n and m be positive integers and let A_1 and B_1 be bounded subsets of $\mathbf{R}^n$ of positive n-demensional Lebesgue measure and let A_2 and B_2 be bounded subsets of $\mathbf{R}^m$ of positive m-dimensional Lebesgue measure. If $A_1 \prec B_1$ and $A_2 \prec B_2$ then $A_1 \times A_2 \prec B_1 \times B_2$.*

A different construction will be discussed next.

Let n be a positive integer with $n \geq 2$. Denote $\mathbf{R}^n$ by E, and consider a proper subspace X of E of dimension k, where $k \geq 1$, and a real number $r \in (0, 1)$. The natural projection of E onto E/X will be denoted by π. Let B be a bounded set of positive Lebesgue measure in $\mathbf{R}^n$. Consider any $Y \in E/X$ such that the intersection $Y \cap B$ has positive k-dimensional Lebesgue measure. Denote by $\pi^\star(B) \subset E/X$ the set of all such cosets of X. Then the barycenter $b(Y) := \overline{x}_{Y\cap B}$ of $Y \cap B$, relative to Y is well defined on such sets, moreover $b(Y) \in Y$. Clearly $\pi^\star(B)$ is a measurable subset of E/X. Note that $\pi^\star$ is the restriction of π to $\pi^{-1}(\pi^\star(B))$. Now, define

$$\begin{aligned} A = C_{X,r}(B) &= \bigcup_{Y\in\pi^\star(B)} \{b(Y) + r((Y \cap B) - b(Y))\} \\ &= \bigcup_{Y\in\pi^\star(B)} \{b(Y) + r(y - b(Y)) : y \in Y \cap B\}. \end{aligned}$$

Observe that $A \cap Y$ and $B \cap Y$ are similar with respect to $b(Y)$ considered as subsets of Y with a similarity factor r. Therefore $m(Y \cap B) = r^k m(Y \cap A)$ when $Y \in \pi^\star(B)$.

Define

$$h_Y(y) := b(Y) + r(y - b(Y)),\ Y \in \pi^\star(B).$$

Then for fixed $Y \in \pi^\star(B)$ the function $h_Y : Y \to Y$ is continuous with a continuous inverse, moreover,

$$h_Y(Y \cap B) = Y \cap A. \tag{27}$$

Furthermore, the function $x \to b(\pi(x)) + r(x - b(\pi(x)))$ is invertible for $x \in \pi^{-1}(\pi^\star(B))$.

Theorem 6.5. *If B is a bounded set of positive Lebesgue measure in $\mathbf{R}^n = E$, then $A = C_{x,r}(B)$ is a bounded measurable set. Furthermore*
(a) $m(A) = \ell_n(A) = m(C_{X,r}(B)) = r^{\dim X}\ell_n(B) = r^{\dim X}m(B)$;
(b) $A \prec B$.

Proof. It is clear that A is a bounded set in $\mathbf{R}^n$. Next, the measurability of A will be shown. One can conclude by Fubini theorem that $b : \pi^\star(B) \to E$ is measurable. By Lusin theorem there is a sequence of compact subsets $F_j \subset \pi^\star(B)$ such that the restriction of b to F_j is continuous and $\ell_{n-k}(\pi^\star(B)\backslash F_\infty) = 0$, where $n = \dim E$, $k = \dim X$

and $F_\infty = \cup_{j=1}^\infty F_j$. As a simple consequence of (27) one obtains that $A \cap \pi^{-1}(F_j) = h_Y(B \cap \pi^{-1}(F_j))$ since $F_j \subset \pi^\star(B)$ for $j = 1, 2, \dots$. The continuity of b on F_j implies that $x \to b(\pi(x)) + r(x - b(\pi(x))$ continuous with a continuous inverse for $x \in \pi^{-1}(F_j)$. Therefore $A \cap \pi^{-1}(F_j)$ is measurable because $B \cap \pi^{-1}(F_j)$ is also measurable. It can be seen readily that A can be written as

$$A = \cup_{j=1}^\infty A \cap \pi^{-1}(F_j) \cup A_\infty$$

where $A_\infty \subset \pi^{-1}(\pi^\star(B) \setminus F_\infty)$. The set $\pi^{-1}(\pi^\star(B) \setminus F_\infty)$ is a measurable set of measure 0 by Fubini theorem. Thus A is measurable.

To establish part (a) notice that one can deduce from Fubini theorem that

$$m(A) = \ell_n(A) = c \int_{\pi^\star(A)} d\ell_{n-k}(Y)(\int_{Y \cap A} 1 d\ell_k), \tag{28}$$

where c is the constant correlating the independent choices of Lebesgue measures ℓ_n, ℓ_{n-k}, ℓ_k on E, E/X and X. Notice that $\pi^\star(A) = \pi^\star(B)$. Now, (28) can be written as

$$\begin{aligned} & c \int_{\pi^\star(B)} d\ell_{n-k}(Y)(\int_{B \cap Y} 1 d(b_k(Y) + r(y - b_k(Y)) \\ = \; & c \int_{\pi^\star(B)} d\ell_{n-k}(Y) \int_{B \cap Y} r^k d\ell_k = r^k \ell_n(B) \end{aligned}$$

To prove part (b) consider an arbitrary convex function $v : E \to \mathbf{R}$. It can be deduced from Theorem 3.2 that for every $Y \in \pi^\star(B)$ the relation $A \cap Y \prec B \cap Y$ holds, i. e. for every $Y \in \pi^\star(B)$ the inequality

$$\frac{1}{\ell_k(A \cap Y)} \int_{A \cap Y} v d\ell_k \le \frac{1}{\ell_k(B \cap Y)} \int_{B \cap Y} v d\ell_k \tag{29}$$

holds. Then one can conclude by Fubini theorem that

$$\begin{aligned} \frac{1}{\ell_n(A)} \int_A v d\ell_n &= \frac{1}{\ell_n(A)} c \int_{\pi^\star(A)} d\ell_{n-k}(Y) \int_{A \cap Y} v d\ell_k \\ &= \frac{c}{\ell_n(A)} \int_{\pi^\star(A)} d\ell_{n-k}(Y) \ell_k(A \cap Y) v_{A \cap Y}. \end{aligned} \tag{30}$$

Using the facts $\pi^\star(A) = \pi^\star(B)$ and $\ell_k(A \cap Y)/\ell_k(B \cap Y) = r^k$ together with (29) and (30) we obtain that

$$\begin{aligned} \frac{1}{\ell_n(A)} \int_A v d\ell_n &\le \frac{1}{\ell_n(A)} c \int_{\pi^\star(B)} d\ell_{n-k}(Y) r^k \ell_k(B \cap Y) v_{B \cap Y} \\ &= \frac{r^k}{\ell_n(A)} c \int_{\pi^\star(B)} d\ell_{n-k}(Y) \int_{B \cap Y} v d\ell_k = \frac{r^k}{\ell_n(A)} \int_B v d\ell_n. \end{aligned} \tag{31}$$

Now by part (a) of this theorem, we have $m(A) = \ell_n(A) = r^k \ell_n(B)$. Therefore one can deduce from (31) that

$$v_A = \frac{1}{\ell_n(A)} \int_A v d\ell_n \le \frac{1}{\ell_n(B)} \int_B v d\ell_n = v_B.$$

□

It is possible to use the previous method repeatedly.

Corollary 6.6. *Suppose that A and B are bounded sets of positive measure in $E = \mathbf{R}^n$ and that there is a chain of sets $B_0, B_1, ..., B_m$ such that $B_0 = B$ and $B_m = A$ and $B_{j+1} = C_{X_j,r_j}(B_j)$ where X_j is a proper subspace of E of dimension at least one and $0 < r_j < 1$ for $j = 0, ..., m-1$. Then $A \prec B$.*

It will be shown, with the aid of an example, that the operation $C_{X,r}$ does not preserve the convexity.

Example 6.7. Let $E = \mathbf{R}^2$ and let B be the isosceles triangle $B = \triangle(P_1P_2P_3)$, where $P_1 = (0,-1), P_2 = (0,1)$ and $P_3 = (1,0)$. It is easy to see that $A = C_{X,1/2}(B)$ is not convex, where the subspace X is determined by the equation $y = 0$ in E.

Conjecture 6.8. *Let $n \geq 2$ and let $E = \mathbf{R}^n$. Let B be a bounded set in E of positive Lebesgue measure. If for all proper subspaces X of E of dimension at least one and for all $0 < r < 1$ the set $C_{X,r}(B)$ is convex, then B is an ellipsoid.*

An application of Corollary 6.6 will be given next.

Theorem 6.9. *Let $e_1, e_2, ..., e_n$ be any basis of $E = \mathbf{R}^n$. Define for any positive number p and for any n-tuple of positive numbers $(t_1, t_2, ..., t_n)$ the set*

$$A_p^{t_1,t_2,...,t_n} := \{x = \sum_{i=1}^{n} x_i e_i : \sum_{i=1}^{n} |x_i|^p t_i^p \leq 1\}. \tag{32}$$

Then

$$A_p^{t_1,t_2,...,t_n} \prec A_p^{s_1,s_2,...,s_n} \quad \text{if} \quad t_1 \geq s_1, t_2 \geq s_2, ..., t_n \geq s_n.$$

Proof. The method of the chain of operations described by Corollary 6.6 will be used. Let $B_0 = A_p^{s_1,s_2,...,s_n}$ and let $B_j = A_p^{t_1,..,t_j,s_{j+1},..s_n}$ for $1 \leq j \leq n-1$ and $B_n = A_p^{t_1,...,t_n}$. The operations C_{X_j,r_j} for $j = 0, ..., n-1$ are determined by the one-dimensional subspaces $X_j = \{xe_{j+1} : x \in \mathbf{R}\}$ and the factors $r_j = s_{j+1}/t_{j+1}$ for $j = 0, 1, ..., n-1$. When $r_j = 1$, it is assumed that $C_{X_j,r_j}(B_j) = B_j = B_{j+1}$. Let Y be a coset of X_j for some $0 \leq j \leq n-1$ such that the one-dimensional Lebesgue measure of $Y \cap B_j$ is positive. This coset Y is a translation of X_j by an at most (n-1)-dimensional vector q_j spanned by the members of the basis but e_{j+1}. It follows from the definition of B_j that the barycenter of $Y \cap B_j$ with respect to Y is the vector q_j. It can be seen by a substitution that $C_{x_j,r_j}(B_j) = B_{j+1}$ for $j = 0, \ldots, n-1$. □

Remark 6.10. Consider the special case $E = \mathbf{R}^2$ and assume that E is equipped with the standard inner product and that a basis has been chosen for E consisting of a pair of orthogonal vectors e_1 and e_2. Let $p = 1$ and let $t_1 > 0$ and $t_2 > 0$. It follows from (32) that the set $A_1^{t_1,t_2}$ is a rhombus. Now, Theorem 6.9 implies that if $A = A_1^{t_1,t_2}$ and $B = A_1^{s_1,s_2}$ are two rhombi such that $A \subset B$, then $A \prec B$. Equivalently, if A and B are rhombi with the same barycenter and if the directions of the diagonals of A are the same as the directions of the diagonal of B, then $A \prec B$ if and only if $A \subset B$. Hence, the relation $A \prec B$ can hold for two rhombi without their sides being parallel.

Now, we shall show the main result of this section.

Theorem 6.11. *Let A and B be compact ellipsoids in $\mathbf{R}^n$ with common barycenter $\overline{x}_A = \overline{x}_B$. Then $A \prec B$, if and only if $A \subset B$.*

Proof. One can deduce from (iii) of Theorem 3.1 that the condition is necessary. To show that the condition is sufficient, assume that $A \subset B$. Let $e_1, e_2, ..., e_n$ be the standard basis of $\mathbf{R}^n$. There is an affine isomorphism $T : \mathbf{R}^n \to \mathbf{R}^n$ such that $T(\overline{x}_A) = T(\overline{x}_B) = 0$, and $T(A)$ is the unit ball. It follows that $T(A) \subset T(B)$. Furthermore, there is an isometry U of $\mathbf{R}^n$ which brings $U \circ T(B)$ into its canonical form, i.e.

$$U \circ T(B) = \{(x_1, \ldots, x_n) : \sum_{j=1}^{n} x_j^2 s_j^2 \leq 1\}.$$

Since $U \circ T(A) \subset U \circ T(B)$, one obtains by Theorem 6.9 that $U \circ T(A) \prec U \circ T(B)$. Theorem 3.12 indicates that T^{-1} and U^{-1} are preserving the relation $\prec$. Therefore, one can conclude that $A \prec B$. □

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